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QUANTITATIVE AND STABILITY STUDY OF THE EVOLUTION OF A VISCOELASTIC BODY

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Abstract. This paper deals with a generalization of the model describing the evolution of a linear viscoelastic body studied by Kirane M. and B.S. Houari in 2011. We prove the existence and uniqueness of the solution of the model using a C_0-semi-group contraction method with a linear operator parameter. Moreover the strong stability of the solution is shown in a particular case.

Keywords: Viscoelasticity; Homogeneous evolution equation; Monotone maximal operator
2020 MSC: Primary 47D06, 92D15, 65J08, 47H05.

1. Introduction

Several mathematical models that come from physics (viscoelasticity motion) lead to the study of partial differential equations (PDEs) and sometimes evolution equations allowing mathematicians to describe the behavior of a quantity that depends on several variables [6–8, 15]. We consider the existence, uniqueness and strong stability of the solutions of the equations of linear viscoelasticity at large time. The simplest model is provided by the one-dimensional inhomogeneous case

$$\begin{aligned}\rho u_{tt}(t,x) &= c u_{xx}(t,x) - \int_{-\infty}^{t} g(t-\tau)u_{xx}(\tau,x)d\tau \\ &\quad +\mu_1 u_t(t,x) + \mu_2 u_t(t-s,x), \end{aligned} \tag{1}$$

where ρ, c are positive constants, μ_1, μ_2 are real numbers, g is independent of x, and integrable, $u(t,x)$ has been assigned to (t,x) in $[0,+\infty[\times[0,\pi]$ and $u_{xx} = \frac{\partial^2}{\partial x^2}u$.
Kirane and Houari [9] studied this equation with suitable initial boundary value conditions with $\rho = 1$ and $c = 1$. Nicaise and Pignotti [11] studied this equation with conditions $\rho = 1$, $c = 1$ and $g = 0$.
In this paper, we prove the existence and uniqueness of a solution to a system with conditions $\rho > 0$, $c > 0$ and g is a positive nonincreasing C^1 and integrable function. Moreover, the stability of the solution is shown in particular case.

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The mathematical model of viscoelasticity aims to take into the behaviour of those materials whose mechanical behaviour is determined not only by the present but also on its past history. Since then a wide variety of results and applications have been obtained in many different fields. In [14] synthetic tissues which mimic human bones are investigated, while in [13] cardiological tissues are considered. Besides a model of apples regarded as viscoelastic bodies is studied in [1, 3–5, 10].
The present investigation concerns viscolastic bodies and their mechanical behaviour aiming to widen the range of applicative cases the theory can be applied. This work is presented in four parts. In the first part, we have presented the mathematical model of the system describing the evolution of a body in viscoelastic motion. In the second part, we have shown the existence and uniqueness of the solution of the system through the methods of semi groups. In the third part, we have shown that the system is strongly stable in a particular case.

2. Mathematical model

Let us consider a one-dimensional homogeneous body of density $\rho > 0$, and having the constitutive equation

$$\sigma(t,x) = cu_x(t,x) - \int_{-\infty}^{t} g(t-\tau)u_x(\tau,x)d\tau, \tag{2}$$

where $g : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a positive, nonincreasing C^1 function and integrable satisfying

$$a = \left(c - \int_0^{+\infty} g(s)ds\right) > 0. \tag{3}$$

The state of the body at the time $t \in \mathbb{R}$ is characterized by the displacement $u(t,x)$, the momentum $v(t,x)$ and the history of the displacement $w(t,s,x)$, which is defined by

$$w(t,s,x) \;=\; u(t-s,x) \quad \text{for } s \in \mathbb{R}_+.$$

The equation of motion is

$$\rho u_{tt}(t,x) = \sigma_x(t,x) + \mu_1 u_t(t,x) + \mu_2 u_t(t-s,x). \tag{4}$$

We have

$$\begin{aligned} \rho u_{tt}(t,x) \;&=\; cu_{xx}(t,x) - \int_{-\infty}^{t} g(t-\tau)u_{xx}(\tau,x)d\tau \\ &\quad + \mu_1 u_t(t,x) + \mu_2 u_t(t-s,x). \end{aligned} \tag{5}$$

We shall study the existence and uniqueness of the equation (5) firstly in the case $\mu_1 = 0 = \mu_2$ and secondly in the case $\mu_1 \neq 0$ and $\mu_2 \neq 0$. We assume that

$u(t,x) \in H_0^1(0,\pi) \cap H^2(0,\pi)$ and we set $v = \rho u_t$. Therefore

$$\begin{aligned} w_t(t,s,x) &= \lim_{h \longrightarrow 0} \frac{w(t+h,s,x) - w(t,s,x)}{h}. \\ &= \lim_{h \longrightarrow 0} \frac{u(t+h-s,x) - u(t-s,x)}{h}. \\ &= \lim_{h \longrightarrow 0} \frac{u(t-s+h,x) - u(t-s,x)}{h}. \end{aligned}$$

$$\begin{aligned} w_s(t,s,x) &= \lim_{h \longrightarrow 0} \frac{w(t,s+h,x) - w(t,s,x)}{h}. \\ &= \lim_{h \longrightarrow 0} \frac{u(t-s-h,x) - u(t-s,x)}{h}. \\ &= -\lim_{\lambda \longrightarrow 0} \frac{u(t-s+\lambda,x) - u(t-s,x)}{\lambda}. \\ &= -w_t(t,s,x). \end{aligned}$$

For $\mu_1 = 0 = \mu_2$, we obtain the system

$$(6) \qquad \begin{cases} u_t = \rho^{-1} v & (t,x) \in Q_\infty, \\ v_t = c u_{xx} - \int_0^{+\infty} g(s) w_{xx}(t,s,x) ds & (t,x) \in Q_\infty, \\ w_t = -w_s & (t,s,x) \in R_\infty, \\ u(t,0) = u(t,\pi) = 0 & t \in \mathbb{R}_+, \\ u(0,x) = u_0(x); v_0(x) = v(0,x) & x \in [0,\pi], \\ w(0,s,x) = w_0(s,x) & (s,x) \in Q_\infty, \end{cases}$$

where $Q_\infty = \mathbb{R}_+ \times [0,\pi]$ and $R_\infty = \mathbb{R}_+^2 \times [0,\pi]$.

Definition 2.1. Let g be an integrable function on the interval $[0,\pi]$.
The subspace $L_g^2(\mathbb{R}_+, H_0^1(0,\pi))$ is defined by

$$L_g^2(\mathbb{R}_+, H_0^1(0,\pi)) = \left\{ u \in L^2(\mathbb{R}_+, H_0^1(0,\pi)) \Big/ \int_0^\infty \int_0^\pi g(s) u_x^2(s,x) dx ds < \infty \right\}.$$

The norm on L_g^2 is defined by

$$\|u\|_{L_g^2(\mathbb{R}_+, H_0^1(0,\pi))} = \left(\int_0^\infty g(s) \, \|u\|_{H_0^1(0,\pi)}^2 \, ds \right)^{\frac{1}{2}},$$

where

$$\|u\|_{H_0^1(0,\pi)}^2 = \|u_x\|_{L^2(0,\pi)}^2 \quad \text{with} \quad \|u\|_{L^2(0,\pi)} = \left(\int_0^\pi |u|^2 \, dx \right)^{\frac{1}{2}}.$$

$H^m(\Omega) = \{u \in L^2(\Omega), D^\alpha u \in L^2(\Omega), \alpha \in \mathbb{N}^n, |\alpha| \leq m\}$
$H_0^1(\Omega) = \{u \in H^1(\Omega), u = 0 \text{ on } \partial\Omega\}$

Proposition 2.1. Let the subspaces $H_0^1(0,\pi)$, $H^2(0,\pi)$ and $L_g^2(\mathbb{R}_+; H_0^1(0,\pi))$ be defined as before. Let $L^2(0,\pi)$ be the set of square integrable functions on the interval $[0,\pi]$. Then

$$< (u,v,w),(\tilde{u},\tilde{v},\tilde{w}) > \;=\; \int_0^\pi \Big(a u_x \tilde{u}_x + \rho^{-1} v\tilde{v} + \int_0^\infty g(s)[u_x - w_x][\tilde{u}_x - \tilde{w}_x]ds \Big) dx, \quad (7)$$

is an inner product on the subspace

$$H \;=\; \Big(H_0^1(0,\pi) \cap H^2(0,\pi)\Big) \times L^2(0,\pi) \times L_g^2(\mathbb{R}_+; H_0^1(0,\pi)),$$

with

$$\|(u,v,w)\|_H = (< (u,v,w),(u,v,w) >)^{\frac{1}{2}},$$

where

$$< (u,v,w),(u,v,w) >= \int_0^\pi \Big(a u_x^2 + \rho^{-1} v^2 + \int_0^\infty g(s)(u_x - w_x)^2 ds \Big) dx.$$

Proof. Let $(u,v,w),(u',v',w'),(\tilde{u},\tilde{v},\tilde{w}) \in H$ and $\lambda \in \mathbb{R}$. From linearity property of the integral operator, we obtain

$$\begin{aligned} < \lambda(u,v,w) + (u',v',w'),(\tilde{u},\tilde{v},\tilde{w}) > \;&=\; \lambda < (u,v,w),(\tilde{u},\tilde{v},\tilde{w}) > \\ &\quad + < (u',v',w'),(\tilde{u},\tilde{v},\tilde{w}) > . \\ < (u,v,w), \lambda(\tilde{u},\tilde{v},\tilde{w}) + (u',v',w') > \;&=\; \lambda < (u,v,w),(\tilde{u},\tilde{v},\tilde{w}) > \\ &\quad + < (u',v',w'),(\tilde{u},\tilde{v},\tilde{w}) > . \end{aligned}$$

Then $< .,. >$ is bilinear.

$$\begin{aligned} < (u,v,w),(\tilde{u},\tilde{v},\tilde{w}) > \;&=\; \int_0^\pi \Big(a u_x \tilde{u}_x + \rho^{-1} v\tilde{v} \Big) dx \\ &\quad + \int_0^\pi \Big(\int_0^\infty g(s)[u_x - w_x][\tilde{u}_x - \tilde{w}_x]ds \Big) dx. \\ &=\; \int_0^\pi \Big(a \tilde{u}_x u_x + \rho^{-1} \tilde{v} v \Big) dx \\ &\quad + \int_0^\pi \Big(\int_0^\infty g(s)[\tilde{u}_x - \tilde{w}_x][u_x - w_x]ds \Big) dx. \\ &=\; < (\tilde{u},\tilde{v},\tilde{w}),(u,v,w) > . \end{aligned}$$

Then $< .,. >$ is symetric. We show that $< .,. >$ is positive definite.

$$< (u,v,w),(u,v,w) >= \int_0^\pi \Big(a u_x^2 + \rho^{-1} v^2 + \int_0^\infty g(s)(u_x - w_x)^2 ds \Big) dx.$$

$au_x^2 \geq 0$, $\rho^{-1}v^2 \geq 0$, $\int_0^{\infty} g(s)(u_x - w_x)^2 ds \geq 0$, then

$$(au_x^2 + \rho^{-1}v^2 + \int_0^{\infty} g(s)(u_x - w_x)^2) \geq 0,$$

and

$$\int_0^{\pi} \left(au_x^2 + \rho^{-1}v^2 + \int_0^{\infty} g(s)(u_x - w_x)^2 ds \right) dx \geq 0.$$

Then $< (u,v,w),(u,v,w) >\geq 0$ and $< .,. >$ is positive.

$$\begin{aligned} < (u,v,w),(u,v,w) >= 0 \quad &\Longleftrightarrow \quad \int_0^{\pi} \Big(au_x^2 + \rho^{-1}v^2 \\ &\qquad + \int_0^{\infty} g(s)(u_x - w_x)^2 ds \Big) dx = 0. \\ &\Longleftrightarrow \quad \begin{cases} u_x = 0 \quad \forall (t,x) \in \mathbb{R}_+ \times [0,\pi], \\ v = 0 \quad \forall (t,x) \in \mathbb{R}_+ \times [0,\pi], \\ u_x - w_x = 0 \quad \forall (t,x) \in \mathbb{R}_+ \times [0,\pi]. \end{cases} \\ &\Longleftrightarrow \quad (u,v,w) = (0,0,0). \end{aligned}$$

Conclusion : $< .,. >$ is an inner product on H. □

Proposition 2.2. H equipped with the inner product $< .,. >$ defined by (7) is a Hilbert space.

Proof. H equipped with the inner product $< .,. >$ defined by (7) is a prehilbertian space. We will show that H is complete.
Let $(u^n, v^n, w^n)_{n\geq 0}$ be a Cauchy sequence in H. Then

$$\forall \epsilon > 0, \exists n_0 \ : \ \forall n,m \geq n_0; \|(u^n,v^n,w^n) - (u^m,v^m,w^m)\|_H \leq \epsilon.$$

$$\begin{aligned} \|(u^n,v^n,w^n) - (u^m,v^m,w^m)\|_H^2 &= \|(u^n - u^m, v^n - v^m, w^n - w^m)\|_H^2 . \\ &= \int_0^{\pi} \left(a(u_x^n - u_x^m)^2 + \rho^{-1}(v^n - v^m)^2 \right) dx \\ &\quad + \int_0^{\pi} \left(\int_0^{\infty} g(s)(u_x^n - u_x^m - w^n + w^m)^2 ds \right) dx. \\ &= a\int_0^{\pi} (u_x^n - u_x^m)^2 dx + \rho^{-1} \int_0^{\pi} (v^n - v^m)^2 dx \\ &\quad + \int_0^{\pi} \int_0^{\infty} g(s)(u_x^n - u_x^m - w^n + w^m)^2 ds dx. \end{aligned}$$

Since $\|(u^n,v^n,w^n) - (u^m,v^m,w^m)\|_H \leq \epsilon$, we have

$$a \|(u_x^n - u_x^m)\|_{L^2(0,\pi)}^2 + \rho^{-1} \|v^n - v^m\|_{L^2(0,\pi)}^2 + \int_0^{\infty} g(s) \|w_x^n - w_x^m\|_{L^2(0,\pi)}^2 ds \leq \epsilon.$$

So

$$\begin{cases} a\,\|(u_x^n - u_x^m)\|^2_{L^2(0,\pi)} \le \epsilon, \\ \rho^{-1}\,\|v^n - v^m\|^2_{L^2(0,\pi)} \le \epsilon, \\ \int_0^\infty g(s)\,\|w_x^n - w_x^m\|^2_{L^2(0,\pi)}\,ds \le \epsilon, \end{cases}$$

and

$$\begin{cases} \|(u_x^n - u_x^m)\|^2_{L^2(0,\pi)} \le \epsilon_1 & \forall \epsilon_1 > 0, \\ \|v^n - v^m\|^2_{L^2(0,\pi)} \le \epsilon_2 & \forall \epsilon_2 > 0, \\ \|w^n - w^m\|^2_{L^2_g(\mathbb{R}_+, H_0^1(0,\pi))} \le \epsilon_3 & \forall \epsilon_3 > 0. \end{cases}$$

Then u^n, v^n and w^n are Cauchy sequences in $H_0^1(0,\pi)$, $L^2(0,\pi)$ and $L^2_g(\mathbb{R}_+, H_0^1(0,\pi))$ respectively.
u^n is a Cauchy sequence in $H_0^1(0,\pi)$ and since $H_0^1(0,\pi)$ is complete, there exists a rank n_1 from which u^n converges to an element l_1 in $H_0^1(0,\pi)$.
v^n is a Cauchy sequence in $L^2(0,\pi)$ and since $L^2(0,\pi)$ is complete, there exists a rank n_2 from which v^n converges to an element l_2 in $L^2(0,\pi)$.
$L^2_g(\mathbb{R}_+, H_0^1(0,\pi))$ is a closed subspace of $L^2(\mathbb{R}_+, H_0^1(0,\pi))$ and $L^2(\mathbb{R}_+, H_0^1(0,\pi))$ complete. Thus $L^2_g(\mathbb{R}_+, H_0^1(0,\pi))$ is complete.
w^n is a Cauchy sequence in $L^2_g(\mathbb{R}_+, H_0^1(0,\pi))$ and since $L^2_g(\mathbb{R}_+, H_0^1(0,\pi))$ is complete, there exists a rank n_3 from which w^n converges to an element l_3 in $L^2_g(\mathbb{R}_+, H_0^1(0,\pi))$.
Taking $n' = max(n_1, n_2, n_3)$, $(u^n, v^n, w^n)_n$ converges to the triplet (l_1, l_2, l_3) belonging to H from rank n'.
Conclusion : H is a Hilbert space. □

Let us define the operator A as follows

$$A : D(A) \subseteq H \longrightarrow H$$

$$A\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \rho^{-1} v \\ cu_{xx}(t,x) - \int_0^\infty g(s) w_{xx}(t,s,x)ds \\ w_s \end{pmatrix},$$

in which $D(A) = \{(u,v,w) \in H; (u,v,w) \text{ satisfy } (8)\}$, and the meaning of condition (8) is that

$$(8) \quad \begin{cases} v \in H_0^1(0,\pi), w_s \in L^2_g(\mathbb{R}_+; H_0^1(0;\pi)), w(t,0,x,) = u(t,x) \\ \\ cu_{xx}(t,x) - \int_0^\infty g(s) w_{xx}(t,s,x)ds \in L^2(0,\pi) \end{cases} .$$

At this point, let us observe that (6) can be equivalently rewritten under the abstract form

$$(9) \quad \begin{cases} Z'(t) = AZ(t), \quad t \ge 0, \\ Z(0) = \xi, \end{cases}$$

where $Z(t)(x) = (u(t,x), v(t,x), w(t,x))$ and $\xi = (u_0, v_0, w_0)$.

3. Main results

3.1. Existence and uniqueness of the solution of problem (9).

The existence and uniqueness of the system (9) depends on the operator A. If A is an infinitesimal generator of a C_0-semi group of contraction, then the system admits a unique solution. In this section, we will show that $-A$ is maximal monotone which means that A is infinitesimal generator. [12]

Proposition 3.1. The operator $-A$ is maximal monotone on H.

Proof. We will show that the operator $I - A$ is surjective.
Let $(\tilde{u}, \tilde{v}, \tilde{w}) \in H_0^1(0,\pi) \times L^2(0,\pi) \times L_g^2(\mathbb{R}_+; H_0^1(0;\pi))$ and $(u, v, w) \in D(A)$.
$(I - A)(u, v, w) = (\tilde{u}, \tilde{v}, \tilde{w}) \Longrightarrow$

$$\begin{cases} u - \rho^{-1} v = \tilde{u}, \\ v - c u_{xx} + \int_0^\infty g(s) w_{xx} ds = \tilde{v}, \\ w + w_s = \tilde{w}, \\ w(t,0,x) = u(t,x). \end{cases} \tag{10}$$

$w_s + w = \tilde{w}$, then

$$w(s,x) = w(0,x)e^{-s} + \int_0^s e^{-(s-\tau)} \tilde{w}(\tau, x) d\tau. \tag{11}$$

$$= u(x)e^{-s} + \int_0^s e^{-(s-\tau)} \tilde{w}(\tau, x) d\tau. \tag{12}$$

$$w_s + w = \tilde{w} \Longrightarrow w_x = -w_{sx} + \tilde{w}_x. \tag{13}$$

$$\Longrightarrow g(s) w_x^2 = -g(s) w_x w_{sx} + g(s) w_x \tilde{w}_x. \tag{14}$$

$$\Longrightarrow g(s) w_x^2 = -\frac{1}{2} g(s) [(w_x)^2]_s + g(s) w_x \tilde{w}_x. \tag{15}$$

$$\Longrightarrow \int_0^\infty \int_0^\pi g(s) w_x^2 dx ds = -\frac{1}{2} \int_0^\infty \int_0^\pi g(s) (w_x^2)_s dx ds + \int_0^\infty \int_0^\pi g(s) w_x \tilde{w}_x dx ds. \tag{16}$$

$$\int_0^\infty \int_0^\pi g(s)(w_x^2)_s dx ds = \int_0^\pi \lim_{X \longrightarrow \infty} [g(s) w_x^2(s,x)]_0^X dx - \int_0^\infty \int_0^\pi g'(s) w_x^2 dx ds. \tag{17}$$

$$= \int_0^\pi -g(0) w_x^2(0,x) dx - \int_0^\infty \int_0^\pi g'(s) w_x^2 dx ds. \tag{18}$$

$$= -g(0) \int_0^\pi u_x^2(t,x) dx - \int_0^\infty \int_0^\pi g'(s) w_x^2 dx ds. \tag{19}$$

$$\int_0^\infty \int_0^\pi g(s) w_x^2 dx ds = \frac{1}{2} g(0) \int_0^\pi u_x^2(t,x) dx \tag{20}$$

$$+\frac{1}{2}\int_0^\infty\int_0^\pi g(s)w_x^2dxds+\int_0^\infty\int_0^\pi g(s)w_x\tilde{w}dxds.$$

$$\forall a,b(a-b)^2=a^2+b^2-2ab\geq 0.$$

$$ab\leq\frac{1}{2}a^2+\frac{1}{2}b^2.$$

Using this relation with $a=w_x, b=\tilde{w}_x$, we obtain

$$w_x\tilde{w}_x\leq\frac{1}{2}w_x^2+\frac{1}{2}\tilde{w}_x^2.$$

So,

(21)

$$\int_0^\infty\int_0^\pi g(s)w_x\tilde{w}_xdxds\leq\int_0^\infty\int_0^\pi g(s)(\frac{1}{2}w_x^2+\frac{1}{2}\tilde{w}_x^2)dxds.$$

$$\leq\frac{1}{2}\int_0^\infty\int_0^\pi g(s)w_x^2dxds+\frac{1}{2}\int_0^\infty\int_0^\pi g(s)\tilde{w}^2dxds. \tag{22}$$

$$\int_0^\infty\int_0^\pi g(s)w_x^2dxds\leq\frac{1}{2}g(0)\int_0^\pi u_x^2dx+\frac{1}{2}\int_0^\infty\int_0^\pi g(s)w_x^2dxds \tag{23}$$

$$+\frac{1}{2}\int_0^\infty\int_0^\pi g(s)\tilde{w}^2dxds+\frac{1}{2}\int_0^\infty\int_0^\pi g'(s)w_x^2dxds.$$

$$\int_0^\infty\int_0^\pi g(s)w_x^2dxds-\frac{1}{2}\int_0^\infty\int_0^\pi g(s)w_x^2dxds\leq\frac{1}{2}g(0)\int_0^\pi u_x^2dx \tag{24}$$

$$+\frac{1}{2}\int_0^\infty\int_0^\pi g(s)\tilde{w}^2dxds+\frac{1}{2}\int_0^\infty\int_0^\pi g'(s)w_x^2dxds.$$

$$\frac{1}{2}\int_0^\infty\int_0^\pi g(s)w_x^2dxds\leq\frac{1}{2}g(0)\int_0^\pi u_x^2dx+\frac{1}{2}\int_0^\infty\int_0^\pi g(s)\tilde{w}^2dxds. \tag{25}$$

$$\int_0^\infty\int_0^\pi g(s)w_x^2dxds\leq g(0)\int_0^\pi u_x^2dx+\int_0^\infty\int_0^\pi g(s)\tilde{w}^2dxds. \tag{26}$$

Since $\tilde{w}\in L_g^2(\mathbb{R}_+;H_0^1(0,\pi))$, we have

$$\int_0^\infty\int_0^\pi g(s)\tilde{w}^2dxds<\infty,$$

and so (26) shows that $\int_0^\infty\int_0^\pi g(s)w_x^2dxds<\infty$ and $w\in L_g^2(\mathbb{R}_+;H_0^1(0,\pi))$.

$$u-\rho^{-1}v=\tilde{u}\Longrightarrow v=\rho u-\rho\tilde{u}. \tag{27}$$

$$v-cu_{xx}+\int_0^\infty g(s)w_{xx}ds=\tilde{v}. \tag{28}$$

then

$$-\rho\tilde{u}+\rho u-cu_{xx}+\int_0^\infty g(s)w_{xx}ds=\tilde{v}. \tag{29}$$

$$w = u(x)e^{-s} + \int_0^s e^{-(s-\tau)}\tilde{w}(\tau, x)d\tau. \tag{30}$$

Then

$$\rho u - cu_{xx} + u_{xx}\int_0^\infty g(s)e^{-s}ds \tag{31}$$
$$+\int_0^\infty g(s)e^{-(s-\tau)}\tilde{w}_{xx}(\tau, x)d\tau ds = \tilde{v} + \rho\tilde{u}.$$

$$\begin{aligned}\rho u - \left(c - \int_0^\infty g(s)e^{-s}ds\right)u_{xx} \quad &= \quad \tilde{v} + \rho\tilde{u} \\ &\quad -\int_0^\infty g(s)e^{-(s-\tau)}\tilde{w}_{xx}(\tau, x)d\tau ds. \end{aligned} \tag{32}$$

We thus obtain a linear differential equation of the second order with second member. The homogeneous equation has characteristic equation

$$-\left(c - \int_0^\infty g(s)e^{-s}ds\right)r^2 + \rho = 0. \tag{33}$$

The main goal at this level is to obtain two real solutions for equation (33). So, the constant c must be greater than $\int_0^\infty g(s)e^{-s}ds$. That is to say

$$\left(c - \int_0^\infty g(s)e^{-s}ds\right) > 0.$$

Under this condition, we have

$$-\left(c - \int_0^\infty g(s)e^{-s}ds\right) < 0.$$

The characteristic equation (33) admits two reals solutions. Consequently the differential equation (32) has a unique solution u. So, for any triplet $(\tilde{u}, \tilde{v}, \tilde{w}) \in H$, there exists a triplet $(u, v, w) \in D(A)$ such that $(I - A)(u, v, w) = (\tilde{u}, \tilde{v}, \tilde{w})$, that is $I - A$ is surjective and $Im(I - A) = H$.

$$\begin{aligned}< (u, v, w), (\tilde{u}, \tilde{v}, \tilde{w}) > \quad &= \quad \int_0^\pi \Bigg(au_x\tilde{u}_x + \rho^{-1}v\tilde{v} \\ &\quad + \int_0^\infty g(s)[u_x - w_x][\tilde{u}_x - \tilde{w}_x]ds\Bigg)dx. \end{aligned} \tag{34}$$

$$A(u, v, w) = (\rho^{-1}v, cu_{xx}(t, x) - \int_0^\infty g(s)w_{xx}(t, s, x)ds, w_s). \tag{35}$$

$$< A(u, v, w), (u, v, w) >= \tag{36}$$
$$\int_0^\pi \left(a\rho^{-1}v_xu_x + \rho^{-1}v(cu_{xx}(t, x) - \int_0^\infty g(s)w_{xx}(t, s, x)ds)\right)dx$$

$$+\int_0^\pi \left(\int_0^\infty g(s)(\rho^{-1}v_x + w_{sx})(u_x - w_x)ds\right)dx.$$

$$\begin{aligned} < A(u,v,w),(u,v,w) > \quad = \quad & \int_0^\pi \left((c - \int_0^\infty g(s)ds)\rho^{-1}v_x u_x\right)dx \\ & + \int_0^\pi \Bigg(\rho^{-1}v(cu_{xx}(t,x) - \int_0^\infty g(s)w_{xx}(t,s,x)ds) \\ & + \int_0^\infty g(s)(\rho^{-1}v_x u_x - \rho^{-1}v_x w_x + w_{sx}u_x - w_{sx}w_x)ds\Bigg)dx. \end{aligned}$$

$$< A(u,v,w),(u,v,w) >= c\rho^{-1}\int_0^\pi v_x u_x dx \tag{37}$$

$$-\int_0^\pi\int_0^\infty g(s)(\rho^{-1}v_x u_x)dsdx + c\rho^{-1}\int_0^\pi v u_{xx}dx - \rho^{-1}\int_0^\pi\int_0^\infty g(s)vw_{xx}dsdx$$
$$+\rho^{-1}\int_0^\pi\int_0^\infty g(s)v_x u_x dsdx - \rho^{-1}\int_0^\pi\int_0^\infty g(s)v_x w_x dsdx$$
$$+\int_0^\pi\int_0^\infty g(s)w_{sx}u_x dsdx - \int_0^\pi\int_0^\infty g(s)w_{sx}w_x dsdx.$$

$$\int_0^\pi v u_{xx}dx = -\int_0^\pi v_x u_x dx. \tag{38}$$

$$\int_0^\pi\int_0^\infty g(s)vw_{xx}dsdx = -\int_0^\pi\int_0^\infty g(s)v_x w_x dsdx. \tag{39}$$

$$\begin{aligned} \int_0^\pi\int_0^\infty g(s)w_{sx}w_x dsdx = & -\frac{1}{2}\int_0^\pi g(0)u_x^2 dx \\ & -\frac{1}{2}\int_0^\pi\int_0^\infty g'(s)w_x^2 dsdx. \end{aligned} \tag{40}$$

$$\begin{aligned} \int_0^\pi\int_0^\infty g(s)u_x w_{sx}dsdx = & -\int_0^\pi g(0)u_x^2 dx \\ & -\int_0^\pi\int_0^\infty g'(s)u_x w_x dsdx. \end{aligned} \tag{41}$$

$$\begin{aligned} < A(u,v,w),(u,v,w) >= & \frac{1}{2}\int_0^\pi\int_0^\infty g'(s)w_x^2 dsdx \\ & -\frac{1}{2}\int_0^\pi\int_0^\infty g'(s)w_x u_x dsdx - \frac{1}{2}\int_0^\pi g(0)u_x^2 dx. \end{aligned} \tag{42}$$

$g(0) = -\int_0^\infty g'(s)ds$ then

$$< A(u,v,w),(u,v,w) >= \frac{1}{2}\int_0^\pi\int_0^\infty g'(s)w_x^2 dsdx \tag{43}$$

$$-\int_0^\pi\int_0^\infty g'(s)w_xu_xdsdx+\frac{1}{2}\int_0^\pi\int_0^\infty g'(s)u_x^2dx.$$

$$< A(u,v,w),(u,v,w) >= \frac{1}{2}\int_0^\pi\int_0^\infty g'(s)(w_x-u_x)^2dsdx. \tag{44}$$

The main goal at this level for equation (44), is to get $< A(u,v,w),(u,v,w) >\leq 0$. So for that, the function g has to be nonincreasing. That is to say $g'(s)\leq 0$. Under this condition, we have $< A(u,v,w),(u,v,w) >\leq 0$ and $< -A(u,v,w),(u,v,w) >\geq 0$. So the operator $-A$ is monotone.
The operator $-A$ is monotone and $Im(I-A)=H$ so the operator $-A$ is maximal monotone. □

We deduce the following theorem.

Theorem 3.1. *The system 9 admits a unique solution $Z(t)(x)=T(t)\xi(x)$ for all $\xi\in D(A)$ and $x\in]0,\pi[$ where $(T(t))_{t\geq 0}$ is a C_0-semi group of contraction on H.*

We then turn to the case of $\mu_1\neq 0$ and $\mu_2\neq 0$, we obtain the system

$$\begin{cases} Z'(t)=AZ(t)+h(t), & t\geq 0,\\ Z(0)=\xi, \end{cases} \tag{45}$$

with

$$h(t)=\begin{pmatrix} 0\\ \mu_1u_t(t,x)+\mu_2u_t(t-s,x)\\ 0\end{pmatrix}.$$

Since $u\in H_0^1(0,\pi)\cap H^2(0,\pi)$, we have

$$(\mu_1u_t(t,x)+\mu_2u_t(t-s,x))\in L^1(0,T;H_0^1(0,\pi)\cap H^2(0,\pi)), \tag{46}$$

so $h\in L^1(0,T;H_0^1(0,\pi)\cap H^2(0,\pi))$. Since A is the infinitesimal generator and $h\in L^1(0,T;H_0^1(0,\pi)\cap H^2(0,\pi))$,
we deduce the following theorem

Theorem 3.2. *The system* (45) *admits a unique solution*

$$Z(t)(x)=T(t)\xi(x)+\int_0^t T(t-s)h(s)ds, \tag{47}$$

for all $\xi\in D(A)$ and $x\in[0,\pi]$ where $(T(t))_{t\geq 0}$ is a C_0-semi group of contraction on H.

3.2. **Stability of the solution of the initial value problem.**
The stability of the solution is shown in a particular case $\mu_1=\mu_2=0$.

A C_0-semigroup $T(t)$ on a Banach space X is called strongly stable if

$$\lim_{t\longrightarrow+\infty}\|T(t)p\|=0,$$

for every $p \in X$ [2].

For the strong stability of the system, we have shown that the mapping $\varphi : t \longmapsto \|T(t)\xi(x)\|$ is nonincreasing and bounded below on $[0, +\infty[$. So its limit gives $\inf_{t\geq 0} \varphi(t)$. Under the conditions $v(0, x) = 0$ and $w(0, s, x) = ku(0, x)$ with $k = \dfrac{c}{\int_0^{+\infty} g(s)ds}$, the function φ is nonincreasing. Thus, for the system to be strongly stable, it is necessary that $\inf_{t\geq 0} \|T(t)\xi(x)\| = 0$.

Proposition 3.2.

$$\lim_{t\longrightarrow+\infty} \|T(t)\xi(x)\| = 0,$$

for all $\xi \in H$.

Proof.

$$\rho u - a u_{xx} = \tilde{v} + \lambda\rho\tilde{u} - \int_0^\infty g(s)e^{-(s-\tau)}\tilde{w}_{xx}(\tau, x)d\tau, \tag{48}$$

with $a = (c - \int_0^\infty g(s)e^{-s}ds) > 0$.

$$u_{xx} - \frac{\rho}{a}u = -\frac{1}{a}(\tilde{v} + \rho\tilde{u} - \int_0^\infty g(s)e^{-(s-\tau)}\tilde{w}_{xx}(\tau, x)d\tau). \tag{49}$$

$$u_{xx} - \omega^2 u = f(t, x), \tag{50}$$

where $\omega = \sqrt{\frac{\rho}{a}}$ and

$$f(t, x) = -\frac{1}{a}(\tilde{v} + \rho\tilde{u} - \int_0^\infty g(s)e^{-(s-\tau)}\tilde{w}_{xx}(\tau, x)d\tau). \tag{51}$$

The characteristic equation admits two real solutions ω and $-\omega$.

$$u(t, x) = c_1e^{\omega x} + c_2e^{-\omega x} + \int_0^x (c_1e^{\omega(x-y)} + c_2e^{-\omega(x-y)})f(t, y)dy. \tag{52}$$

$u(t, 0) = 0 \Longrightarrow c_1 + c_2 = 0$ then $c_1 = -c_2$.

$$u(t, x) = c_1(e^{\omega x} - e^{-\omega x}) + c_1 \int_0^x (e^{\omega(x-y)} - e^{-\omega(x-y)})f(t, y)dy. \tag{53}$$

$$u(t, x) = c_1[e^{\omega x} - e^{-\omega x} + \int_0^x (e^{\omega(x-y)} - e^{-\omega(x-y)})f(t, y)dy]. \tag{54}$$

$$v(t, x) = \rho u_t = c_1\frac{d}{dt}\int_0^x (e^{\omega(x-y)} - e^{-\omega(x-y)})f(t, y)dy. \tag{55}$$

$$\begin{aligned} w(t, s, x) &= u(t - s, x), \\ &= c_1[e^{\omega x} - e^{-\omega x} + \int_0^x (e^{\omega(x-y)} - e^{-\omega(x-y)})f(t - s, y)dy]. \end{aligned} \tag{56}$$

$csh(x) = \frac{e^x+e^{-x}}{2}$ and $sh(x) = \frac{e^x-e^{-x}}{2}$. Then

$$u(t,x) = 2c_1 sh(\omega x) + 2c_1 \int_0^x sh(\omega(x-y))f(t,y)dy. \tag{57}$$

$$v(t,x) = 2c_1 \frac{d}{dt}\int_0^x sh(\omega(x-y))f(t,y)dy. \tag{58}$$

$$w(t,s,x) = 2c_1 sh(\omega x) \tag{59}$$
$$+2c_1 \int_0^x sh(\omega(x-y))f(t-s,y)dy.$$

$$T(t)\xi(x) = Z(t)(x) = (u(t,x), v(t,x), w(t,x)), \tag{60}$$

with $\xi(x) = Z(0)(x) = (u_0(x), v_0(x), w_0(x))$.

$$\frac{d^+}{dt}\|T(t)\xi(x)\| = \lim_{h\longrightarrow 0^+} \frac{\|T(t+h)\xi(x)\| - \|T(t)\xi(x)\|}{h}. \tag{61}$$

$$\forall x, y, \|x\| - \|y\| \leq \|x-y\|.$$

$$\|T(t+h)\xi(x)\| - \|T(t)\xi(x)\| \leq \|T(t+h)\xi(x) - T(t)\xi(x)\|.$$

$$\forall h > 0, \frac{\|T(t+h)\xi(x)\| - \|T(t)\xi(x)\|}{h} \leq \frac{\|T(t+h)\xi(x) - T(t)\xi(x)\|}{h}.$$

$$\forall h > 0, \frac{\|T(t+h)\xi(x)\| - \|T(t)\xi(x)\|}{h} \leq \frac{\|T(t).T(h)\xi(x) - T(t)\xi(x)\|}{h}.$$

$$\forall h > 0, \frac{\|T(t+h)\xi(x)\| - \|T(t)\xi(x)\|}{h} \leq \|T(t)\| \left\|(\frac{T(h)-I}{h})\xi(x)\right\|.$$

$$\lim_{h\longrightarrow 0^+} \left\|(\frac{T(h)-I}{h})\xi(x)\right\| = \|A\xi(x)\|. \tag{62}$$

$$\frac{d^+}{dt}\|T(t)\xi(x)\| \leq \|A\xi(x)\|, \tag{63}$$

since $(T(t))_{t\geq 0}$ is a C_0-semi group of contraction.

$$A\xi(x) = \frac{d}{dt}\Big|_{t=0} Z(t)(x) = \begin{pmatrix} \rho^{-1}v(0,x) \\ cu_{xx}(0,x) - \int_0^\infty g(s)w_{xx}(0,s,x)ds \\ w_s(0,s,x) \end{pmatrix}. \tag{64}$$

We have

$$w_s(0,s,x) = -u_t(0,x) = \rho^{-1}v(0,x). \tag{65}$$

$v(0,x) = 0$, then $w_s(o,s,x) = 0$ and so $w(0,s,x)$ is constant with respect to s.

$$\int_0^\infty g(s)w_{xx}(0,s,x)ds = w_{xx}(0,s,x)\int_0^\infty g(s)ds. \tag{66}$$

$w(0,s,x) = ku(0,x)$, then $w_{xx}(0,s,x) = ku_{xx}(0;x)$. So

(67)
$$cu_{xx}(0,x) - \int_0^\infty g(s)w_{xx}(0,s,x)ds = cu_{xx}(0,x) - w_{xx}(0,s,x)\int_0^\infty g(s)ds.$$

(68) $$= cu_{xx}(0,x) - ku_{xx}(0,x)\int_0^\infty g(s)ds.$$

(69) $$= cu_{xx}(0,x) - cu_{xx}(0,x).$$

(70) $$= 0.$$

With $k = \frac{c}{\int_0^{+\infty} g(s)ds}$. We have

(71) $$cu_{xx}(0,x) - \int_0^\infty g(s)w_{xx}(0,s,x)ds = 0,$$

and

$$v(0,x) = w_s(0,s,x) = 0,$$

then $A\xi(x) = 0$ and $\|A\xi\| = 0$.
So,

(72) $$\frac{d^+}{dt}\|T(t)\xi(x)\| \leq 0.$$

Let $\varphi(t) = \|T(t)\xi(x)\|$. Then φ is nonincreasing and bounded below so,

(73) $$\lim_{t\longrightarrow+\infty}\varphi(t) = \inf_{t\geq 0}\varphi(t) = 0,$$

and $(T(t))_{t\geq 0}$ is strongly stable. □

4. Conclusion

We have proved in our work, the existence, uniqueness and stability of the solution to integro differentiel system (6) that is motivated evolution linear viscoelasticity body. Our results coincide with conditions $\rho = 1$, $c = 1$ of Kirane and Houari.

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8. Conflict of interest

The authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership,, or other equity interest, and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationship, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

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