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JENSEN'S INEQUALITY AND tgs-CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In recent years, many researches have been done on the tgs-convex functions and their applications. In this article, we present some properties of the tgs-convex functions by interesting examples. Then we investigate the non-positive property of the tgs-convex functions. Also, we derive types of the Jensen's inequality for the tgs-convex functions and obtain several inequalities with respect to the Jensen's inequality. Finally, we give some applications of these inequalities.

Keywords: Jensen's inequality, tgs-convex function, Global bounds.
2020 MSC: Primary 26A51, 26B25, 26D20. 26D15, 26D10.

1. Introduction and Preliminaries

Recently, many researchs and surveis have been published on mathematical inequalities and their applications in ergodic theory (see [11, 12, 14, 20–22])and convex analysis (see [1, 3–8, 13, 15–19, 23, 24]). In the last few decades, mathematical inequalities and their generalization for convex functions have attracted wide attention. Convex analysis has an important role in the development of inequalities theory. Applying the convexity property of functions, researchers have extracted many inequality theories. In fact, the convexity property of functions is base of some inequalities such as the arithmetic mean, harmonic mean inequality also in inequality with respect to entropies including Shannon's inequality, Ky Fan's inequality and etc. In applied literature of mathematical inequalities, the Jensen inequality is a well-known, paramount, and extensively used inequality. This inequality is as follows: Let $f : I \longrightarrow \mathbb{R}$ be a convex function. Then the inequality

$$f(\sum_{i=1}^{n} p_i x_i) \leq \sum_{i=1}^{n} p_i f(x_i)$$

holds for every convex combination $\sum_{i=1}^{n} p_i x_i$ of points $x_i \in I$. Recently, generalizations and improvements of Jensen's inequality have been considered by many researchers. It has been generalized to some functions including, s-convex, m-convex, etc. The concept of the tgs-convex functions was introduced

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by the authors in [25]. They have obtained some results on these functions. Here we examine this definition and conclude the only non-negative tgs-convex function is zero. In fact, we prove that any tgs-convex function is non-positive. Hence, we assume that f is an arbitrary function that means that f is not necessarily non-negative. In this paper, we try to find Jensen's inequality for tgs-convex functions. Also, we obtain some inequalities with respect to Jensen's inequality with some applications.

Definition 1.1. Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \longrightarrow \mathbb{R}$ is said to be convex if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

holds for every $x, y \in I$ and every $\lambda \in [0, 1]$.

The following definition for non-negative functions can be found in ([9], [16], [23], [24] [25]).

Definition 1.2. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a function. The function f is called a tgs-convex function on I if the inequality

$$f(tx + (1-t)y) \leq t(1-t)(f(x) + f(y))$$

holds for all $x, y \in I$ and $t \in (0, 1)$.
Notice that in this definition, the non-negativity costraint on f is removed.

Definition 1.3. Let $x_1, ..., x_n \in I$ be n points, and let $p_1, ..., p_n \in [0, 1]$ be n coefficients such that $\sum_{i=1}^{n} p_i = 1$. The summmation $\sum_{i=1}^{n} p_i x_i$ is called the convex combination of points x_i (with coefficients p_i).

2. Results and proofs

First, we prove that tgs-convex functions cannot be positive.

Proposition 2.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a tgs-convex function. Then $f \leq 0$.*

Proof. Suppose that $x = y \in I$ and $t = \frac{1}{2}$. Since f is a tgs-convex, we have

$$f(\frac{1}{2}x + \frac{1}{2}x) \leq \frac{1}{4}(2f(x))$$

that is, $f(x) \leq 0$ for every x. □

Remark 2.2. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative tgs-convex function. Then, $f = 0$.

Lemma 2.1 introduces an example of tgs-convex functions.

Lemma 2.3. *For $a \geq 1$, let $f : [a, b] \to \mathbb{R}$ be a function defined by $f(x) = -\log x$. Then f is a tgs-convex function.*

Proof. Let $x, y \in [a, b]$ and $0 < t < 1$ be arbitrary. Since $a \geq 1$, we have $x, y \geq 1$. Hence, $x^{t(1-t)} \leq x^t$ and $y^{t(1-t)} \leq y^{1-t}$. Thus

$$(xy)^{t(1-t)} = x^{t(1-t)} y^{t(1-t)} \leq x^t y^{t(1-t)} \leq x^t y^{1-t}. \tag{1}$$

On the other hand, by Young's inequality, we have

$$x^t y^{1-t} \leq tx + (1-t)y.$$

Then by Equation (2.1), we obtain

$$(xy)^{t(1-t)} \leq tx + (1-t)y.$$

Hence,

$$\log((xy)^{t(1-t)}) \leq \log(tx + (1-t)y),$$

we get

$$t(1-t)(\log x + \log y) \leq \log(tx + (1-t)y),$$

and also,

$$-\log(tx + (1-t)y) \leq t(1-t)(-\log x - \log y).$$

□

In the following, we consider the relationship between convex functions and tgs-convex functions in view of Definition 1.2.

Theorem 2.4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a negative convex function. Then f is a tgs-convex function.*

Proof. Let $t \in [0, 1]$ and $x, y \in I$. Then $t(1-t) \leq t,\ t(1-t) \leq 1-t$. Since $f \leq 0$, we get $tf(x) \leq t(1-t)f(x)$ and $(1-t)f(y) \leq t(1-t)f(y)$. Hence

$$\begin{aligned} f(tx + (1-t)y) &\leq tf(x) + (1-t)f(y) \\ &\leq t(1-t)f(x) + t(1-t)f(y) \\ &= t(1-t)(f(x) + f(y)). \end{aligned}$$

□

Example 2.5. *The function $f(x) = -\sqrt{x}$ on $[0, +\infty)$ is tgs-convex.*

Example 2.6. *The function $f(x) = -\ln x$ on $[1, +\infty)$ is tgs-convex.*

In the following, we present a function f such that f is a tgs-convex which it is not convex.

Example 2.7. *Let $f : [0, 4] \longrightarrow \mathbb{R}$ be a function defined by*

$$f(x) = \begin{cases} -1 & \text{if } 0 \leq x < 2 \\ -2 & \text{if } 2 \leq x \leq 4 \end{cases}.$$

Then f is tgs-convex. In fact, let $x, y \in [0, 4]$ and $t \in [0, 1]$. We have

$$f(tx + (1-t)y) \leq -1 \leq -4t(1-t) \leq t(1-t)(f(x) + f(y)).$$

Note that $t(1-t) \leq \frac{1}{4}$ *for each* $t \in [0,1]$. *The function* f *is not convex because for* $x = 0, y = 2, t = \frac{1}{2}$ *we have*

$$f(1) \nleq \frac{f(0)+f(2)}{2}.$$

Theorem 2.8. *Let* $f : I \to \mathbb{R}$ *be a tgs-convex function and* r *be a root of* f *such that* $r \in int(I)$. *Then* f *is identically 0.*

Proof. Let $x, y \in I$ be arbitrary such that $x < r < y$. There exists $t \in (0,1)$ such that $r = tx + (1-t)y$. Hence

$$0 = f(r) = f(tx + (1-t)y) \leq t(1-t)(f(x) + f(y)) \leq 0.$$

Thus, $f(x) = f(y) = 0$ (since $f \leq 0$) which implies that f is zero on I. □

Remark 2.9. The condition $r \in int(I)$ in Theorem 2.8 is essential, see the following example,

Example 2.10. *Let* $f : [0,1] \longrightarrow \mathbb{R}$ *be a function defined by*

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ -1 & \text{if } 0 < x \leq 1 \end{cases}.$$

Then f *is tgs-convex which it is not identically 0. In fact, let* $x, y \in [0,1]$ *and* $t \in [0,1]$. *We have*

$$f(tx + (1-t)y) = -1 \leq -\frac{1}{2} \leq t(1-t)(f(x) + f(y)).$$

Note that $t(1-t) \leq \frac{1}{4}$ *for each* $t \in [0,1]$.

Remark 2.11. Let $f : I \to \mathbb{R}$ be a function, $b > 0$ and $-2b \leq f(x) \leq -b$. Then f is a tgs-convex function.

Proof. Let $x, y \in I$ and $t \in [0,1]$. Then we have

$$f(tx + (1-t)y) \leq -b \leq -4bt(1-t) \leq t(1-t)(f(x) + f(y)).$$

□

In the following, we present some results on Jensen's inequality.

Theorem 2.12. *Let* $f : [a,b] \to \mathbb{R}$ *be a tgs-convex function and* $t_i \in [0,1]$ *such that* $\sum_{i=1}^{n} t_i = 1$. *Then*

$$f(\sum_{i=1}^{n} t_i x_i) \leq \frac{1}{n} \sum_{i=1}^{n} t_i(1-t_i) f(x_i).$$

Proof. Let $t_1, t_2 \in [0,1]$ such that $t_1 + t_2 = 1$. We have

$$\begin{aligned} f(t_1x_1 + t_2x_2) &\leq t_1t_2(f(x_1) + f(x_2)) \\ &= t_1t_2f(x_1) + t_1t_2f(x_2) \\ &\leq t_1t_2f(x_1) \ (\textbf{and} \ \leq t_1t_2f(x_2)) \text{ since } f(x) \leq 0 \\ &= t_1(1-t_1)f(x_1) \ (\textbf{and} \ \leq t_2(1-t_2)f(x_2)). \end{aligned}$$

So $2f(t_1x_1+t_2x_2) \leq t_1t_2(f(x_1)+f(x_2))$. Hence, $f(t_1x_1+t_2x_2) \leq \frac{1}{2}t_1t_2(f(x_1)+f(x_2))$. Now, we prove that the result holds for n. Let $\alpha_j = \sum_{k=1,k\neq j}^{n} t_k$. We have,

$$f(\sum_{k=1}^{n} t_kx_k) = f(\alpha_j(\sum_{k=1,k\neq j}^{n} \frac{t_k}{\alpha_j}x_k) + t_jx_j) \leq t_j(1-t_j)f(x_j),\ j = 1, 2, \ldots, n.$$

In fact, we prove that $f(\sum_{i=1}^{n} t_ix_i)$ is less than $t_i(1-t_i)f(x_i)$ for each $1 \leq i \leq n$. By summing the above inequalities, we have

$$nf(\sum_{i=1}^{n} t_ix_i) \leq \sum_{i=1}^{n} t_i(1-t_i)f(x_i).$$

□

Since $f(\sum_{i=1}^{n} t_ix_i)$ is less than $t_i(1-t_i)f(x_i)$ for each $1 \leq i \leq n$, we can improve the result of Theorem2.12 as follows:

Example 2.13. *Assume that $x_1, \ldots, x_n \in \mathbb{R}$. Then we have*

(1) *if $x_i \geq 1$, then we*

$$\sum_{i=1}^{n} t_ix_i \geq \prod_{i=1}^{n} x_i^{\frac{t_i(1-t_i)}{n}};$$

(2) *if $x_i \geq 1$, then we have*

$$\frac{\sum_{i=1}^{n} x_i}{n} \geq (\prod_{i=1}^{n} x_i)^{\frac{n-1}{n^3}};$$

(3) *if $n \in \mathbb{N}$, then we have*

$$n + 1 \geq 2[n!]^{\frac{n-1}{n^3}}.$$

Proof. (1): In Theorem 2.12, put $f(x) = -\log x$.
(2): In (1), put $t_i = \frac{1}{n}$.
(3): In (2), put $x_i = i$. □

Remark 2.14. Let $f : I \to \mathbb{R}$ be a tgs-convex function $n \geq 2$ and $t_i \in [0, 1]$ such that $\sum_{i=1}^{n} t_i = 1$. Then we have

$$f(\sum_{i=1}^{n} t_ix_i) \leq \min_{1\leq j\leq n} \{t_j(1-t_j)f(x_j)\}.$$

We give another relation with respect to Jensen's inequality.

Theorem 2.15. *Let $f : I \to \mathbb{R}$ be a tgs-convex function $n \geq 2$ and $t_i \in [0, 1]$ such that $\sum_{i=1}^{n} t_i = 1$. Then we have*

Proof.

$$
\begin{aligned}
f(\sum_{i=1}^{n} t_i x_i) \leq \prod_{i=1}^{n} t_i f(x_1) + t_1 \prod_{i=2}^{n} t_i f(x_2) + (t_1 + t_2) \prod_{i=3}^{n} t_i f(x_4) \\
+ (t_1 + t_2 + t_3) \prod_{i=4}^{n} t_i f(x_i) + \cdots + (\sum_{i=1}^{n-2} t_i) t_{n-1} t_n f(x_{n-1}) \\
+ (\sum_{i=1}^{n-1} t_i) t_n f(x_n).
\end{aligned}
$$

Proof by induction. For $n = 2$, the result is trivially held. Assume that for $n - 1$, the result holds. Specifically, we have

$$
\begin{aligned}
f(\sum_{j=1}^{n-1} t_i x_j) \leq \prod_{j=1}^{n-1} t_j f(x_1) + t_1 \prod_{j=2}^{n-1} t_j f(x_2) + (t_1 + t_2) \prod_{j=3}^{n-1} t_j f(x_4) \\
+ (t_1 + t_2 + t_3) \prod_{j=4}^{n-1} t_j f(x_j) + \cdots + (\sum_{j=1}^{n-2} t_j) t_{n-1} f(x_{n-1}),
\end{aligned}
$$

which $\sum_{j=1}^{n-1} t_j = 1$. Now, we prove that the result holds for n. For this purpose, assume that $\sum_{i=1}^{n} t_i = 1,\ t_i \in [0, 1]$ and $\alpha = \sum_{i=1}^{n-1} t_i$. We

$$
\begin{aligned}
f(\sum_{i=1}^{n} t_i x_i) &= f(\sum_{i=1}^{n-1} t_i x_i + t_n x_n) = f(\alpha \sum_{i=1}^{n-1} \frac{t_i}{\alpha} x_i + t_n x_n) \\
&\leq \alpha t_n (f(\sum_{i=1}^{n-1} \frac{t_i}{\alpha} x_i) + f(x_n)) \\
&= \alpha t_n f(\sum_{i=1}^{n-1} \frac{t_i}{\alpha} x_i) + \alpha t_n f(x_n) \\
&\leq \alpha t_n [\prod_{i=1}^{n-1} \frac{t_i}{\alpha} f(x_1) + \frac{t_1}{\alpha} \prod_{i=2}^{n-1} \frac{t_i}{\alpha} f(x_2) + (\frac{t_1}{\alpha} + \frac{t_2}{\alpha}) \prod_{i=3}^{n-1} \frac{t_i}{\alpha} f(x_4)
\end{aligned}
$$

$$+ (\frac{t_1}{\alpha} + \frac{t_2}{\alpha} + \frac{t_3}{\alpha}) \prod_{i=4}^{n-1} \frac{t_i}{\alpha} f(x_i) + \cdots + (\sum_{i=1}^{n-2} \frac{t_i}{\alpha}) \frac{t_{n-1}}{\alpha} f(x_{n-1})] + \alpha t_n f(x_n)$$

$$\leq \frac{t_n}{\alpha^{n-2}} \prod_{i=1}^{n-1} t_i f(x_1) + \frac{t_n}{\alpha^{n-2}} t_1 \prod_{i=2}^{n-1} t_i f(x_2) + \frac{t_n}{\alpha^{n-3}} (t_1 + t_2) \prod_{i=3}^{n-1} t_i f(x_4)$$

$$+ \frac{t_n}{\alpha^{n-4}} (t_1 + t_2 + t_3) \prod_{i=4}^{n-1} t_i f(x_i) + \cdots + \frac{t_n}{\alpha} (\sum_{i=1}^{n-2} t_i) t_{n-1} f(x_{n-1})$$

$$+ (\sum_{i=1}^{n-1} t_i) t_n f(x_n)$$

$$\leq \prod_{i=1}^{n} t_i f(x_1) + t_1 \prod_{i=2}^{n} t_i f(x_2) + (t_1 + t_2) \prod_{i=3}^{n} t_i f(x_4)$$

$$+ (t_1 + t_2 + t_3) \prod_{i=4}^{n} t_i f(x_i) + \cdots + (\sum_{i=1}^{n-2} t_i) t_{n-1} t_n f(x_{n-1})$$

$$+ (\sum_{i=1}^{n-1} t_i) t_n f(x_n).$$

Note that since $0 < \alpha \leq 1$ and $1 \leq j \leq n$, we have $\frac{1}{\alpha} \geq 1$ and $f(x) \leq 0$. We get

$$\frac{1}{\alpha^j} (\sum_{k=1}^{m} t_k) \Pi_{s=m+1}^{n} t_s f(x_s) \leq (\sum_{k=1}^{m} t_k) \Pi_{s=m+1}^{n} t_s f(x_s).$$

□

Example 2.16. *Let $f : I \to \mathbb{R}$ be a tgs-convex function and $x_1, \ldots, x_n \in \mathbb{R}$. Then*

(1) *we have*

$$f(\frac{\sum_{i=1}^{n} x_i}{n}) \leq \frac{f(x_1) + \sum_{i=2}^{n} (i-1) n^{i-2} f(x_i)}{n^n};$$

(2) *if $x_i \geq 1$, then we*

$$(\frac{\sum_{i=1}^{n} x_i}{n})^{n^n} \geq x_1 \prod_{i=2}^{n} x_i^{(i-1)n^{i-2}}.$$

Proof. (1): In Theorem 2.15, put $t_i = \frac{1}{n}$.
(2): In (1), put $f(x) = -\log x$. □

Theorem 2.17. *Let $f : [a, b] \to \mathbb{R}$ be a tgs-convex function and $t_i \in [0, 1]$ such that $\sum_{i=1}^{n} t_i = 1$. Then*

$$f(\sum_{i=1}^{n} t_i x_i) \leq \prod_{i=1}^{n} t_i (\sum_{i=1}^{n} f(x_i)).$$

Proof. Proof by induction. For $m=2$, the result is trivially held. Assume that for m, the result holds Specifically, we have

$$f(\sum_{i=1}^{m} t_i x_i) \leq \prod_{i=1}^{m} t_i (\sum_{i=1}^{m} f(x_i)),$$

which $\sum_{i=1}^{m} t_i = 1$. Now, we prove that the result holds for $m+1$. For this purpose, assume that $\sum_{k=1}^{m+1} t_k = 1$, we have

$$\begin{aligned}
f(\sum_{k=1}^{m+1} t_k x_k) &= f(\sum_{k=1}^{m-1} t_k x_k + (t_m + t_{m+1}) \frac{t_m x_m + t_{m+1} x_{m+1}}{t_m + t_{m+1}}) \\
&\leq \prod_{k=1}^{m-1} t_k (t_m + t_{m+1}) (\sum_{k=1}^{m-1} f(x_k) + f(\frac{t_m x_m + t_{m+1} x_{m+1}}{t_m + t_{m+1}})) \\
&\leq \prod_{k=1}^{m-1} t_k (t_m + t_{m+1}) (\sum_{k=1}^{m-1} f(x_k) + \frac{t_m t_{m+1}}{(t_m + t_{m+1})^2} [f(x_m) + f(x_{m+1})] \\
&= \prod_{k=1}^{m-1} t_k (t_m + t_{m+1}) (\sum_{k=1}^{m-1} f(x_k)) + \frac{1}{t_m + t_{m+1}} \prod_{k=1}^{m+1} t_k [f(x_m) + f(x_{m+1})].
\end{aligned}$$

Note that since $f(x) \leq 0$, we have

$$(t_m + t_{m+1}) \prod_{k=1}^{m-1} t_k \geq \prod_{k=1}^{m+1} t_k,$$

$$\frac{1}{t_m + t_{m+1}} \prod_{k=1}^{m+1} t_k \geq \prod_{k=1}^{m+1} t_k.$$

□

3. Applications

In this section, we present an application of our results.

Proposition 3.1. *Let $a_0, a_1, \ldots, a_n \in \mathbb{R}$ where $a_0 \neq 0$, and suppose that for $i = 1, 2, \ldots, n, x_i \in \mathbb{R}$ and $t_i \in [0,1]$ such that $\sum_{i=1}^{n} t_i = 1$. Then we have*

(1) *if $x_i \geq 1$, then we*

$$\Big(\prod_{i=1}^{n} x_i\Big)^{\prod_{i=1}^{n} t_i} \leq \sum_{i=1}^{n} t_i x_i;$$

(2) *if $x_i \geq 1$, then we*

$$\sqrt[n^n]{\prod_{i=1}^{n} x_i} \leq \frac{\sum_{i=1}^{n} x_i}{n};$$

(3) *if* $0 < a_0 \leq a_1 \leq \cdots \leq a_n$ *then*

$$\sqrt[n^n]{\frac{a_n}{a_0}} \leq \frac{\sum_{i=1}^{n} \frac{a_i}{a_{i-1}}}{n}.$$

Proof. (1): Using Lemma 2.3, it is proved that the function $f(x) = -\log x$ is tgs-convex on $[1, +\infty]$. Now, appling Theorem 2.17 for $-\log x$, we have

$$-\log(\sum_{i=1}^{n} t_i x_i) \leq (\prod_{i=1}^{n} t_i)(-\sum_{i=1}^{n} \log x_i),$$

hence

$$\log(\sum_{i=1}^{n} t_i x_i) \geq (\prod_{i=1}^{n} t_i)(\sum_{i=1}^{n} \log x_i) = \log\big((\prod_{i=1}^{n} x_i)^{\Pi_{i=1}^{n} t_i}\big).$$

(2): In (1), put $t_i = \frac{1}{n}$.
(3): In (2), put $x_i = \frac{a_i}{a_{i-1}}$. □

4. Conclusion

This paper investigated the tgs-convex functions. It was proven that if we consider the tgs-convex function as a non-negative function, it must be the zero function. So we conclude that any tgs-convex function is non-positive. Also, we presented three versions of Jensen inequality for tgs-convex functions.

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