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A SUBCLASS OF BI-UNIVALENT FUNCTIONS BY TREMBLAY DIFFERENTIAL OPERATOR SATISFYING SUBORDINATE CONDITIONS

S. Fadaei, Sh. Najafzadeh ✉, and A. Ebadian

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Abstract. In this paper, we introduce a newly defined subclass $\mathcal{S}_\Sigma(\vartheta,\gamma,\eta;\varphi)$ of bi-univalent functions by using the Tremblay differential operator satisfying subordinate conditions in the unit disk. Moreover, we use the *Faber polynomial* expansion to derive bounds for the *Fekete-Szegö* problem and first two *Taylor-Maclaurin coefficients* $|a_2|$ and $|a_3|$ for functions of this class.

Keywords: Analytic function, Bi-univalent function, Coefficient estimates, Faber polynomial expansion, Tremblay fractional derivative operator.
2020 MSC: 26E05.

1. Introduction

Let $\mathcal{T}$ denote the class of analytic functions in the unit disk $\Delta=\{z\in\mathbb{C}: |z|<1\}$ of the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots=z+\sum_{k=2}^{\infty}a_kz^k, \tag{1}$$

and let S be the subclass of $\mathcal{T}$ consisting of univalent functions in Δ. The *Koebe-One Quarter Theorem* ensures that every function $f\in S$ has an inverse f^{-1}, which

$$f^{-1}(f(z))=z,\ (z\in\Delta)\ \text{ and } f(f^{-1}(w))=w,\ \left(|w|<r_0(f);\ r_0(f)\geq\frac{1}{4}\right)$$

with the power series:

$$g(w)=f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots. \tag{2}$$

A function $f\in\mathcal{T}$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. The class of bi-univalent functions are denoted by Σ. For a brief of notable investigations of function in Σ, see the pioneering works of Srivastava et al., [25] (see also [8] and [17]). In a considerable number of consequences of the aforementioned work by Srivastava et al., [25] several different subclasses

✉ najafzadeh1234@yahoo.ie, ORCID: 0000-0002-8124-8344
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of Σ, were introduced and studied homogeneously by many authors (see, for example, [22], [24]).

Previously, Brannan and Taha [9] introduced certain subclasses of bi-univalent functions in Σ, namely bi-starlike functions of order α indicated by $S^*_\Sigma(\alpha)$ and bi-convex functions of order α indicated by $K_\Sigma(\alpha)$ corresponding to the classes of starlike and convex functions denoted by $S^*(\alpha)$ and $K(\alpha)$, respectively. For each of the classes, $S^*_\Sigma(\alpha)$ and $K_\Sigma(\alpha)$, non-sharp estimates on the first two *Taylor-Maclaurin coefficients* $|a_2|$ and $|a_3|$ were found in [9]. However, one of the remarkable problem in *Geometric Function Theory* is determination of the bounds of the general coefficients $|a_n|$, which is still an open problem.

Definition 1.1. (see [26]) For two analytic functions f and g in Δ, we say that the function f is subordinate to g in Δ, written

$$f(z) \prec g(z) \quad (z \in \Delta)$$

if there exists a Schwarz function $w(z)$ which is analytic in Δ with

$$w(0) = 0 \quad \textit{and} \quad w(z) < 1 \quad (w \in \Delta)$$

such that

$$f(z) = g(w(z)) \quad (z \in \Delta).$$

In particular, if g is univalent in Δ, then

$$f \prec g \Longleftrightarrow \big\{f(0) = g(0) \quad \textit{and} \quad f(\Delta) \subseteq g(\Delta)\big\}.$$

Lemma 1.2. *(see [5]) Let*

$$\varphi = 1 + \sum_{n=1}^{\infty} \varphi_n z^n$$

be an analytic function with positive real part in the unit disk Δ, satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$ and symmetric with respect to the real axis. By the Caratheodory's lemma (see e.g., [12]), we have $|\varphi_n| \leq 2$.

Lemma 1.3. *(see [12]) Let $u(z)$ be analytic in the unit disk Δ with*

$$u(0) = 0 \qquad \textit{and} \qquad |u(z)| < 1$$

and suppose that

$$u(z) = \sum_{n=1}^{\infty} c_n z^n \qquad (z \in \mathbb{U}).$$

Then

$$|c_n| \leq 1 \qquad (n \in \mathbb{N}).$$

Lemma 1.4. *(see [14]) Let*

$$\varphi(z) = \sum_{n=1}^{\infty} \varphi_n z^n \in \mathcal{T}$$

be a Schwarz function, so that $|\varphi(z)| < 1$ *for* $|z| < 1$*. If* $\alpha \geq 0$*, then*

$$|\varphi_2 + \alpha\varphi_1^2| \leq 1 + (\alpha - 1)|\varphi_1|^2.$$

Definition 1.5. (see [23]) For a function f, the fractional derivative of order γ is defined by

$$D_z^{\gamma} f(z) = \frac{1}{\Gamma(\gamma)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\gamma}} d\zeta \quad (\gamma > 0),$$

where $f(z)$ is an analytic function in a simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-\zeta)^{\gamma-1}$ is removed by requiring, $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

Definition 1.6. (see [23]) For a function f, the fractional derivative of order γ is defined by

$$D_z^{\gamma} f(z) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\gamma}} d\zeta \quad (0 \leq \gamma < 1),$$

where the function $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\gamma}$ is removed, as in Definition 1.5.

Definition 1.7. (see [23]) Under the hypotheses of Definition 1.6, the fractional derivative of order $(n+\gamma)$ is defined by

$$D_z^{n+\gamma} f(z) = \frac{d^n}{dz^n}\{D_z^{\gamma} f(z)\} \quad (0 \leq \gamma < 1; \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

As consequences of Definitions 1.5-1.7, we note that

$$D_z^{-\gamma} z^n = \frac{\Gamma(n+1)}{\Gamma(n+\gamma+1)} z^{n+\gamma} \quad (\gamma > 0; \quad n \in \mathbb{N})$$

and

$$D_z^{\gamma} z^n = \frac{\Gamma(n+1)}{\Gamma(n-\gamma+1)} z^{n-\gamma} \quad (0 \leq \gamma < 1; \quad n \in \mathbb{N}).$$

Definition 1.8. (see [23]) The Tremblay fractional derivative operator $\mathcal{O}_z^{\eta,\gamma}$ of a function $f \in \mathcal{T}$ is defined, for all $z \in \Delta$, by

$$\mathcal{O}_z^{\eta,\gamma} f(z) = \frac{\Gamma(\gamma)}{\Gamma(\eta)} z^{1-\gamma} D_z^{\eta-\gamma} z^{\eta-1} f(z) \quad (0 < \gamma \leq 1; 0 < \eta \leq 1; \eta > \gamma; 0 < \eta - \gamma < 1).$$

It is clear from Definition 1.8 that, for $\eta = \gamma = 1$, we have $O_z^{1,1} f(z) = f(z)$ and we can easily see that

$$\mathcal{O}_z^{\eta,\gamma} f(z) = \frac{\eta}{\gamma} z + \sum_{n=2}^{\infty} \frac{\Gamma(\gamma)\Gamma(n+\eta)}{\Gamma(\eta)\Gamma(n+\gamma)} a_n z^n. \tag{3}$$

The role of Faber polynomials that introduced by Faber, [13], in Geometric Function Theory is significant. Several authors used Faber polynomial expansions to find coefficient bounds for bi-univalent functions, see [7,10,14,15,21,23].

By using the Faber polynomial expansion of $f \in \mathcal{T}$ of the form (1), it's inverse map $g = f^{-1}$ is expressed as, [3],

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ..., a_n) w^n,$$

where

$$\begin{aligned} K_{n-1}^{-n} = & \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!(n-3)!} a_2^{n-3} a_3 \\ & + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 + \frac{(-n)!}{[2(-n+2)]!(n-5)!} a_2^{n-5} \left[a_5 + (-n+2) a_3^2\right] \\ & + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} \left[a_6 + (-2n+5) a_3 a_4\right] + \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned}$$

such that V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $a_2, a_3, ..., a_n$, [4], and expressions such as (for example) $(n)!$ are symbolically interpreted as follows:

$$-(n)! \equiv \Gamma(1-n) := (-n)(-n-1)(-n-2) \cdots \quad (n \in \mathbb{N}_0).$$

In particular, the first three terms of K_{n-1}^{-n} are given by

$$\begin{aligned} \frac{1}{2} K_1^{-2} &= -a_2, \\ \frac{1}{3} K_2^{-3} &= 2a_2^2 - a_3, \\ \frac{1}{4} k_3^{-4} &= -(5a_2^3 - 5a_2 a_3 + a_4). \end{aligned}$$

In general, for any $p \in \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, ...\}$, an expansion of K_n^p is as, [3,26],

$$K_n^p = p a_{n+1} + \frac{p(p-1)}{2} E_n^2 + \frac{p!}{(p-3)!3!} E_{n-1}^3 + \cdots + \frac{p!}{(p-n)!(n)!} E_n^n,$$

where $E_n^p = E_n^p(a_2, a_3, ...)$ and, [1],

$$E_n^m(a_2, a_3, \cdots, a_{n+1}) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\eta_1} \cdots (a_{n+1})^{\eta_n}}{\eta_1! \cdots \eta_n!} \quad for \quad m \leq n,$$

which $a_1 = 1$, and the sum is taken over all non-negative integers $\eta_1, ..., \eta_n$ satisfying

$$\begin{aligned} \eta_1 + \eta_2 + \cdots + \eta_n &= m, \\ \eta_1 + 2\eta_2 + \cdots + n\eta_n &= n. \end{aligned}$$

Evidently, $E_n^n(a_2, ..., a_{n+1}) = a_2^n$, [2,26].

Definition 1.9. (see [11]) A function $f \in \Sigma$ is said to be in the class $\varphi_\Sigma(\vartheta, \delta, \varphi)$, $0 \leq \vartheta \leq 1, \delta \in \mathbb{C} \setminus \{0\}$ if the following subordinations hold:

$$1 + \frac{1}{\delta}\left(\frac{zf'(z) + \vartheta z^2 f''(z)}{\vartheta z f'(z) + (1-\vartheta)f(z)} - 1\right) \prec \varphi(z)$$

$$\textit{and} \quad 1 + \frac{1}{\delta}\left(\frac{wg'(w) + \vartheta w^2 g''(w)}{\vartheta w g'(w) + (1-\vartheta)g(w)} - 1\right) \prec \varphi(w), \quad (g(w) := f^{-1}(w)).$$

A function in the class $\varphi_\Sigma(\vartheta, \delta, \varphi)$ is called both bi-ϑ-convex function and bi-ϑ-starlike function of complex order δ of Ma-Minda type, as we know $\varphi_\Sigma(1, \delta, \varphi)$ unifies the Starlike class of order δ, and also $\varphi_\Sigma(0, \delta, \varphi)$ unifies Convex class of order δ. (This class was introduced in [11])

2. Results

In this section, we introduce a new subclass of bi-univalent functions satisfying subordinate conditions and defined by Tremblay differential operator. Also we obtain the coefficient estimates for $|a_2|$ and $|a_3|$ and the Fekete-Szegö problem for functions of the new class.

Definition 2.1. For $0 \leq \vartheta \leq 1,\ 0 < \gamma \leq 1\ ,\ 0 < \eta \leq 1,\ \eta > \gamma$ *and* $0 < \eta - \gamma < 1$, a function $f \in \Sigma$ is said to be in the subclass $\mathcal{S}_\Sigma(\vartheta, \gamma, \eta; \varphi)$ if the following subordination conditions hold true:

$$1 + \frac{1}{\delta}\left(\frac{z\big(\mathcal{O}_z^{\eta,\gamma} f(z)\big)' + \vartheta z^2 \big(\mathcal{O}_z^{\eta,\gamma} f(z)\big)''}{\vartheta z \big(\mathcal{O}_z^{\eta,\gamma} f(z)\big)' + (1-\vartheta)\mathcal{O}_z^{\eta,\gamma} f(z)} - 1\right) \prec \varphi(z)$$

and

$$1 + \frac{1}{\delta}\left(\frac{w\big(\mathcal{O}_z^{\eta,\gamma} g(w)\big)' + \vartheta w^2 \big(\mathcal{O}_z^{\eta,\gamma} g(w)\big)''}{\vartheta w \big(\mathcal{O}_z^{\eta,\gamma} g(w)\big)' + (1-\vartheta)\mathcal{O}_z^{\eta,\gamma} g(w)} - 1\right) \prec \varphi(w),\ (g(w) = f^{-1}(w)).$$

Theorem 2.2. *For* $0 \leq \vartheta \leq 1, 0 < \gamma \leq 1,\ 0 < \eta \leq 1, \eta > \gamma$ *and* $0 < \eta - \gamma < 1$, *let the function* $f \in \mathcal{S}_\Sigma(\vartheta, \gamma, \eta; \varphi)$ *be given by* (1) *Also let*

$$\varphi_2 = \alpha \varphi_1 \qquad (0 < \alpha \leq 1).$$

Then the following coefficient inequalities hold true:

$$|a_2| \leq \min\left\{\frac{2\delta(1+\gamma)}{(1+\eta)(1+\vartheta)},\right.$$
$$\left.\sqrt{\frac{2\delta(\gamma+2)(\gamma+1)^2}{2(1+2\vartheta)(\eta+2)(\eta+1)(\gamma+1) - (\eta+1)^2(1+\vartheta)^2(\gamma+2)}}\right\}$$
$$= \sqrt{\frac{2\delta(\gamma+2)(\gamma+1)^2}{2(1+2\vartheta)(\eta+2)(\eta+1)(\gamma+1) - (\eta+1)^2(1+\vartheta)^2(\gamma+2)}}$$

and

$$|a_3| \leq \min\left\{\frac{4\delta^2(\gamma+1)^2}{(1+\vartheta)^2(\eta+1)^2}+\frac{\delta(\gamma+2)(\gamma+1)}{2(1+2\vartheta)(\eta+2)(\eta+1)}\right.$$
$$,\frac{2\delta(\gamma+2)(\gamma+1)^2}{2(1+2\vartheta)(\eta+2)(\eta+1)(\gamma+1)-(\eta+1)^2(1+\vartheta)^2(\gamma+2)}$$
$$\left.+\frac{\delta(\gamma+2)(\gamma+1)}{2(1+2\vartheta)(\eta+2)(\eta+1)}\right\}=\frac{4\delta^2(\gamma+1)^2}{(1+\vartheta)^2(\eta+1)^2}+\frac{\delta(\gamma+2)(\gamma+1)}{2(1+2\vartheta)(\eta+2)(\eta+1)}$$

Proof. Let $f \in \mathcal{S}_\Sigma(\vartheta,\gamma,\eta;\varphi)$ given by (1). From (3) we have

$$1+\frac{1}{\delta}\left(\frac{z\big(\mathcal{O}_z^{\eta,\gamma}f(z)\big)'+\vartheta z^2\big(\mathcal{O}_z^{\eta,\gamma}f(z)\big)''}{\vartheta z\big(\mathcal{O}_z^{\eta,\gamma}f(z)\big)'+(1-\vartheta)\mathcal{O}_z^{\eta,\gamma}f(z)}-1\right)$$
$$=1+\frac{\gamma\Gamma(\gamma)\Gamma(2+\eta)(1+\vartheta)}{\delta\eta\Gamma(\eta)\Gamma(2+\gamma)}a_2z+\left(\frac{2\gamma(1+2\vartheta)\Gamma(\gamma)\Gamma(3+\eta)}{\delta\eta\Gamma(\eta)\Gamma(3+\gamma)}a_3\right.$$
$$\left.-\frac{\gamma^2(1+\vartheta)^2\Gamma(\gamma)^2\Gamma(2+\eta)^2}{\delta\eta^2\Gamma(\eta)^2\Gamma(2+\gamma)^2}a_2^2\right)z^2+\cdots$$

and for it's inverse map, $g=f^{-1}$, by (2) we have

$$1+\frac{1}{\delta}\left(\frac{w\big(\mathcal{O}_z^{\eta,\gamma}g(w)\big)'+\vartheta w^2\big(\mathcal{O}_z^{\eta,\gamma}g(w)\big)''}{\vartheta w\big(\mathcal{O}_z^{\eta,\gamma}g(w)\big)'+(1-\vartheta)\mathcal{O}_z^{\eta,\gamma}g(w)}-1\right)=1-\frac{\gamma\Gamma(\gamma)\Gamma(2+\eta)(1+\vartheta)}{\delta\eta\Gamma(\eta)\Gamma(2+\gamma)}a_2z$$
$$+\left(\frac{2\gamma(1+2\vartheta)\Gamma(\gamma)\Gamma(3+\eta)}{\delta\eta\Gamma(\eta)\Gamma(3+\gamma)}(2a_2^2-a_3)-\frac{\gamma^2(1+\vartheta)^2\Gamma(\gamma)^2\Gamma(2+\eta)^2}{\delta\eta^2\Gamma(\eta)\Gamma(2+\gamma^2)}a_2^2\right)z^2$$
$$+\cdots.$$

On the other hand, by Lemma 1.3 for $f \in \mathcal{S}_\Sigma(\vartheta,\gamma,\eta;\varphi)$ and $\varphi \in P$ there are two Schwarz functinos

$$u(z)=\sum_{n=1}^{\infty}c_nz^n$$

and

$$v(w)=\sum_{n=1}^{\infty}d_nw^n$$

such that

$$1+\frac{1}{\delta}\left(\frac{z\big(\mathcal{O}_z^{\eta,\gamma}f(z)\big)'+\vartheta z^2\big(\mathcal{O}_z^{\eta,\gamma}f(z)\big)''}{\vartheta z\big(\mathcal{O}_z^{\eta,\gamma}f(z)\big)'+(1-\vartheta)\mathcal{O}_z^{\eta,\gamma}f(z)}-1\right)=\varphi(u(z)) \tag{4}$$

and

$$1+\frac{1}{\delta}\left(\frac{w\big(\mathcal{O}_z^{\eta,\gamma}g(w)\big)'+\vartheta w^2\big(\mathcal{O}_z^{\eta,\gamma}g(w)\big)''}{\vartheta w\big(\mathcal{O}_z^{\eta,\gamma}g(w)\big)'+(1-\vartheta)\mathcal{O}_z^{\eta,\gamma}g(w)}-1\right)=\varphi(v(w)) \tag{5}$$

where

$$\varphi(u(z)) = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \varphi_k E_n^k(c_1, c_2, ..., c_n) z^n \tag{6}$$

and

$$\varphi(v(w)) = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \varphi_k E_n^k(d_1, d_2, ..., d_n) w^n. \tag{7}$$

By compairing (4) and (6) we can find that

$$\frac{\gamma\Gamma(\gamma)\Gamma(2+\eta)(1+\vartheta)}{\delta\eta\Gamma(\eta)\Gamma(2+\gamma)} a_2 = \varphi_1 c_1, \tag{8}$$

(9)

$$\left(\frac{2\gamma(1+2\vartheta)\Gamma(\gamma)\Gamma(3+\eta)}{\delta\eta\Gamma(\eta)\Gamma(3+\gamma)} a_3 - \frac{\gamma^2(1+\vartheta)^2\Gamma(\gamma)^2\Gamma(2+\eta)^2}{\delta\eta^2\Gamma(\eta)^2\Gamma(2+\gamma)^2} a_2^2\right) = \varphi_1 c_2 + \alpha\varphi_1 c_1^2,$$

and similarly from (5) and (7) we obtain

$$-\frac{\gamma\Gamma(\gamma)\Gamma(2+\eta)(1+\vartheta)}{\delta\eta\Gamma(\eta)\Gamma(2+\gamma)} a_2 = \varphi_1 d_1, \tag{10}$$

$$\left(\frac{2\gamma(1+2\vartheta)\Gamma(\gamma)\Gamma(3+\eta)}{\delta\eta\Gamma(\eta)\Gamma(3+\gamma)}(2a_2^2 - a_3) - \frac{\gamma^2(1+\vartheta)^2\Gamma(\gamma)^2\Gamma(2+\eta)^2}{\delta\eta^2\Gamma(\eta)^2\Gamma(2+\gamma)^2} a_2^2\right)$$

$$= \varphi_1 d_2 + \alpha\varphi_1 d_1^2. \tag{11}$$

From (8) or (10) we obtain

$$|a_2| = \left|\frac{\varphi_1 c_1 \delta(1+\gamma)}{(1+\eta)(1+\vartheta)}\right| = \left|\frac{\varphi_1 d_1 \delta(1+\gamma)}{(1+\eta)(1+\vartheta)}\right| \leq \frac{2\delta(1+\gamma)}{(1+\eta)(1+\vartheta)}. \tag{12}$$

Adding (9) and (2) implies

$$\begin{aligned}\left(\frac{4(1+2\vartheta)(\eta+2)(\eta+1)}{\delta(\gamma+2)(\gamma+1)} - \frac{2(\eta+1)^2(1+\vartheta)^2}{\delta(\gamma+1)^2}\right) a_2^2 &= \varphi_1(c_2+d_2) \\ &+ \alpha\varphi_1(c_1^2+d_1^2),\end{aligned} \tag{13}$$

which, upon taking the moduli of both sides, yields

$$\begin{aligned}\left(\frac{4(1+2\vartheta)(\eta+2)(\eta+1)}{\delta(\gamma+2)(\gamma+1)} - \frac{2(\eta+1)^2(1+\vartheta)^2}{\delta(\gamma+1)^2}\right)|a_2|^2 = \varphi_1\big[&|c_2+\alpha c_1^2| \\ &+ |d_2+\alpha d_1^2|\big].\end{aligned} \tag{14}$$

Thus, by using Lemma 1.4, we obtain

$$\left(\frac{4(1+2\vartheta)(\eta+2)(\eta+1)}{\delta(\gamma+2)(\gamma+1)}-\frac{2(\eta+1)^2(1+\vartheta)^2}{\delta(\gamma+1)^2}\right)|a_2|^2 = \varphi_1\big[1+(\alpha-1)|c_1|^2 + 1+(\alpha-1)|d_1|^2\big] \le 2\varphi_1. \tag{15}$$

Therefore,

$$|a_2| \le \sqrt{\frac{2\delta(\gamma+2)(\gamma+1)^2}{2(1+2\vartheta)(\eta+2)(\eta+1)(\gamma+1)-(\eta+1)^2(1+\vartheta)^2(\gamma+2)}}. \tag{16}$$

Next, by substract (2) from (9), we get

$$\frac{4(1+2\vartheta)(\eta+2)(\eta+1)}{\delta(\gamma+2)(\gamma+1)}(a_3-a_2^2) = \varphi_1(c_2-d_2)+\varphi_2(c_1^2-d_1^2) \tag{17}$$

or

$$\begin{aligned}|a_3| =& |a_2|^2 + \frac{|\varphi_1(c_2-d_2)|\delta(\gamma+2)(\gamma+1)}{(1+2\vartheta)(\eta+2)(\eta+1)}\\ &\le |a_2|^2 + \frac{\delta(\gamma+2)(\gamma+1)}{2(1+2\vartheta)(\eta+2)(\eta+1)}.\end{aligned} \tag{18}$$

Upon substituting the value of $|a_2|$ from (12) and (16) into (18), it follows that

$$|a_3| \le \frac{4\delta^2(\gamma+1)^2}{(1+\vartheta)^2(\eta+1)^2} + \frac{\delta(\gamma+2)(\gamma+1)}{2(1+2\vartheta)(\eta+2)(\eta+1)}$$

and

$$|a_3| \le \frac{2\delta(\gamma+2)(\gamma+1)^2}{2(1+2\vartheta)(\eta+2)(\eta+1)(\gamma+1)-(\eta+1)^2(1+\vartheta)^2(\gamma+2)} + \frac{\delta(\gamma+2)(\gamma+1)}{2(1+2\vartheta)(\eta+2)(\eta+1)}.$$

□

Theorem 2.3. *For $0 \le \vartheta \le 1, 0 < \gamma \le 1,\ 0 < \eta \le 1, \eta > \gamma$ and $0 < \eta - \gamma < 1$, let the function $f \in \mathcal{S}_\Sigma(\vartheta, \gamma, \eta; \varphi)$ be given by (1). Also let*

$$\varphi_2 = \alpha\varphi_1 \qquad (0 < \alpha \le 1),$$

we have

$$|a_3 - 2a_2^2| \le \frac{2\delta(1+\delta)(\gamma+1)(\gamma+2)}{(1+2\vartheta)(\eta+2)(\eta+1)}$$

and also,

$$|a_3 - a_2^2| \le \frac{\delta(\gamma+2)(\gamma+1)}{2(1+2\vartheta)(\eta+2)(\eta+1)}.$$

Proof. If we rewrite (2) as

$$\left(\frac{2\gamma(1+2\vartheta)\Gamma(\gamma)\Gamma(3+\eta)}{\delta\eta\Gamma(\eta)\Gamma(3+\gamma)}(a_3-2a_2^2)+\frac{\gamma^2(1+\vartheta)^2\Gamma(\gamma)\Gamma(2+\eta)}{\delta\eta^2\Gamma(\eta)^2\Gamma(2+\gamma)^2}a_2^2\right)$$
$$=-(\varphi_1 d_2+\varphi_2 d_1^2)$$

and therefore

$$|a_3-2a_2^2|\leq\frac{2\delta(1+\delta)(\gamma+1)(\gamma+2)}{(1+2\vartheta)(\eta+2)(\eta+1)}.$$

Solving (17) for $(a_3-a_2^2)$, we obtain

$$|a_3-a_2^2|=\frac{|\varphi_1(c_2-d_2)+\varphi_2(c_1^2-d_1^2)|\delta(\gamma+2)(\gamma+1)}{4(1+2\vartheta)(\eta+2)(\eta+1)}\leq\frac{\delta(\gamma+2)(\gamma+1)}{2(1+2\vartheta)(\eta+2)(\eta+1)}.$$

□

Corollary 2.4. *By letting $\eta=\gamma=\delta=1$ in Theorem 2.2, we obtain estimates for $|a_2|$ given in [11] and [6].*

Corollary 2.5. *By taking $\eta=\gamma=\delta=1$ in Theorem 2.2, we get an improvment of the estimates for $|a_3-a_2^2|$ given in [19] and [18].*

3. Conclusion

The operator defined was motivated by various work studied earlier by the researchers. This operator can be generalized further and many other geometric properties can be obtained. Also by using other polynomials such as Chebyshev, Hohlo, Gegenbauer, Lucas polynomials many other results can be find. [16, 20, 27]

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Somayeh Fadaei
Orcid number:0000-0001-6483-2680
Department of Mathematics
Payame Noor University
P.O.Box 19395-3697, Tehran, Iran
Email address: fadaei.somayye@student.pnu.ac.ir

Shahram Najafzadeh
Orcid number: 0000-0002-8124-8344
Department of Mathematics
Payame Noor University
P.O.Box 19395-3697, Tehran, Iran
Email address: najafzadeh1234@yahoo.ie, shnajafzadeh44@pnu.ac.ir

Ali Ebadian
Orcid number:0000-0003-4067-6729
Department of Mathematics
Urmia University
Urmia, Iran
Email address: A.ebadian@urmia.ac.ir