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NEW GENERALIZATION OF THE BEST PROXIMITY POINT PROBLEM

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Abstract. Let (C, D) be a nonempty pair of disjoint subsets of a metric space. Main purpose of this paper is to present a range of a convergence sequence to $u \in C \cup D$ such that $d(Tu, fu) = dist(C, D)$, for mappings $T, f : C \cup D \to C \cup D$. In fact, we give a generalization of best proximity point results for cyclic contractive mappings. To this end, we consider an example is presented to support the main result.

Keywords: Common best proximity point, coincidence point, metric space, fixed point.
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1. Introduction

Let X be a metric space and C and D nonempty subsets of X. Put

$$C_\circ = \{u \in C : d(u, v) = d(C, D) \ for \ some \ v \in D\},$$
$$D_\circ = \{u \in D : d(u, v) = d(C, D) \ for \ some \ v \in C\}.$$

If there is a pair $(\vartheta_\circ, \varsigma_\circ) \in C \times D$ for which $d(\vartheta_\circ, \varsigma_\circ) = d(C, D)$,that $d(C, D)$ is distance of C and D, then the pair $(\vartheta_\circ, \varsigma_\circ)$ is said to a best proximity pair for (C, D). Best proximity pair evolves as a expansion of the concept of best approximation.

We can find the best proximity points of (C, D), by considering a map $T : C \cup D \to C \cup D$. We say that the point $u \in C \cup D$ is a best proximity point of the pair (C, D), if $d(u, Tu) = d(C, D)$ and we denote the set of all best proximity points of (C, D) by $P_T(C, D)$, that is

$$P_T(C, D) = \{u \in C \cup D : d(u, Tu) = d(C, D)\}.$$

Best proximity point also evolves as a expansion of the concept of fixed point of mappings, because if $C \cap D \neq \emptyset$ every best proximity point is a fixed point of T.

Eldred et al. [3] and Sankar Raj et al. [10] gave a best proximity point theorem for relatively nonexpansive mappings. . A best proximity point theorem for contraction has been obtained by Basha [6]. Best proximity point theorems

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for various variants of contractions have been explored in [4, 5]

There have been a large number of publications that contribute to the subject common fixed point theorems (see [1, 2]). Common best proximity points, have become one of the most studied topics in the field of common fixed point theory. These notions generalize common fixed points and allow us to deal with nonself-mappings. There are many more results on common best proximity points in the literature that for instance we can see [7–9].

In the following we give some theorems that we need they in main results. We recall that the mapping $T : C \cup D \to C \cup D$ is relatively nonexpansive if $d(T\vartheta, T\varsigma) \leq d(\vartheta, \varsigma)$ for any $(\vartheta, \varsigma) \in C \times D$.

Definition 1.1. A Banach space X is said to be uniformly convex if for any ε, $0 < \varepsilon \leq 2$, the inequalities $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$ imply there exists a $\delta = \delta(\varepsilon) > 0$ such that $\frac{\|x+y\|}{2} \leq 1 - \delta$.

Before stating the first existence result of best proximity points (pairs), we recall that the mapping $\mathcal{T} : C \cup D \to C \cup D$ is *relatively nonexpansive* if $\|\mathcal{T}x - \mathcal{T}y\| \leq \|x - y\|$ for any $(x, y) \in C \times D$.

Theorem 1.2. *[3] Let (C, D) be a nonempty, bounded, closed and convex pair of disjoint subsets of a uniformly convex Banach space X. If $\mathcal{T} : C \cup D \to C \cup D$ is a cyclic relatively nonexpansive mapping, then $\mathcal{T}$ has a best proximity point.*

Theorem 1.3. *[3] Let (C, D) be a nonempty, bounded, closed and convex pair of disjoint subsets of a uniformly convex Banach space X. If $\mathcal{T} : C \cup D \to C \cup D$ is a relatively nonexpansive mapping and*

$$\mathcal{T}(C) \subset C, \ \mathcal{T}(D) \subset D,$$

then $\mathcal{T}$ has a best proximity point.

Definition 1.4. [4] Let X be a complete metric space and C and D subsets of X. A map $\mathcal{T} : C \cup D \to C \cup D$ is a cyclic contraction map if it satisfies:

(i) $\mathcal{T}(C) \subset D, \ \mathcal{T}(C) \subset D$
(ii) $d(\mathcal{T}x, \mathcal{T}y) \leq kd(x, y) + (1 - k)dist(C, D)$ for all $x \in C$ and $y \in D$.

Theorem 1.5. *[4] Let (C, D) be a nonempty closed convex pair of disjoint subsets of a uniformly convex Banach space X. If $\mathcal{T} : C \cup D \to C \cup D$ is a cyclic contraction mapping and either C or D is boundedly compact, then $\mathcal{T}$ has a unique best proximity point. Further, if $\tau_\circ \in C$ and $\tau_{n+1} = \mathcal{T}\tau_n$, then $\{\tau_{2n}\}$ converges to the best proximity point.*

The best proximity point problem is an unconstrained optimization problem, which is an interesting generalization of the fixed point problem and has many applications in engineering problems. Main purpose of this paper gives a new generalization of the best proximity point problem for cyclic contractive mappings. We present a range of a convergence sequence to $u \in C \cup D$ such that $d(Tu, fu) = dist(C, D)$, for mappings $T, f : C \cup D \to C \cup D$. In fact, we give a generalization of best proximity point results. It is notable that if the mapping f be onto, then we find the best proximity point.

2. Generalization of the common best proximity point

We start our main results by a generalization of the common best proximity point theorem.

Theorem 2.1. *Let C and D be closed disjoint subsets of complete metric space X where C is compact. Suppose the mappings $S, T, f : C \cup D \to C \cup D$ that f is continuous satisfy:*

$$d(Su, Tv) \leqslant \alpha[d(fu, Tv) + d(fv, Su)] + \gamma d(fu, fv) + \mu d(C, D),$$

for all $u \in C$ and $v \in D$, where α, γ, γ are positive real numbers with

$$2\alpha + \gamma + \mu = 1.$$

If $S(C) \cup T(C) \subseteq f(D)$, $S(D) \cup T(D) \subseteq f(C)$, $f(C) \subseteq A$, $f(D) \subseteq D$, then there exist a point $u \in C \cup D$ such that $d(fu, Su) = d(C, D)$ and $d(fu, Tu) = d(C, D)$.

Proof. Let ϑ_0 be an arbitrary point in C. Choose a point ϑ_1 in D such that $f\vartheta_1 = S\vartheta_0$. Similarly, choose a point ϑ_2 in C such that such $f\vartheta_2 = T\vartheta_1$. Continue this process until selected ϑ_n in X such that $\vartheta_{2n} \in C$ $\vartheta_{2n+1} \in D$ we obtain ϑ_{2n+1} in D such that

$$\begin{aligned} &f\vartheta_{2k+1} = S\vartheta_{2k}, \\ &f\vartheta_{2k+2} = T\vartheta_{2k+1}, \quad (k \geqslant 0). \end{aligned}$$

Then,

$$\begin{aligned} d(f\vartheta_{2k+1}, f\vartheta_{2k+2}) &= d(S\vartheta_{2k}, T\vartheta_{2k+1}) \\ &\leqslant \alpha d(f\vartheta_{2k}, T\vartheta_{2k+1}) + \alpha d(f\vartheta_{2k+1}, S\vartheta_{2k}) \\ &\quad + \gamma d(f\vartheta_{2k}, f\vartheta_{2k+1}) + \mu d(C, D) \\ &\leqslant [\alpha + \gamma] d(f\vartheta_{2k}, f\vartheta_{2k+1}) + \alpha d(f\vartheta_{2k+1}, f\vartheta_{2k+2}) + \mu d(C, D). \end{aligned}$$

This implies:

$$[1 - \alpha] d(f\vartheta_{2k+1}, f\vartheta_{2k+2}) \leqslant [\alpha + \gamma] d(f\vartheta_{2k}, f\vartheta_{2k+1}) + \mu d(C, D).$$

Thus,

$$d(f\vartheta_{2k+1}, f\vartheta_{2k+2}) \leqslant \left[\frac{\alpha + \gamma}{1 - \alpha}\right] d(f\vartheta_{2k}, f\vartheta_{2k+1}) + \big(1 - \frac{\alpha + \gamma}{1 - \alpha}\big) d(C, D).$$

Now, by induction, we obtain:

$$\begin{aligned} d(f\vartheta_{2k+1}, f\vartheta_{2k+2}) &\leqslant \left[\frac{\alpha+\gamma}{1-\alpha}\right] d(f\vartheta_{2k}, f\vartheta_{2k+1}) + \big(1 - \frac{\alpha+\gamma}{1-\alpha}\big) d(C,D) \\ &\leqslant \left[\frac{\alpha+\gamma}{1-\alpha}\right]^2 d(f\vartheta_{2k-1}, f\vartheta_{2k}) + \big(1 - (\frac{\alpha+\gamma}{1-\alpha})^2\big) d(C,D) \\ &\leqslant \cdots \leqslant \left[\frac{\alpha+\gamma}{1-\alpha}\right]^{k+1} d(f\vartheta_0, f\vartheta_1) + 1 - \big(\frac{\alpha+\gamma}{1-\alpha}\big)^{k+1} d(C,D), \end{aligned}$$

for each $k \geqslant 0$. Let $\lambda = \left[\frac{\alpha+\gamma}{1-\alpha}\right]$, $0 < \lambda < 1$. Hence $d(f\vartheta_{2k+1}, f\vartheta_{2k+2}) \to d(C,D)$ and so

$$d(f\vartheta_{2k+1}, T\vartheta_{2k+1}) \to d(C,D), d(f\vartheta_{2k}, S\vartheta_{2k}) \to d(C,D).$$

Because $\{\vartheta_{2n}\} \subseteq C$ and C is compact, so $\vartheta_{2n_k} \to u$, therefore $f\vartheta_{2n_k} \to fu$. Moreover, we have

$$d(C,D) \leq d(fu, f\vartheta_{2n_k+1}) \leq d(fu, f\vartheta_{2n_k}) + d(f\vartheta_{2n_k}, f\vartheta_{2n_k+1})$$

Then, $d(fu, Su) = d(C,D)$ and similarly $d(fu, Tu) = d(C,D)$. □

In the following we give a new form of Theorem 2.1 where compactness of C is omitted.

Theorem 2.2. *Let C and D be closed disjoint subsets of complete metric space X . Suppose the mappings $S, T, f : C \cup D \longrightarrow C \cup D$ that f is continuous satisfy:*

$$d(Su, Tv) \leqslant \alpha[d(fu, Tv) + d(fv, Su)] + \gamma d(fu, fv) + \mu d(C,D),$$

for all $u \in C$ and $v \in D$, also

$$d(Tu, Ty) \leqslant \lambda d(fu, fv), \ d(Su, Sy) \leqslant \lambda d(fu, fv),$$

for all $u, v \in C$ or $u, v \in D$, where $\alpha, \beta, \gamma, \lambda, \zeta$ are positive real numbers with

$$2\alpha + \gamma + \mu = 1, \ \lambda < 1.$$

If $S(C) \cup T(C) \subseteq f(D)$, $S(D) \cup T(D) \subseteq f(C)$, $f(C) \subseteq C$, $f(D) \subseteq D$, then there exist a point $u \in C \cup D$ such that $d(fu, Su) = d(C,D)$ and $d(fu, Tu) = d(C,D)$.

Proof. Let ϑ_0 be an arbitrary point in C. Choose a point ϑ_1 in D such that $f\vartheta_1 = S\vartheta_0$. Similarly, choose a point ϑ_2 in C such that such $f\vartheta_2 = T\vartheta_1$. Continuing this process till having chosen ϑ_n in X such that $\vartheta_{2n} \in C$ $\vartheta_{2n+1} \in D$ we obtain ϑ_{2n+1} in D such that

$$\begin{aligned} f\vartheta_{2k+1} &= S\vartheta_{2k}, \\ f\vartheta_{2k+2} &= T\vartheta_{2k+1}, \quad (k \geqslant 0). \end{aligned}$$

By a similar proof of the Theorem 2.1:

$$d(f\vartheta_{2k+1}, f\vartheta_{2k+2}) \to d(C,D)$$

and so

$$d(f\vartheta_{2k+1}, T\vartheta_{2k+1}) \to d(C,D), d(f\vartheta_{2k}, S\vartheta_{2k}) \to d(C,D).$$

Now, we show that $\lim_{n\to\infty} d(f\vartheta_{2n}, f\vartheta_{2n+2}) = 0$. Note

$$\begin{aligned} d(f\vartheta_{2n}, f\vartheta_{2n+2}) &= d(T\vartheta_{2n-1}, T\vartheta_{2n+1}) \\ &\leq \lambda d(f\vartheta_{2n-1}, f\vartheta_{2n+1}) \\ &= \lambda d(S\vartheta_{2n-1}, S\vartheta_{2n+1}) \\ &= \lambda^2 d(f\vartheta_{2n-2}, f\vartheta_{2n}) \\ &... \\ &\leq \lambda^{2n} d(f\vartheta_0, f\vartheta_2) \end{aligned}$$

Hence $\lim_{n\to\infty} d(f\vartheta_{2n}, f\vartheta_{2n+2}) = 0$ and so $\{f\vartheta_{2n}\}$ is a Cauchy sequence in D. Because $\{f\vartheta_{2n}\}$ is Cauchy, X is complete and D is closed, $\lim_{n\to\infty} f\vartheta_{2n} = fx \in f(C)$. Now

$$d(C,D) \leq d(fx, S\vartheta_{2n}) = d(fx, f\vartheta_{2n-1}) \leq d(fx, f\vartheta_{2n}) + d(f\vartheta_{2n}, f\vartheta_{2n-1}).$$

Thus we have $d(fx, S\vartheta_{2n})$ converges to $d(C,D)$. and so $d(fx, Sx) = d(C,D)$. Similarly $d(fx, Tx) = d(C,D)$. □

In the following we give a corollary of Theorem 2.2 that it has omitted compactness of C with respect to Theorem 1.5.

Corollary 2.3. *Let C and D be closed disjoint subsets of complete metric space X . Suppose the mappings $T, f : C \cup D \longrightarrow C \cup D$ that $T(C) \subseteq f(D) \subseteq D$, $T(D) \subseteq f(C) \subseteq C$ and*

$$d(Tu, Tv) \leqslant \gamma d(fu, fv) + \mu d(C,D),$$

for all $u \in C$ and $v \in D$, also

$$d(Tu, Tv) \leqslant \lambda d(fu, fy),$$

for all $u, v \in C$ or $u, v \in D$, where γ, μ are positive real numbers with

$$\gamma + \mu = 1,\ 0 < \lambda < 1.$$

If f is continuous , then there exist a point $u \in C \cup D$ such that $d(fu, Tu) = d(C,D)$. Moreover, if f be onto, then T has a best proximity point.

Corollary 2.4. *Let C and D be closed disjoint subsets of complete metric space X . Suppose the mappings $T, f : C \cup D \longrightarrow C \cup D$ that $T(C) \subseteq f(D) \subseteq D$, $T(D) \subseteq f(C) \subseteq C$ and*

$$d(Tx, Ty) \leqslant \gamma d(fx, fy) + \mu d(C,D),$$

for all $x \in C$ and $y \in D$, where $\gamma, , \mu$ are positive real numbers with $\gamma + \mu = 1$. If f is continuous and C is compact, then there exist a point $u \in C \cup D$ such that $d(fu, Tu) = d(C,D)$. Moreover, if f is onto, then T has a best proximity point.

Let C and D be nonempty subsets of a normed linear space X. We shall say that a pair (C,D) satisfies a property if both C and D satisfy that property. For example, (C,D) is convex if and only if both C and D are convex. Let $f : C \cup D \longrightarrow C \cup D$ that $f(D) \subseteq D$, $f(C) \subseteq C$. Now, we will define $(C_\circ^f, D_\circ^f)$.

$$C_\circ^f := \{x \in C : \|x - fy\| = \operatorname{dist}(C,D), \quad \text{for some} \;\; y \in D\},$$

$$D_\circ^f := \{y \in D : \|fx - y\| = \operatorname{dist}(C,D), \quad \text{for some} \;\; x \in C\}.$$

Notice that, by Theorem 1.3, suppose (C,D) is a nonempty, bounded, closed and convex pair in a uniformly convex Banach space X, if f is a relatively nonexpansive then $(C_\circ^f, D_\circ^f)$ is nonempty, closed and convex pair in X.

Theorem 2.5. *Let X be a normed space, C and D be nonempty subsets of X and $c_\circ \in C_\circ^f$ Suppose the mappings $T, f : C \cup D \longrightarrow C \cup D$ that $T(C) \subseteq f(D) \subseteq D$, $T(D) \subseteq f(C) \subseteq C$ and*

$$\|Tu - Tv\| \leqslant \|fu - fv\| \;\; u \in C, \;\; v \in D,$$

If C is a compact set, then there exist $u \in C \cup D$ with $\|fu - Tu\| = dist(C,D)$.

Proof. Suppose $c_\circ \in C_\circ^f$. Then there is $d_\circ \in D$ such that $\|c_\circ - fd_\circ\| = dist(C,D)$. For arbitrary $\vartheta_\circ \in C$, put

$$T_n(x) = \begin{cases} \frac{1}{n} fd_\circ + (1 - \frac{1}{n})Tx & x \in C \\ \\ \frac{1}{n} c_\circ + (1 - \frac{1}{n})Tx & x \in D \end{cases} \tag{1}$$

$$\|T_n(x) - T_n(y)\| \leqslant \frac{1}{n} dist(C,D) + (1 - \frac{1}{n})\|Tx - Ty\|,$$

and so

$$\|T_n(x) - T_n(y)\| \leqslant \frac{1}{n} dist(C,D) + (1 - \frac{1}{n})\|x - y\|.$$

Hence by Corollary 2.4 for every $n \in \mathbb{N}$ there is $\vartheta_n \in C$ such that

$$\|T_n(\vartheta_n) - f(\vartheta_n)\| = dist(C,D).$$

Since C is compact, there is subsequence $\{\vartheta_{n_k}\}$ and $u \in C$ such that $\vartheta_{n_k} \to u$. Therefore we have

$$\begin{aligned} \|Tu - fu\| &= \lim_{k\to\infty} \|T\vartheta_{n_k} - f\vartheta_{n_k}\| \\ &= \lim_{k\to\infty} \|T_{n_k}\vartheta_{n_k} - f\vartheta_{n_k}\| = dist(C,D). \end{aligned}$$

□

In the following we give a generalization of Theorem 1.2. Hence we first define the proximal normal structure.

Definition 2.6. [3] A convex pair (K_1, K_2) in a Banach space is said to have proximal normal structure if for any closed, bounded, convex proximal pair $(H_1, H_2) \subseteq (K_1, K_2)$ for which $dist(H_1, H_2) = dist(K_1, K_2)$ and $\delta(H_1, H_2) > dist(H_1, H_2)$, there exists $(x_1, x_2) \in H_1 \times H_2$ such that $\delta(x_1, H_2) < \delta(H_1, H_2)$ and $\delta(x_2, H_1) < \delta(H_1, H_2)$.

Theorem 2.7. *Let (C, D) be a nonempty, bounded, closed and convex pair of disjoint subsets of a uniformly convex Banach space X and let (C, D) has proximal normal structure. Suppose the mappings $T, f : C \cup D \longrightarrow C \cup D$ that T is relatively nonexpansive, f is continuous and $T(C) \subseteq f(D) \subseteq D$, $T(D) \subseteq f(C) \subseteq C$. If $Tfu = fTu$ for all $u \in C \cup D$, then there exist a point $u \in C \cup D$ such that $d(fu, Tu) = d(C, D)$.*

Proof. Since X is a uniformly convex Banach space, f is continuous and $(C_\circ^f, D_\circ^f)$ are closed and convex, hence $C_\circ^f, D_\circ^f$ are weakly compact and convex. Also, we have $dist(C_\circ^f, D_\circ^f) = dist(C, D)$ from the definition of $C_\circ^f$ and $D_\circ^f$. Let $u \in C \cup D$. Then there exists $z \in D$ such that $\|u - fz\| = dist(C, D)$. Thus

$$\|Tu - fTz\| = \|Tu - Tfz\| \leqslant \|u - fz\| = dist(C, D).$$

This implies $Tu \in D_\circ^f$ and so $T(C_\circ^f) \subseteq D_\circ^f$. Similarly, $T(D_\circ^f) \subseteq C_\circ^f$. Also

$$\|Tu - Tv\| \leqslant \|u - v\|, \ \forall u \in C_\circ^f, v \in D_\circ^f.$$

Hence $(C_\circ^f, D_\circ^f)$ also has proximal normal structure. Now, by same proof of the Theorem 1.2 there is $u \in C$ such that $\|fu - Tu\| = dist(C, D)$.

□

In the following, we will present an example that shows that the point resulting from the generalization of the best approximate point is not necessarily the best approximate point.

Example 2.8. *Consider $X = \mathbb{R}^2$ with the usual norm, $C = [1, 2] \times [1, 2]$, $D = [-2, -1] \times [-2, -1]$ and $x_\circ \in C$. Define mappings $T, f : C \cup D \to C \cup D$ by*

$$T(x, y) = \begin{cases} (-\frac{x}{3} - \frac{2}{3}, -\frac{y}{3} - \frac{2}{3}) & (x, y) \in C \\ (-\frac{x}{3} + \frac{2}{3}, -\frac{y}{3} + \frac{2}{3}) & (x, y) \in D \end{cases} \tag{2}$$

$$f(x, y) = \begin{cases} (\frac{x}{2} + \frac{1}{2}, \frac{y}{2} + \frac{1}{2}) & (x, y) \in C \\ (\frac{x}{2} - \frac{1}{2}, \frac{y}{2} - \frac{1}{2}) & (x, y) \in D \end{cases} \tag{3}$$

Obviously, $T(C) \subseteq f(D) \subseteq D$, $T(D) \subseteq f(C) \subseteq C$. Initially, we show that

$$\|T(x, y) - T(u, w)\| \leqslant \|f(x, y) - f(u, w)\|,$$

for $(x, y) \in C$ *and* $(u, w) \in D$. *We have*

$$\begin{aligned}\|T(x,y) - T(u,w)\| &= \|(-\frac{x}{3} - \frac{2}{3} - [-\frac{u}{3} + \frac{2}{3}], -\frac{y}{3} - \frac{2}{3} - [-\frac{w}{3} + \frac{2}{3}])\| \\ &\leqslant \sqrt{|\frac{u-x}{3} - \frac{4}{3}|^2 + |\frac{w-y}{3} - \frac{4}{3}|^2} \\ &\leqslant \sqrt{|\frac{u-x}{2} - 1|^2 + |\frac{w-y}{2} - 1|^2} \\ &\leqslant \|f(x,y) - f(u,w)\|,\end{aligned}$$

and in $f(x^*, y^*) = (x^*, y^*) = (1, 1)$ *such that*

$$\|T(x^*, y^*) - f(x^*, y^*)\| = \|(1,1) - (-1,-1)\| = 2\sqrt{2} = dist(C, D).$$

3. Conclusion

The best proximity point problem is an unconstrained optimization problem, which is an interesting generalization of the fixed point problem. In this paper gives a new generalization of the best proximity point problem. We conculude that if f be continuous and $T(C) \subseteq f(D) \subseteq D$, $T(D) \subseteq f(C) \subseteq C$ and $Tfu = fTu$ for all $u \in C \cup D$, then there exist a point $u_0 \in C \cup D$ such that $d(fu_0, Tu_0) = d(C, D)$. It is notable that if the mapping f be onto, then we find the best proximity point. Also, we present an example that shows that the point resulting from the generalization of the best proximity point is not necessarily the best proximity point.

4. Conflict of interest

The author declare no conflict of interest. Authors identify and declare any personal circumstances or interest that may be perceived as inappropriately influencing the representation or interpretation of reported research results.

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