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SOME CRITERIA FOR SOLVABILITY AND SUPERSOLVABILITY

Z. Habibi ✉ and M. Hezarjaribi

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Abstract. Denote by G a finite group, by $\mathrm{hsn}(G)$ the harmonic mean Sylow number (eliminating the Sylow numbers that are one) in G and by $\mathrm{gsn}(G)$ the geometric mean Sylow number (eliminating the Sylow numbers that are one) in G. In this paper, we prove that if either $\mathrm{hsn}(G) < 45/7$ or $\mathrm{gsn}(G) < \sqrt[3]{300}$, then G is solvable. Also, we show that if either $\mathrm{hsn}(G) < 24/7$ or $\mathrm{gsn}(G) < \sqrt{12}$, then G is supersolvable.

Keywords: Finite group, Sylow subgroup, solvable groups.
2020 MSC: 20D20, 20D10.

1. Introduction

Let G be a finite group, $\mathrm{S}(G) = \{p \textit{ prime} \mid v_p(G) > 1\}$ and $\pi(G) = \{p \textit{ prime} \mid p \mid |G|\}$, where $v_p(G)$ stands for the number of Sylow p-subgroups of G. We know that $v_p(G) = |G : N_G(P)| = 1 + kp$, where $P \in \mathrm{Syl}_p(G)$ and $k \in \mathbb{N}$. In 1995, Zhang was the first person to observe that knowing the set of Sylow numbers could also restrict the structure of G (see [11]). By using the number of Sylow subgroups, several criteria for p-nilpotency and also solvability of finite groups have been also determined (see [4,8,11]). For example, Zhang and Chigira found an equivalent condition for p-nilpotency (see [4,11]). Robati proved the following theorem:

Theorem A. [10, Corollary 3.4] Let G be a finite group. If $v_p(G) \leq p^2 - p + 1$, for each prime p, then G is solvable.

Anabanti et al. proved the following theorem:

Theorem B. [1, Theorem B] Let G be a finite group. Assume that $v_p(G) \leq p^2 - p + 1$ for $p \in \{3, 5\}$. Then G is solvable.

We consider the function $\mathrm{asn}(G) = \sum_{p \in \mathrm{S}(G)} v_p(G)/|\mathrm{S}(G)|$ introduced in [9].

Theorem C. [9, Theorem 1.1] Let G be a finite group such that $\mathrm{asn}(G) < 7$. Then G is solvable.

✉ z_habibi@pnu.ac.ir, ORCID: 0000-0003-3144-0394
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We encourage the readers to check [2, 3, 7] for more results concerning solvability and supersolvability.

We define the harmonic mean Sylow number and geometric mean Sylow number of G respectively as follow:

$$\mathrm{hsn}(G) = \frac{|\mathrm{S}(G)|}{\sum_{p\in \mathrm{S}(G)} \frac{1}{v_p(G)}},$$

$$\mathrm{gsn}(G) = \sqrt[|\mathrm{S}(G)|]{\prod_{p\in S(G)} v_p(G)}.$$

In this paper, we prove the following theorems:

Theorem 1.1. *Let G be a finite group.*
(a) *If $\mathrm{hsn}(G) < 45/7$, then G is solvable.*
(b) *If $\mathrm{gsn}(G) < \sqrt[3]{300}$, then G is solvable.*

Theorem 1.2. *Let G be a finite group. Then G is a supersolvable group, if one of the following statements holds.*

(1) $\mathrm{gsn}(G) < \sqrt{12}$;
(2) $\mathrm{hsn}(G) < 24/7$.

Observe that

$$\mathrm{asn}(A_5) = 7, \quad \mathrm{gsn}(A_5) = \sqrt[3]{300}, \quad \mathrm{hsn}(A_5) = 45/7,$$
$$\mathrm{asn}(S_4) = 7/2, \quad \mathrm{gsn}(S_4) = \sqrt{12}, \quad \mathrm{hsn}(S_4) = 24/7,$$

so the hypotheses of the above theorems cannot be weakened.

The following lemmas are necessary for the rest of article.

Lemma 1.3. *[6, Theorem 2.1] Let $M \trianglelefteq G$ and $P \in \mathrm{Syl}_p(G)$. Then*

$$v_p(G) = v_p(M)v_p(\frac{G}{M})v_p(\frac{N_{PM}(P\cap M)}{P\cap M}).$$

Lemma 1.4. *[9, Lemma 2.1] Let S be a composition factor of a finite group G. Then $v_p(S) \le v_p(G)$ for every prime p.*

2. Some criteria for solvability and supersolvability of a finite group

Theorem 2.1. *Let G be a finite group.*
(a) *If $\mathrm{hsn}(G) < 45/7$, then G is solvable.*
(b) *If $\mathrm{gsn}(G) < \sqrt[3]{300}$, then G is solvable.*

Proof. On the contrary, let G be a non-solvable group. Therefore G has a non-abelian composition factor S.

We claim that, for any prime divisor p of $|S|$, we have $v_p(S) > 6$.

If $v_p(S) \leq 6$, for some prime divisor p of $|S|$, then S has a proper subgroup of index less than or equal to 6. This leads that S is isomorphic to a subgroup of symmetric group S_6. Consequently, $S = A_5$ or $S = A_6$. Therefore by GAP system [5], we have

$$v_2(A_5) = 5, \quad v_2(A_6) = 45, \quad v_3(A_5) = v_3(A_6) = 10, \quad v_5(A_5) = 6, \quad v_5(A_6) = 36.$$

Using Lemma 1.4, we have

$$v_2(G) \geq 5, \quad v_3(G) \geq 10, \quad v_5(G) \geq 6.$$

On the other hand, for every prime divisor r of $|G|$ such that $r \geq 7$ and $v_r(G) \neq 1$, we have $v_r(G)$ greater than or equal to 8. Thus we obtain that $\mathrm{hsn}(G) \geq 45/7$ and $\mathrm{gsn}(G) \geq \sqrt[3]{300}$, which are contradictions.

Therefore, we can suppose that $v_p(S) \geq 7$, where p is a prime divisor of $|S|$. Burnside's p^aq^b-theorem implies that there exist at least 3 different prime divisors of $|S|$, since S is a nonabelian simple group. The Feit-Thompson theorem implies that 2 is a prime divisor of $|S|$.

If 5 divides $|S|$, then $v_5(G) \geq 7$ and therefore $v_5(G)$ must be at least 11. Because 2 must divide $|S|$ we also have $v_2(G) \geq 7$, and S has at least one more prime divisor by Burnside's p^aq^b-theorem.

If 3 divids $|S|$, then $\mathrm{S}(S)$ is a set of numbers all greater than 7 and clearly both $hsn(G)$ and $gsn(G)$ exceed 7. We may therefore assume 3 does not divide $|S|$ so there is a prime $p \geq 7$ dividing $|S|$. The values for $\mathrm{hsn}(G)$ and $\mathrm{gsn}(G)$ are at least the values of the harmonic and geometric means of the set $\{7, 4, 11, 8\}$. The harmonic and geometric means of this set are larger than the proposed $45/7$ and $\sqrt[3]{300}$.

We may therefore assume 5 does not divide $|S|$ and that $v_5(G) \geq 6$. Again $v_2(G) \geq 7$ and we focus on the prime 3. If 3 does not divide $|S|$, then there are at least two primes other than 2 dividing $|S|$. The values for $\mathrm{hsn}(G)$ and $\mathrm{gsn}(G)$ are then larger than the harmonic and geometric means of the set $\{4, 6, 8, 12\}$, and the harmonic and geometric means of this set are larger than the proposed $45/7$ and $\sqrt[3]{300}$.

If 3 divides $|S|$, then $2 \mid |S|$, $3 \mid |S|$, $5 \nmid |S|$ and there exists $p \geq 7$ such that $p \mid |S|$. Therefore $v_2(G) \geq v_2(S) \geq 7$, $v_3(G) \geq v_3(S) \geq 7$ and $v_p(G) \geq v_p(S) \geq p + 1 \geq 8$.

- If $v_5(G) = 1$, then we will consider the harmonic and geometric means of the set $\{7, 7, 8\}$ as a lower bound for $\mathrm{hsn}(G)$ and $\mathrm{gsn}(G)$.
- If $v_5(G) \neq 1$, then $v_5(G) \geq 6$ we will consider the harmonic and geometric means of the set $\{7, 7, 6, 8\}$ as a lower bound for $\mathrm{hsn}(G)$ and $\mathrm{gsn}(G)$.

Again in the above cases we see the harmonic and geometric means of these sets exceed $45/7$ and $\sqrt[3]{300}$.

The proof is now complete. □

In the following examples, we see that there are some solvable groups which satisfy the hypothesis of Theorem 1.1, but does not satisfy the hypothesis in Theorems A, B and C.

Example 2.2. *Let $G = D_6 \times (C_{13} : C_3)$. Then $v_2(G) = 3$, $v_3(G) = 13$ and $v_{13}(G) = 1$. Therefore*

$$v_p(G) \nleq p^2 - p + 1, \text{ where } p = 3; \qquad \operatorname{asn}(G) = \frac{3+13}{2} = 8 \nless \operatorname{asn}(A_5);$$

$$\operatorname{hsn}(G) = \frac{2}{\frac{1}{3} + \frac{1}{13}} = \frac{39}{8} < \operatorname{hsn}(A_5); \qquad \operatorname{gsn}(G) = \sqrt{3 \cdot 13} < \operatorname{gsn}(A_5).$$

Example 2.3. *Let $G = D_{14} \times A_4 \times (C_{11} : C_5)$. Then $v_2(G) = 7$, $v_3(G) = 4$, $v_5(G) = 11$, $v_7(G) = 1$ and $v_{11}(G) = 1$. Therefore $v_p(G) \nleq p^2 - p + 1$, where $p = 2$ and also*

$$\operatorname{asn}(G) = \frac{7+4+11}{3} = \frac{22}{3} \nless \operatorname{asn}(A_5); \qquad \operatorname{hsn}(G) = \frac{3}{\frac{1}{7} + \frac{1}{4} + \frac{1}{11}} = \frac{924}{149} < \operatorname{hsn}(A_5).$$

Theorem 2.4. *Let G be a finite group. Then G is a supersolvable group, if one of the following statements holds.*

(1) $\operatorname{gsn}(G) < \sqrt{12}$;
(2) $\operatorname{hsn}(G) < 24/7$.

Proof. We consider the following Cases:

- Let $|\mathrm{S}(G)| \geq 2$. Then there exist at least two primes p, q such that $v_p(G) \neq 1$ and $v_q(G) \neq 1$. Without loss of generality we assume that $v_p(G) < v_q(G) < v_r(G)$, for all $r \in \mathrm{S}(G) \setminus \{p, q\}$. Therefore $v_p(G) \geq 3$ and $v_q(G) \geq 4$. Then
  - the harmoic mean v_p and v_q is greater than or equal to 24/7. Then
  $$\operatorname{hsn}(G) \geq 45/7 = \operatorname{hsn}(S_4).$$
  - the geometric mean v_p and v_q is greater than or equal to $\sqrt{12}$. Therefore
  $$\operatorname{gsn}(G) \geq \sqrt{12} = \operatorname{gsn}(S_4).$$

  These are contradiction
- Let $|\mathrm{S}(G)| = 1$. Then there exists exactly one prime p such that $v_p(G) \neq 1$. If $p \geq 3$, then $v_p(G) \geq 4 > 7/2$, we get a contradiction. Therefore $p = 2$ and $v_2 = 3$. By [11, Corollary 7] we get the result.
- Let $|\mathrm{S}(G)| = 0$. Then G is a nilpotent group. Therefore G is a supersolvable group.

The proof is now complete. □

Remark 2.5. The converse of Theorem 2.4 is not hold. For example we know that $S_3 \times S_3$ is a supersolvable group but

$$\operatorname{gsn}(S_3 \times S_3) = \operatorname{hsn}(S_3 \times S_3) = 9 > \max\{\sqrt{12}, 24/7\}.$$

3. Conclusion

We give some new criteria for solvability and supersolvability of finite groups. We proved that if $\mathrm{hsn}(G) < 45/7$ or $\mathrm{gsn}(G) < \sqrt[3]{300}$, then G is solvable; and also we showed that if $\mathrm{gsn}(G) < \sqrt{12}$ or $\mathrm{hsn}(G) < 24/7$, then G is supersolvable.

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Zohreh Habibi
Orcid number: 0000-0003-3144-0394
Department of Mathematics
Payame Noor University (PNU)
Tehran, Iran
Email address: z_habibi@pnu.ac.ir

Masoomeh Hezarjaribi
Orcid number: 0000-0002-3861-8995
Department of Mathematics
Payame Noor University (PNU)
Tehran, Iran
Email address: masoomeh.hezarjaribi@pnu.ac.ir