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ON THE GRAPHS WITH DISTINGUISHING NUMBER EQUAL LIST DISTINGUISHING NUMBER

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Abstract. The distinguishing number $D(G)$ of a graph G is the least integer d such that G has a vertex labeling with d labels that is preserved only by the trivial automorphism. A list assignment to G is an assignment $L = \{L(v)\}_{v\in V(G)}$ of lists of labels to the vertices of G. A distinguishing L-labeling of G is a distinguishing labeling of G where the label of each vertex v comes from $L(v)$. The list distinguishing number of G, denoted by $D_l(G)$, is the minimum k such that every list assignment to G in which $|L(v)| = k$ for all $v \in V(G)$ yields a distinguishing L-labeling of G. In this paper, we determine the list-distinguishing number for two families of graphs. We also characterize all graphs with the distinguishing number equal the list distinguishing number. Finally, we show that this characterization works for other list numbers of a graph.

Keywords: Distinguishing number; list-distinguishing labeling; list distinguishing chromatic number.
2020 MSC: Primary 05C15, 05E18.

1. Introduction

Let $G = (V, E)$ be a simple graph. We use the standard graph notation ([11]). The graph coloring problem (GCP) models a wide range of planning problems such as timetabling, scheduling, electronic bandwidth allocation and sequencing. Some applications impose additional constraints to the GCP, giving rise to known variants such as equitable coloring, precoloring extension, and list coloring, see e.g. [3,16]. Actually, list coloring generalizes GCP and precoloring extension and has several specific applications such as channel allocation in wireless networks [18].

The set of all *automorphisms* of G, with the operation of composition of permutations, is a permutation group on V and is denoted by $\mathrm{Aut}(G)$. A labeling of G, $\phi : V \to \{1, 2, \ldots, r\}$, is r-*distinguishing*, if no non-trivial automorphism of G preserves all of the vertex labels. In other words, ϕ is r-distinguishing if for every non-trivial $\sigma \in \mathrm{Aut}(G)$, there exists x in V such that $\phi(x) \neq \phi(\sigma(x))$. The *distinguishing number* of a graph G is the minimum number r such that

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G has a labeling that is r-distinguishing; this was defined in [1]. The introduction of the distinguishing number was a great success; by now about one hundred papers have been written motivated by this seminal paper. The core of the research has been done on the invariant itself, either on finite [4, 12, 15] or infinite graphs [17]; see also the references therein. Similar to the distinguishing number, Kalinowski and Pilśniak [14] have defined the distinguishing index $D'(G)$ of G which is the least integer d such that G has an edge colouring with d colours that is preserved only by a trivial automorphism. If a graph has no nontrivial automorphisms, its distinguishing number is 1. In other words, $D(G) = 1$ for the asymmetric graphs. The other extreme, $D(G) = |V(G)|$, occurs if and only if $G = K_n$. The distinguishing index of some examples of graphs was exhibited in [14]. For instance, $D(P_n) = D'(P_n) = 2$ for every $n \geqslant 3$, and $D(C_n) = D'(C_n) = 3$ for $n = 3, 4, 5$, $D(C_n) = D'(C_n) = 2$ for $n \geqslant 6$. It is easy to see that the value $|D(G) - D'(G)|$ can be large. For example $D'(K_{p,p}) = 2$ and $D(K_{p,p}) = p + 1$, for $p \geq 4$.

In 1979, Erdős, Rubin and Taylor [8] introduced a beautiful new direction of graph labeling. We say that a graph G is *k-choosable* if for any assignment of a set (or "list") L_v of k labels to each vertex v of G, it is possible to select a label $\lambda_v \in L_v$ for each v so that $\lambda_u \neq \lambda_v$ if u and v are adjacent. The *list chromatic number* $\chi_l(G)$ is defined to be the least k such that G is k-choosable.

Motivated by list coloring, Ferrara et al. [9] extended the notion of the distinguishing labeling to the list distinguishing labeling. A *list assignment* to G is an assignment $L = \{L(v)\}_{v \in V(G)}$ of lists of labels to the vertices of G. A *distinguishing L-labeling* of G is a distinguishing labeling of G where the label of each vertex v comes from $L(v)$. The *list distinguishing number* of G, $D_l(G)$ is the minimum k such that every list assignment to G in which $|L(v)| = k$ for all $v \in V(G)$ yields a distinguishing L-labeling of G. Since all of the lists can be identical, we observe that $D(G) \leq D_l(G)$. In some cases, it is easy to show that the list distinguishing number can be equal to the distinguishing number. For example, it is not difficult to see that $D(K_n) = n = D_l(K_n)$, $D(K_{n,n}) = n + 1 = D_l(K_{n,n})$ and $D_l(C_n) = D(C_n) = 2$ ([9]). In particular, Ferrara et al. [10] extended an enumerative technique of Cheng [6], to show that for any tree T, $D_l(T) = D(T)$. Ferrara et al. [9] asked the following question at the end of their paper.

Question Does there exist a graph G such that $D(G) \neq D_l(G)$?

Immel and Wenger in [12] proved the following result:

Theorem 1.1. *If G is an interval graph, then $D_l(G) = D(G)$.*

Recently, authors in [5] proved that when a connected graph G is prime with respect to the Cartesian product then $D_l(G^r) = D(G^r)$ for $r \geq 3$, where G^r is the Cartesian product of the graph G taken r times.

In this paper we first compute the list-distinguishing number for friendship and book graphs. We also state a necessary and sufficient condition for a graph

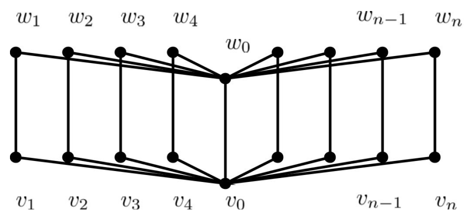

FIGURE 1. Book graph B_n.

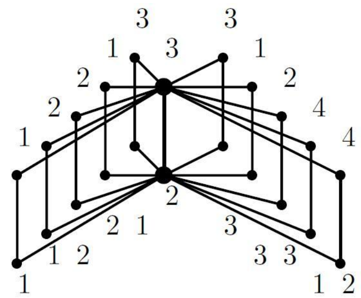

FIGURE 2. The vertex labeling of B_{10}.

G satisfying $D_l(G) = D(G)$ in Section 3. Finally in Section 4, we extend the results of Section 3 for other list numbers of a graph.

2. List distinguishing number of friendship and book graphs

In this section, we consider the friendship and book graphs and compute their list-distinguishing number. First we consider the book graph. The n-book graph, denoted by B_n, is defined as the Cartesian product of $K_{1,n}$ and P_2, i.e. $K_{1,n}\Box P_2$ (Figure 1). Every cycle C_4 in B_n is called a page and all pages have a common edge, namely v_0w_0. The vertices v_0 and w_0 are called central vertices. The distinguishing number of B_n was computed in [2], and we shall show that $D(B_n) = D_l(B_n)$.

Theorem 2.1. [2] *For every $n \geq 2$, $D(B_n) = \lceil \sqrt{n} \rceil$.*

Theorem 2.2. *For every $n \geq 2$, $D_l(B_n) = D(B_n) = \lceil \sqrt{n} \rceil$.*

Proof. It is sufficient to prove that $D_l(B_n) \leq D(B_n)$. For this purpose, we suppose that $L = \{L(v)\}_{v \in V(B_n)}$ is an arbitrary list assignment to B_n in which $|L(v)| = D(B_n)$ for all $v \in V(B_n)$. We find a distinguishing labeling of B_n such that the label of each vertex v comes from $L(v)$. In Figure 1, the set of all ordered pairs (a_i, b_i) such that $a_i \in L(v_i)$ and $b_i \in L(w_i)$ is denoted by

$(L(v_i), L(w_i))$ for every $1 \leq i \leq n$. In fact,

$$(L(v_i), L(w_i)) = \{(a_i, b_i) \mid a_i \in L(v_i) \text{ and } b_i \in L(w_i)\}.$$

It is clear that $|(L(v_i), L(w_i))| \geq (D(B_n))^2$. On the other hand $D(B_n) = \lceil\sqrt{n}\rceil$, and so $n \leq (D(B_n))^2$. Hence for any $1 \leq i \leq n$, we can assign an element of $(L(v_i), L(w_i))$, say (a_i, b_i), to the (v_i, w_i) such that $(a_i, b_i) \neq (a_j, b_j)$ for every $i, j \in \{1, \ldots, n\}$ with $i \neq j$. We label the vertices v_0 and w_0 with two different labels in $L(v_0)$ and $L(w_0)$, respectively. Hence, these two vertices cannot be mapped to each other which shows that they should be fixed under all automorphisms of B_n. In fact, if f is an automorphism of B_n preserving the labeling, then $f(v_0) = v_0$ and $f(w_0) = w_0$, because the label of v_0 and w_0 are distinct. Thus f maps the set $\{(v_i, w_i) : 1 \leq i \leq n\}$ to itself. But since $(a_i, b_i) \neq (a_j, b_j)$ for every $i, j \in \{1, \ldots, n\}$ with $i \neq j$, so the image of the set (v_i, w_i) by f is (v_i, w_i), for every $1 \leq i \leq n$. Since f preserves the adjacency relation and $f(v_0) = v_0$ and $f(w_0) = w_0$, so $f(v_i) = v_i$ and $f(w_i) = w_i$ for every $1 \leq i \leq n$. Then f is the identity automorphism of B_n and the labeling is distinguishing. Therefore we have the result. □

An example for the distinguishing coloring of B_{10} has shown in Figure 2.

Now we consider the friendship graph. The friendship graph F_n $(n \geq 2)$ can be constructed by intersecting n copies of C_3 at a common vertex (Figure 3).

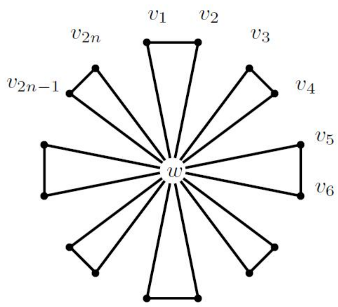

FIGURE 3. Friendship graph F_n.

Theorem 2.3. [2] *For every $n \geq 2$, $D(F_n) = \left\lceil \frac{1+\sqrt{8n+1}}{2} \right\rceil$.*

Note that the friendship graph F_n is an interval graph and so by Theorem 1.1, $D_l(F_n) = D(F_n)$. Also using the same techniques of the proof of Theorem 2.2 we can have the following result.

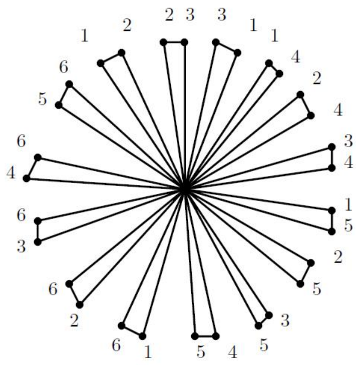

FIGURE 4. The vertex labeling of F_{15}.

Theorem 2.4. *For every $n \geq 2$, $D_l(F_n) = D(F_n) = \left\lceil \frac{1+\sqrt{8n+1}}{2} \right\rceil$.*

An example for the distinguishing coloring of F_{15} has shown in Figure 4.

3. Characterization of graphs G with $D(G) = D_l(G)$

In this section we shall obtain a necessary and sufficient condition for a graph G such that $D(G) = D_l(G)$. To do this, first we need to state some notations and results from set theory in subsection 3.1. In Subsection 3.2 we characterize all graphs G whose distinguishing and list distinguishing number are equal.

3.1. **Some notations and results from set theory.** We begin this subsection with the following definition:

Definition 3.1. Let $f : \{a_1, \dots, a_n\} \to \{b_1, \dots, b_d\}$, $d \leq n$, be a function. An (m, d)-related sequence to f is $L^{(f)} = \{L_i\}_{i=1}^n$ such that $f(a_i) \in L_i$, $|L_i| = d$ and $L_i \subseteq \{b_1, \dots, b_m\}$ for every $1 \leq i \leq n$ where $m \geq d$.

It is clear that we have $\binom{m-1}{d-1}^n$, (m, d)-related sequences to f. If the set of all these sequences is denoted by $\mathcal{L}_{(m,d)}^{\{a_1,\dots,a_n\}}(f)$, then $|\mathcal{L}_{(m,d)}^{\{a_1,\dots,a_n\}}(f)| = \binom{m-1}{d-1}^n$.

Definition 3.2. Let f_i, $1 \leq i \leq t$, be functions of $\{a_1, \dots, a_n\}$ into d-subsets of $\{b_1, \dots, b_m\}$, where $d \leq n$ and $d \leq m$. If $B_{(m,d)}^{\{a_1,\dots,a_n\}}(f_1, \dots, f_t) = \bigcup_{i=1}^t \mathcal{L}_{(m,d)}^{\{a_1,\dots,a_n\}}(f_i)$, then we say that $B_{(m,d)}^{\{a_1,\dots,a_n\}}(f_1, \dots, f_t)$ is the set of all possible different (m, d)-related sequences to $f_1, \dots, f_t$.

It is clear that $|B_{(d,d)}^{\{a_1,\ldots,a_n\}}(f_1,\ldots,f_t)| = 1$. To obtain a characterization of graphs G with $D(G) = D_l(G)$, we need to know the number of elements in $B_{(m,d)}^{\{a_1,\ldots,a_n\}}(f_1,\ldots,f_t)$ for any $m \geq d$. We introduce the following notation to simplify this calculation.

Notation. Let f_i, $1 \leq i \leq t$, be functions of $\{a_1,\ldots,a_n\}$ into d-subsets of $\{b_1,\ldots,b_m\}$, where $d \leq n$ and $d \leq m$. Let $f_{j_1}, f_{j_2},\ldots, f_{j_{i-1}}, f_{j_i}$ be i different functions such that $1 \leq j_1 < j_2 < \cdots < j_{i-1} < j_i \leq t$. For every $1 \leq p \leq i$, we define

$$(1)\quad n^{(p)}_{\{j_1,\ldots,j_{i-1},j_i\}} = \big|\{a \in \{a_1,\ldots,a_n\} :\ |\{f_{j_1}(a),\ldots,f_{j_{i-1}}(a), f_{j_i}(a)\}| = p\}\big|.$$

Proposition 3.3. *Let $f_1,\ldots,f_t$ be functions of $\{a_1,\ldots,a_n\}$ into d-subsets of $\{b_1,\ldots,b_m\}$, where $d \leq n$ and $d \leq m$. Then $|B_{(m,d)}^{\{a_1,\ldots,a_n\}}(f_1,\ldots,f_t)| = \sum_{i=1}^{t} S_i$, in which $S_1 = \binom{m-1}{d-1}^n$, and for every $i \geq 2$ we have*

$$S_i = \binom{m-1}{d-1}^n -$$
$$\sum_{j=1}^{i-1} \binom{m-2}{d-2}^{n^{(2)}_{\{j,i\}}} \binom{m-1}{d-1}^{n^{(1)}_{\{j,i\}}} +$$
$$\sum_{1\leq j<k<i} \binom{m-3}{d-3}^{n^{(3)}_{\{j,k,i\}}} \binom{m-2}{d-2}^{n^{(2)}_{\{j,k,i\}}} \binom{m-1}{d-1}^{n^{(1)}_{\{j,k,i\}}} -$$
$$\sum_{1\leq j<k<h<i} \binom{m-4}{d-4}^{n^{(4)}_{\{j,k,h,i\}}} \binom{m-3}{d-3}^{n^{(3)}_{\{j,k,h,i\}}} \binom{m-2}{d-2}^{n^{(2)}_{\{j,k,h,i\}}} \binom{m-1}{d-1}^{n^{(1)}_{\{j,k,h,i\}}} +$$
$$\vdots$$
$$+ (-1)^{i+1} \binom{m-i}{d-i}^{n^{(i)}_{\{1,2,\ldots,i-1,i\}}} \binom{m-(i-1)}{d-(i-1)}^{n^{(i-1)}_{\{1,\ldots,i-1,i\}}} \cdots \binom{m-1}{d-1}^{n^{(1)}_{\{1,\ldots,i-1,i\}}}.$$

Proof. It is clear that the number of different sequences related to the function f_1 is $S_1 = \binom{m-1}{d-1}^n$. For the function f_2, there are $\binom{m-1}{d-1}^n$ related sequences, but some of these sequences are exactly the same as related sequences to f_1, i.e., $\mathcal{L}_{(m,d)}^{\{a_1,\ldots,a_n\}}(f_1) \cap \mathcal{L}_{(m,d)}^{\{a_1,\ldots,a_n\}}(f_2) \neq \emptyset$. Using notation (1), we conclude that

$$|\mathcal{L}_{(m,d)}^{\{a_1,\ldots,a_n\}}(f_1) \cap \mathcal{L}_{(m,d)}^{\{a_1,\ldots,a_n\}}(f_2)| = \binom{m-2}{d-2}^{n^{(2)}_{\{1,2\}}} \binom{m-1}{d-1}^{n^{(1)}_{\{1,2\}}}.$$

Thus there exist S_2 new sequences related to f_2 by the inclusion-exclusion principle, where

$$S_2 = \binom{m-1}{d-1}^n - \binom{m-2}{d-2}^{n^{(2)}_{\{1,2\}}} \binom{m-1}{d-1}^{n^{(1)}_{\{1,2\}}}.$$

By a similar argument, for the function f_3, there are $\binom{m-1}{d-1}^n$ related sequences, but some of them are exactly the same as related sequences to f_1 or f_2. Using notation (1), we obtain that

- $|\mathcal{L}^{\{a_1,\dots,a_n\}}_{(m,d)}(f_1) \cap \mathcal{L}^{\{a_1,\dots,a_n\}}_{(m,d)}(f_3)| = \binom{m-2}{d-2}^{n^{(2)}_{\{1,3\}}} \binom{m-1}{d-1}^{n^{(1)}_{\{1,3\}}}$.
- $|\mathcal{L}^{\{a_1,\dots,a_n\}}_{(m,d)}(f_2) \cap \mathcal{L}^{\{a_1,\dots,a_n\}}_{(m,d)}(f_3)| = \binom{m-2}{d-2}^{n^{(2)}_{\{2,3\}}} \binom{m-1}{d-1}^{n^{(1)}_{\{2,3\}}}$.
- $|\mathcal{L}^{\{a_1,\dots,a_n\}}_{(m,d)}(f_1) \cap \mathcal{L}^{\{a_1,\dots,a_n\}}_{(m,d)}(f_2) \cap \mathcal{L}^{\{a_1,\dots,a_n\}}_{(m,d)}(f_3)| = \binom{m-3}{d-3}^{n^{(3)}_{\{1,2,3\}}} \binom{m-2}{d-2}^{n^{(2)}_{\{1,2,3\}}} \binom{m-1}{d-1}^{n^{(1)}_{\{1,2,3\}}}$.

Hence there exist S_3 new sequences related to f_3 by the inclusion-exclusion principle, where

$$S_3 = \binom{m-1}{d-1}^n - \left(\binom{m-2}{d-2}^{n^{(2)}_{\{1,3\}}} \binom{m-1}{d-1}^{n^{(1)}_{\{1,3\}}} + \binom{m-2}{d-2}^{n^{(2)}_{\{2,3\}}} \binom{m-1}{d-1}^{n^{(1)}_{\{2,3\}}} \right) + \binom{m-3}{d-3}^{n^{(3)}_{\{1,2,3\}}} \binom{m-2}{d-2}^{n^{(2)}_{\{1,2,3\}}} \binom{m-1}{d-1}^{n^{(1)}_{\{1,2,3\}}}.$$

In general, for the function f_i, there are $\binom{m-1}{d-1}^n$ related sequences, but some of them are exactly the same as related sequences to $f_1, \dots, f_{i-1}$. Using notation (1) and the inclusion-exclusion principle, we conclude that there exist S_i new sequences related to f_i where

$$S_i = \binom{m-1}{d-1}^n -$$
$$\sum_{j=1}^{i-1} \binom{m-2}{d-2}^{n^{(2)}_{\{j,i\}}} \binom{m-1}{d-1}^{n^{(1)}_{\{j,i\}}} +$$
$$\sum_{1\le j<k<i} \binom{m-3}{d-3}^{n^{(3)}_{\{j,k,i\}}} \binom{m-2}{d-2}^{n^{(2)}_{\{j,k,i\}}} \binom{m-1}{d-1}^{n^{(1)}_{\{j,k,i\}}} -$$
$$\sum_{1\le j<k<h<i} \binom{m-4}{d-4}^{n^{(4)}_{\{j,k,h,i\}}} \binom{m-3}{d-3}^{n^{(3)}_{\{j,k,h,i\}}} \binom{m-2}{d-2}^{n^{(2)}_{\{j,k,h,i\}}} \binom{m-1}{d-1}^{n^{(1)}_{\{j,k,h,i\}}} +$$
$$\vdots$$
$$+ (-1)^{i+1} \binom{m-i}{d-i}^{n^{(i)}_{\{1,\dots,i-1,i\}}} \binom{m-(i-1)}{d-(i-1)}^{n^{(i-1)}_{\{1,\dots,i-1,i\}}} \cdots \binom{m-1}{d-1}^{n^{(1)}_{\{1,\dots,i-1,i\}}}.$$

Therefore, $|B^{\{a_1,\dots,a_n\}}_{(m,d)}(f_1,\dots,f_t)| = \sum_{i=1}^t S_i$. □

Here, we present another method for computing $|B^{\{a_1,\dots,a_n\}}_{(m,d)}(f_1,\dots,f_t)|$ by a recurrence relation.

Proposition 3.4. *Let $f_1,\dots,f_t$ be functions of $\{a_1,\dots,a_n\}$ into d-subsets of $\{b_1,\dots,b_m\}$, where $d \le n$ and $d \le m$. Then $|B^{\{a_1,\dots,a_n\}}_{(m+1,d)}(f_1,\dots,f_t)|$ is equal to*

$$\sum_{i=0}^{n} \sum_{q=1}^{\binom{n}{i}} |B^{\{a_{q_1},\dots,a_{q_i}\}}_{(m,d-1)}(f'_{q1},\dots,f'_{qt})||B^{\{a_{q_{i+1}},\dots,a_{q_n}\}}_{(m,d)}(f''_{q1},\dots,f''_{qt})|,$$

where $f'_{q1},\dots,f'_{qt}$ are the restrictions of $f_1,\dots,f_t$ to the q-th i-subset of $\{a_1,\dots a_n\}$, say $\{a_{q_1},\dots,a_{q_i}\}$, and $f''_{q1},\dots,f''_{qt}$ are the restrictions of $f_1,\dots,f_t$ to the set $\{a_1,\dots a_n\} \setminus \{a_{q_1},\dots,a_{q_i}\} = \{a_{q_{i+1}},\dots,a_{q_n}\}$.

Proof. Let $f_1,\dots,f_t$ be functions of $\{a_1,\dots,a_n\}$ into d-subsets of $\{b_1,\dots,b_{m+1}\}$, where $d \le n$ and $d \le m+1$. Let $A_q = \{a_{q_1},\dots,a_{q_i}\}$ be q-th i-subset of $\{a_1,\dots,a_n\}$ where $0 \le i \le n$ and $1 \le q \le \binom{n}{i}$ for which $\{a_1,\dots,a_n\} \setminus A_q = \{a_{q_{i+1}},\dots,a_{q_n}\}$. If $f'_{q_1},\dots,f'_{q_t}$ are the restriction of $f_1,\dots,f_t$ to the set A_q, then by Definition 3.2, there exist $|B^{\{a_{q_1},\dots,a_{q_i}\}}_{(m,d-1)}(f'_{q_1},\dots,f'_{q_t})|$, (m,d)-related sequences $L' = \{L'_k\}_{k=q_1}^{q_i}$ with $|L'_k| = d$ and $L'_k \subseteq \{b_1,\dots,b_{m+1}\}$ such that $b_{m+1} \in L'_k$ for every $k \in \{q_1,\dots,q_i\}$. If $f''_{q_1},\dots,f''_{q_t}$ are the restriction of $f_1,\dots,f_t$ to the set $\{a_{q_{i+1}},\dots,a_{q_n}\}$, then there exist $|B^{\{a_{q_{i+1}},\dots,a_{q_n}\}}_{(m,d)}(f''_{q_1},\dots,f''_{q_t})|$, (m,d)-related sequences $L'' = \{L''_k\}_{k=q_{i+1}}^{q_n}$ with $|L''_k| = d$ and $L''_k \subseteq \{b_1,\dots,b_m\}$

for every $k \in \{q_{i+1}, \ldots, q_n\}$. Since $L = \{L_k\}_{k=1}^n$ where $L_k = L'_k$ for $k \in \{q_1, \ldots, q_i\}$, and $L_k = L''_k$ for $k \in \{q_{i+1}, \ldots, q_n\}$, is an element of $B_{(m+1,d)}^{\{a_1,\ldots,a_n\}}(f_1, \ldots, f_t)$, and every element of $B_{(m+1,d)}^{\{a_1,\ldots,a_n\}}(f_1, \ldots, f_t)$ is obtained in such a way, we have the result by the rules of product and sum. □

3.2. **Graph G with $D(G) = D_l(G)$.** In this subsection, we present a necessary and sufficient condition for a graph G with $D(G) = D_l(G)$.

Let G be a graph with $V(G) = \{a_1, \ldots, a_n\}$ and $D(G) = d$. Suppose that $L = \{L_i\}_{i=1}^n$ is an arbitrary sequence such that $|L_i| = d$ and $L_i \subseteq \{1, \ldots, m\}$ for some $m \geq d$ and every $1 \leq i \leq n$. If L is a distinguishing L-labeling of G then there exists a distinguishing labeling C of vertices of G such that $C(a_i) \in L_i$ for all $1 \leq i \leq n$. On the other hand, for every distinguishing labeling C, we can construct $\binom{m-1}{d-1}^n$ sequences $L^{(C)} = \{L_i^{(C)}\}_{i=1}^n$ such that $C(a_i) \in L_i^{(C)}$, $|L_i^{(C)}| = d$ and $L_i^{(C)} \subseteq \{1, \ldots, m\}$ for every $1 \leq i \leq n$. We call such sequences the (m, d)-*related sequences* to C. If we denote the set of all related sequences to C by $\mathcal{L}_{(m,d)}^{\{a_1,\ldots,a_n\}}(C)$, then $|\mathcal{L}_{(m,d)}^{\{a_1,\ldots,a_n\}}(C)| = \binom{m-1}{d-1}^n$. Let $\mathcal{L}(G, m)$ be the set of all distinguishing labeling of G with at most m labels $\{1, \ldots, m\}$. Set $\mathcal{L}(G, m) = \{C_1, \ldots, C_{t_m}\}$. We suppose that $B_{(m,d)}^{\{a_1,\ldots,a_n\}}(C_1, \ldots, C_{t_m})$ is the set of all those sequences $L = \{L_i\}_{i=1}^n$ such that $|L_i| = d$ and $L_i \subseteq \{1, \ldots, m\}$ which are constructed using the distinguishing labelings in $\mathcal{L}(G, m)$, i.e., $B_{(m,d)}^{\{a_1,\ldots,a_n\}}(C_1, \ldots, C_{t_m}) = \bigcup_{i=1}^{t_m} \mathcal{L}_{(m,d)}^{\{a_1,\ldots,a_n\}}(C_i)$. By these statements we have the following proposition:

Proposition 3.5. *An arbitrary sequence $L = \{L_i\}_{i=1}^n$ with $|L_i| = d$ and $L_i \subseteq \{1, \ldots, m\}$, is a distinguishing L-labeling of G, if and only if $L \in B_{(m,d)}^{\{a_1,\ldots,a_n\}}(C_1, \ldots, C_{t_m})$.*

If the set of all sequences $L = \{L_i\}_{i=1}^n$ with $|L_i| = d$ and $L_i \subseteq \{1, \ldots, m\}$ for $m \geq d$ and $1 \leq i \leq n$ is denoted by A, then $|A| = \binom{m}{d}^n$. It is clear that $B_{(m,d)}^{\{a_1,\ldots,a_n\}}(C_1, \ldots, C_{t_m}) \subseteq A$. We can compute $|B_{(m,d)}^{\{a_1,\ldots,a_n\}}(C_1, \ldots, C_{t_m})|$, by Propositions 3.3 and 3.4. The following theorem gives the characterization of graph G with $D(G) = D_l(G)$.

Theorem 3.6. *For a graph G with $V(G) = \{a_1, \ldots, a_n\}$ and $D(G) = d$, we have:*

(i) $D_l(G) = \min\{d : |B_{(m,d)}^{\{a_1,\ldots,a_n\}}(C_1, \ldots, C_{t_m})| \geq \binom{m}{d}^n$ *for all* $m \geq d\}$.

(ii) $D_l(G) = D(G) = d$ *if and only if* $|B_{(m,d)}^{\{a_1,\ldots,a_n\}}(C_1, \ldots, C_{t_m})| \geq \binom{m}{d}^n$ *for all* $m \geq d$.

Example 3.7. *Let P_n $(n \geq 4)$ be a path graph. We know that the two end-vertices of P_n either have different colors or the same color. Consider the path P_4. The set of all distinguishing labeling of this graph is $L = \{L_1 = \{1,1,1,2\}, L_2 = \{1,2,2,2\}, L_3 = \{1,1,2,1\}, L_4 = \{1,2,1,1\}, L_5 =$*

$\{2,1,2,2\}, L_6 = \{2,2,1,2\}\}$. *If* $m=2$, *then here* $|B_{(2,d)}| = 6$ *and by Theorem 3.6, we have* $6 \geq \binom{2}{d}^4$. *So* $(d!(2-d)!)^4 \geq \frac{8}{3}$. *Therefore* $D_l(P_4) = \min\{d : (d!(2-d)!)^4 \geq \frac{8}{3}\}$ *and so* $D_l(P_4) = 2$.

4. List chromatic number and list distinguishing chromatic number

In this section, we show that we can apply the results of Section 3 for other list numbers, including the list chromatic number χ_l, and list distinguishing chromatic number χ_{D_l}. Clearly, $\chi_l(G) \geq \chi(G)$ for any G (by taking $L_v = \{1,2,\ldots,\chi(G)\}$ for all v). However, the difference between $\chi_l(G)$ and $\chi(G)$ can be arbitrarily large (note that $\chi_l(K_{a,a^a}) = a+1$, but $\chi(K_{a,a^a}) = 2$) (see e.g., [13]). In general, χ_l can not be bounded in terms of χ. In 2006 Collins and Trenk [7] introduced the *distinguishing chromatic number* $\chi_D(G)$ of a graph G, as the minimum number of labels in a distinguishing labeling of G that is also a proper labeling. While not explicitly introduced in [9], it is natural to consider a list analogue of the distinguishing chromatic number. Ferrara et al. [10] say that G has a *proper distinguishing L-labeling* if there is a distinguishing labeling f of G chosen from the lists such that f is also a proper labeling of G. The *list distinguishing chromatic number* $\chi_{D_l}(G)$ of G is the minimum integer k such that G is properly L-distinguishable for any assignment L of lists with $|L_v| = k$ for all v.

Let G be a graph with $V(G) = \{a_1,\ldots,a_n\}$ and the distinguishing chromatic number (resp. chromatic number) $\chi_D(G) = d$ (resp. $\chi(G) = d$). Let $L = \{L_i\}_{i=1}^n$ be an arbitrary sequence such that $|L_i| = d$ and $L_i \subseteq \{1,\ldots,m\}$ for some $m \geq d$ and every $1 \leq i \leq n$. If L is a proper distinguishing L-labeling (resp. proper L-labeling) of G then there exists a proper distinguishing labeling (resp. proper labeling) C of vertices of G such that $C(a_i) \in L_i$ for all $1 \leq i \leq n$. On the other hand, for every proper distinguishing labeling (resp. proper labeling) C with at most m labels, we can construct $\binom{m-1}{d-1}^n$ sequences $L^{(C)} = \{L_i^{(C)}\}_{i=1}^n$ such that $C(v_i) \in L_i^{(C)}$, $|L_i^{(C)}| = d$ and $L_i^{(C)} \subseteq \{1,\ldots,m\}$ for every $1 \leq i \leq n$. We call such sequences the (m,d)-*related sequences* to C. If we denote the set of all related sequences to C by $\mathcal{L}_{(m,d)}^{\{a_1,\ldots,a_n\}}(C)$, then $|\mathcal{L}_{(m,d)}^{\{a_1,\ldots,a_n\}}(C)| = \binom{m-1}{d-1}^n$. Let $\mathcal{L}(G,m)$ be the set of all proper distinguishing labelings (resp. proper labelings) of G with at most m labels $\{1,\ldots,m\}$. Let $B_{(m,d)}^{\{a_1,\ldots,a_n\}}(C_1,\ldots,C_{t_m})$ be the set of all those sequences $L = \{L_i\}_{i=1}^n$ such that $|L_i| = d$ and $L_i \subseteq \{1,\ldots,m\}$ which are constructed using the proper distinguishing labelings (resp. proper labelings) in $\mathcal{L}(G,m)$, i.e., $B_{(m,d)}^{\{a_1,\ldots,a_n\}}(C_1,\ldots,C_{t_m}) = \bigcup_{i=1}^{t_m} \mathcal{L}_{(m,d)}^{\{a_1,\ldots,a_n\}}(C_i)$. Therefore we can conclude the following results by the same argument as Section 3:

Theorem 4.1. *For a graph G with $V(G) = \{a_1,\ldots,a_n\}$ and $\chi_D(G) = d$, we have:*

(i) *an arbitrary sequence* $L = \{L_i\}_{i=1}^{n}$ *with* $|L_i| = d$ *and* $L_i \subseteq \{1, \dots, m\}$ *for every* $1 \leq i \leq n$*, is a proper distinguishing* L*-labeling of* G*, if and only if* $L \in B_{(m,d)}^{\{a_1,\dots,a_n\}}(C_1, \dots, C_{t_m})$*,*

(ii) $\chi_{D_l}(G) = \min\{d : |B_{(m,d)}^{\{a_1,\dots,a_n\}}(C_1, \dots, C_{t_m})| \geq \binom{m}{d}^n$ *for all* $m \geq d\}$*,*

(iii) $\chi_{D_l}(G) = \chi_D(G) = d$ *if and only if* $|B_{(m,d)}^{\{a_1,\dots,a_n\}}(C_1, \dots, C_{t_m})| \geq \binom{m}{d}^n$ *for all* $m \geq d$*.*

As an example for Theorem 4.1, one can consider C_5. Note that $\chi_D(C_5) = 3$. We end this paper with the following theorem:

Theorem 4.2. *For a graph* G *with* $V(G) = \{a_1, \dots, a_n\}$ *and* $\chi(G) = d$*, we have:*

(i) *an arbitrary sequence* $L = \{L_i\}_{i=1}^{n}$ *with* $|L_i| = d$ *and* $L_i \subseteq \{1, \dots, m\}$ *for every* $1 \leq i \leq n$*, is a proper* L*-labeling of* G*, if and only if* $L \in B_{(m,d)}^{\{a_1,\dots,a_n\}}(C_1, \dots, C_{t_m})$*,*

(ii) $\chi_l(G) = \min\{d : |B_{(m,d)}^{\{a_1,\dots,a_n\}}(C_1, \dots, C_{t_m})| \geq \binom{m}{d}^n$ *for all* $m \geq d\}$*,*

(iii) $\chi_l(G) = \chi(G) = d$ *if and only if* $|B_{(m,d)}^{\{a_1,\dots,a_n\}}(C_1, \dots, C_{t_m})| \geq \binom{m}{d}^n$ *for all* $m \geq d$*.*

5. Conclusion

In this paper, we determined the list-distinguishing number for the friendship graph F_n and the book graph B_n. We also characterized all graphs with the distinguishing number equal the list distinguishing number. Our method is dependent to the the number of all (m, d)-related sequences which seems that are not easy to compute. Finding an easier characterization for graphs with the distinguishing number equal the list distinguishing number can be an interesting subject. Also study of the list distinguishing number of graph products are interesting. For instance finding the list distinguishing number of corona product and lexicographic product of two graphs seems as a good area of study.

6. Author Contributions

S.A. presented the presented idea. S.S. developed the theory and performed the computations. All authors discussed the results and contributed to the final manuscript.

7. Data Availability Statement

Not applicable.

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9. Conflict of interest

The authors declare no conflict of interest.

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