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CLASSES OF F-HYPERIDEALS IN A KRASNER $F^{(m,n)}$-HYPERRING

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Abstract. Krasner $F^{(m,n)}$-hyperrings were introduced and investigated by Farshi and Davvaz. In this paper, our purpose is to define and characterize three particular classes of F-hyperideals in a Krasner $F^{(m,n)}$-hyperring, namely prime F-hyperideals, maximal F-hyperideals and primary F-hyperideals, which extend similar concepts of ring context. Furthermore, we examine the relations between these structures. Then a number of major conclusions are given to explain the general framework of these structures.

Keywords: Prime F-hyperideal, Maximal F-hyperideal, Primary F-hyperideal, Krasner $F^{(m,n)}$-hyperring.
2020 MSC: 03E72, 20N20, 05C50.

1. Introduction

As it is well known, the notion of a fuzzy set was introduced by Zadeh in 1965 [45]. After the pioneering work of Zadeh, the fuzzy sets have been used in the reconsideration of classical mathematics. A number of articles have applied fuzzy concepts to algebraic structures. Rosenfeld introduced and studied fuzzy sets in the context of group theory and formulated the notion of a fuzzy subgroup of a group [41]. The notions of fuzzy subrings and ideals were defined by Liu [33]. A considerable amount of work has been done on fuzzy ideals. In particular, many papers were written on prime fuzzy ideals. There are many applications of fuzzy algebra, such as in coding theory and automata theory [15].

Hyperstructure theory was born in 1934 when Marty [35], a French mathematician, defined the concept of a hypergroup as a generalization of groups. Many papers and books concerning hyperstructure theory have appeared in literature. For instance, you can see the papers [9, 15, 19, 39, 43]. The simplest algebraic hyperstructures which possess the properties of closure and associativity are called semihypergroups. n-ary semigroups and n-ary groups are algebras with one n-ary operation which is associative and invertible in a generalized sense. The idea of investigations of n-ary algebras goes back to Kasner's

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lecture [26] at the 53rd annual meeting of the American Association of the Advancement of Science in 1904. In 1928, Dorente wrote the first paper concerning the theory of n-ary groups [22]. Later on, Crombez and Timm [7, 8] defined the notion of the (m, n)-rings and their quotient structures. The n-ary hyperstructures have been studied in [29–31, 34, 40]. In [20], Davvaz and Vougiouklis introduced a generalization of the notion of a hypergroup in the sense of Marty and a generalization of an n-ary group, which is called n-ary hypergroup. Mirvakili and Davvaz [37] defined (m, n)-hyperrings and obtained several results in this respect. One important class of hyperrings was introduced by Krasner, where the addition is a hyperoperation, while the multiplication is an ordinary binary operation, which is called Krasner hyperring. In [36], a generalization of the Krasner hyperrings, which is a subclass of (m, n)-hyperrings, was defined by Mirvakili and Davvaz. It is called Krasner (m, n)-hyperring. A Krasner (m, n)-hyperring is an algebraic hyperstructure (R, f, g), or simply R, which satisfies the following axioms: (1) (R, f) is a canonical m-ary hypergroup; (2) (R, g) is a n-ary semigroup; (3) the n-ary operation g is distributive with respect to the m-ary hyperoperation f , i.e., for every $a_1^{i-1}, a_{i+1}^n, x_1^m \in R$, and $1 \leq i \leq n$, $g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), ..., g(a_1^{i-1}, x_m, a_{i+1}^n))$; (4) 0 is a zero element (absorbing element) of the n-ary operation g, i.e., for every $x_2^n \in R$ we have $g(0, x_2^n) = g(x_2, 0, x_3^n) = ... = g(x_2^n, 0) = 0$.
Ameri and Norouzi in [3] introduced some important hyperideals such as Jacobson radical, n-ary prime and primary hyperideals, nil radical, and n-ary multiplicative subsets of Krasner (m, n)-hyperrings. For more study on Krasner (m, n)-hyperring refer to [4–6, 25, 36, 38, 40, 44].

The connections between hyperstructures and fuzzy sets have been studied by a variety of authors. Fuzzy hyperstructures can be classified into three groups. A first group of works studies crisp hyperoperations introduced through fuzzy sets [10, 11, 13, 14]. A second group is about the fuzzy hyperalgebras [1, 2, 16–18]. A third group of works also concerns fuzzy hyperstructures [12, 27, 28, 42].

The concept of F-polygroups was introduced by Zahedi and Hasankhani in [47, 48]. The fuzzy hyperring notion was defined and studied in [32]. In this regards, Motameni and et al. continued the study of the notion of fuzzy hyperideals of a fuzzy hyperring. They defined and characterized prime fuzzy hyperideals and maximal fuzzy hyperideals and studied the hyperideal transfer through a fuzzy hyperring homomorphism. Zhan and et al. in [49] concentrated on the quasi-coincidence of a fuzzy interval value with an interval valued fuzzy set. Davvaz in [39] introduced the notion of a fuzzy hyperideal of a Krasner (m, n)-hyperring and extended the fuzzy results to Krasner (m, n)-hyperring. Let G be an arbitrary set and $L = [0, 1]$ be the unit interval. Let L^G (resp. L_*^G) be the set of all fuzzy subsets of G. An F-hyperoperation on G is a function $\circ$ from $G \times G$ into L_*^G. If $\mu, \gamma \in L_*^G$ and $x \in G$, then $x \circ \mu = \bigcup_{a \in supp(\mu)} x \circ \mu$ and $\mu \circ \gamma = \bigcup_{a \in supp(\mu), b \in supp(\gamma)} a \circ b$ such that $supp(\mu) = \{a \in G \mid \mu(a) \neq 0\}$. The couple $(G, \circ)$ is called an F-polygroup if the following conditions are satisfied: (1) $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in G$, (2) there exists $e \in G$ with $a \in supp(a \circ e \cap e \circ a)$, for all $a \in G$, (3) for each $a \in G$, there exists a unique element $a^{-1} \in G$ with $e \in supp(a \circ a^{-1} \cap a^{-1} \circ a)$, (4) $c \in supp(a \circ b)$ implies that $a \in supp(c \circ b^{-1})$ implies that $b \in supp(a^{-1} \circ c)$, for all $a, b, c \in G$. Indeed, a fuzzy hyperoperation assigns to each pair of elements of G a non-zero fuzzy subset of G, while a hyperoperation assigns to each pair of elements of G a non-empty subset of G.

The concepts of Krasner $F^{(m,n)}$-hyperrings and F-hyperideals were defined in [24] by Farshi and Davvaz. In this paper, we continue the study of F-hyperideals of a Krasner $F^{(m,n)}$-hyperring, initiated in [24]. We define and analyze three particular types of F-hyperideals in a Krasner $F^{(m,n)}$-hyperring, namely prime F-hyperideals, maximal F-hyperideals and primary F-hyperideals. We investigate the connections between them. Moreover, we introduce the concepts of F-radical, quotient Krasner $F^{(m,n)}$-hyperring and Jacobson radical. The overall framework of these structures is then explained. It is shown (Theorem 4.7) that if $\mathfrak{Q}$ is an primary F-hyperideal of a Krasner $F^{(m,n)}$-hyperring $(\mathfrak{R}, f, g)$, then $\sqrt{\mathfrak{Q}}^F$ is a prime F-hyperideal of $\mathfrak{R}$.

2. Preliminaries

In this section we recall some basic terms and definitions from [24] which we need to develop our paper.

A fuzzy subset of G is a function $\mu : G \longrightarrow L$ such that L is the unit interval $[0, 1] \subseteq \mathbb{R}$. The set of all fuzzy subsets of G is denoted by L^G . Let $\mu, \gamma \in L^G$ and $\{\mu_\alpha \mid \alpha \in \Lambda\} \subseteq L^G$. We define the fuzzy subsets $\mu \cup \gamma$ and $\bigcup_{\alpha\in\Lambda} \mu_\alpha$ as follows:

$$(\mu \cup \gamma)(a) = max\{\mu(a), \gamma(a)\}$$

and

$$(\bigcup_{\alpha\in\Lambda} \mu_\alpha)(a) = \bigvee_{\alpha\in\Lambda} \{\mu_\alpha(a)\}$$

for all $a \in G$. The set, $\{a \in G \mid \mu(a) \neq 0\}$ is called the support of μ and is denoted by $supp(\mu)$. When $H \subseteq G$ and $t \in L$, we define $H_t \in L^G$ as follows:

$$H_t(a) = \begin{cases} t & \text{if } a \in H, \\ 0 & \text{if } a \notin H. \end{cases}$$

In particular, if H is a singleton, say $\{x\}$, then $\{x\}_t$ is referred to as fuzzy point and, sometimes, denoted by x_t. The characteristic function of set H is denoted by χ_H. Let $L^G_* = L^G - \{0\}$. For a positive integer n, an F^n-hyperoperation on G is a mapping f from G^n to L^G_*. This means that for any $a_1, \cdots, a_n \in G$, $f(a_1, \cdots, a_n)$ is a non- zero fuzzy subset of G. If for all $a_1, \cdots, a_n \in G$, $supp(f(a_1, \cdots, a_n))$ is singleton, then f is called an F^n-operation.

Notice that the sequence $a_i, a_{i+1}, ..., a_j$ will be denoted by a_i^j. For $j < i$, a_i^j is the empty symbol. Using this notation,

$$f(a_1, ..., a_i, b_{i+1}, ..., b_j, c_{j+1}, ..., c_n)$$

will be written as $f(a_1^i, b_{i+1}^j, c_{j+1}^n)$. The expression will be written in the form $f(a_1^i, b^{(j-i)}, c_{j+1}^n)$, when $b_{i+1} = ... = b_j = b$.

For $\mu_1^n \in L_*^G$, we define $f(\mu_1^n)$ as follows:

$$f(\mu_1^n) = \bigcup_{a_i \in supp(\mu_i)} f(a_1^n)$$

. Let $a_1^n, a \in G$, $H \in P^*(G)$ and $\mu_1^n, \mu \in L_*^G$. Then, for $1 \leq i \leq n$

(1) $f(a_1^{i-1}, \mu, a_{i+1}^n)$ denotes $f(\chi_{\{a_1\}}, \ldots, \chi_{\{a_{i-1}\}}, \mu, \chi_{\{a_{i+1}\}}, \ldots, \chi_{\{a_n\}})$,

(2) $f(a_1^{i-1}, H, a_{i+1}^n)$ denotes $f(\chi_{\{a_1\}}, \ldots, \chi_{\{a_{i-1}\}}, \chi_H, \chi_{\{a_{i+1}\}}, \ldots, \chi_{\{a_n\}})$,

(3) $f(\mu_1^{i-1}, a, \mu_{i+1}^n)$ denotes $f(\mu_1^{i-1}, \chi_{\{a\}}, \mu_{i+1}^n)$,

(4) $f(\mu_1^{i-1}, H, \mu_{i+1}^n)$ denotes $f(\mu_1^{i-1}, \chi_H, \mu_{i+1}^n)$.

If for every $1 \leq i < j \leq n$ and all $a_1^{2n-1} \in G$,

$$f(a_1^{i-1}, f(a_i^{n+i-1}), a_{n+i}^{2n-1}) = f(a_1^{j-1}, f(a_j^{n+j-1}), a_{n+j}^{2n-1}),$$

then the F^n-hyperoperation (F^n-operation) f is called associative. G with the associative F^n-hyperoperation (F^n-operation) is called F^n-semihypergroup (F^n-semigroup).

Definition 2.1. Let (G, f) be a F^m-semihypergroup. Suppose that G is equipped with a unitary operation $^{-1} : G \longrightarrow G$. The couple (G, f) is called a canonical F^m-hypergroup, if

(1) G has an F-identity element, i.e., there exists an element $e \in G$ such that for every $a \in G$, $supp(f(a, e^{(n-1)})) = \{a\}$,

(2) $a \in supp(f(a_1^m))$ implies $a_i \in supp(f(a_1^{-1}, \ldots, a_{i-1}^{-1}, a, a_{i+1}^{-1}, \ldots, a_n^{-1}))$, for all $a_1^m, a \in G$ and $1 \leq i \leq n$,

(3) for all a_1^m and for all $\sigma \in \mathbb{S}_m$, $f(a_1^m) = f(a_{\sigma(1)}^{\sigma(m)})$.

Definition 2.2. A Krasner $F^{(m,n)}$-hyperring is an algebraic hyperstructure $(\mathfrak{R}, f, g)$, or simply $\mathfrak{R}$, which satisfies the following axioms:

(i) $(\mathfrak{R}, f)$ is a canonical F^m-hypergroup,

(ii) $(\mathfrak{R}, g)$ is an F^n-semigroup,

(iii) for every $x_1^{i-1}, x_{i+1}^n, a_1^m \in \mathfrak{R}$ and $1 \leq i \leq n$,

$$g(x_1^{i-1}, f(a_1^m), x_{i+1}^n) = f(g(x_1^{i-1}, a_1, x_{i+1}^n), \ldots, g(x_1^{i-1}, a_m, x_{i+1}^n)),$$

(iv) for every $a_2^n \in \mathfrak{R}$, $supp(g(e, a_2^n)) = \{e\}$ where e is the F-identity element of $(\mathfrak{R}, f)$.

$\mathfrak{R}$ is called commutative Krasner $F^{(m,n)}$-hyperring if $g(a_1^n) = g(x_{\sigma(1)}^{\sigma(n)})$, for all $a_1^n \in \mathfrak{R}$ and for every $\sigma \in \mathbb{S}_n$. In the sequel, we assume that all Krasner $F^{(m,n)}$-hyperrings are commutative. We say that $\mathfrak{R}$ is with scalar F-identity if there exists an element e' such that $supp(g(a, e'^{(n-1)})) = \{a\}$ for all $a \in \mathfrak{R}$.

Definition 2.3. Let $(\mathfrak{R}, f, g)$ be a Krasner $F^{(m,n)}$-hyperring. A non-empty subset S of $\mathfrak{R}$ is support closed under f and g whenever for all $a_1^m, b_1^n \in S$, $supp(f(a_1^m)) \subseteq S$ and $supp(g(b_1^n)) \subseteq S$. S is called a Krasner F-subhyperring of $(\mathfrak{R}, f, g)$ if (S, f, g) is itself a Krasner $F^{(m,n)}$-hyperring. A Krasner F-subhyperring $\mathfrak{I}$ of $(\mathfrak{R}, f, g)$ is said to be an F-hyperideal if $supp(g(a_1^{i-1}, \mathfrak{I}, a_{i+1}^n)) \subseteq \mathfrak{I}$ for all $a_1^n \in \mathfrak{R}$ and $1 \leq i \leq n$.

Definition 2.4. Let $(\mathfrak{R}_1, f_1, g_1)$ and $(\mathfrak{R}_2, f_2, g_2)$ be two Krasner (m, n)-hyperrings. A mapping $h : \mathfrak{R}_1 \longrightarrow \mathfrak{R}_2$ is called a homomorphism if for all $x_1^m \in \mathfrak{R}_1$ and $y_1^n \in \mathfrak{R}_1$ we have

(1) $h(e_{\mathfrak{R}_1}) = e_{\mathfrak{R}_2}$ such that $e_{\mathfrak{R}_1}$ and $e_{\mathfrak{R}_2}$ are F-identity elements of $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively,

(2) $h(supp(f_1(x_1, ..., x_m))) = supp(f_2(h(x_1), ..., h(x_m)))$

(3) $h(supp(g_1(y_1, ..., y_n))) = supp(g_2(h(y_1), ..., h(y_n)))$.

3. Prime F-hyperideals and maximal F-hyperideals

We start this section by introducing the concept of prime F-hyperideals of a Krasner $F^{(m,n)}$-hyperring $(\mathfrak{R}, f, g)$.

Definition 3.1. An F-hyperideal $\mathfrak{P}$ of a Krasner $F^{(m,n)}$-hyperring $(\mathfrak{R}, f, g)$ is called a prime F-hyperideal if for all $\mu_1^n \in L_*^{\mathfrak{R}}$, $supp(g(\mu_1^n)) \subseteq \mathfrak{P}$ implies that $supp(\mu_i) \subseteq \mathfrak{P}$ for some $1 \leq i \leq n$.

Our first theorem characterizes the prime F-hyperideals of a Krasner $F^{(m,n)}$-hyperring $(\mathfrak{R}, f, g)$.

Theorem 3.2. *Let $\mathfrak{P}$ be an F-hyperideal of a Krasner $F^{(m,n)}$-hyperring $(\mathfrak{R}, f, g)$. Then $\mathfrak{P}$ is a prime F-hyperideal if and only if for all $a_1^n \in \mathfrak{R}$, $supp(g(a_1^n)) \subseteq \mathfrak{P}$ implies that $a_i \in \mathfrak{P}$ for some $1 \leq i \leq n$.*

Proof. $\Longrightarrow$ Let $\mathfrak{P}$ be a prime F-hyperideal of $\mathfrak{R}$. Let for $a_1^n \in \mathfrak{R}$, $supp(g(a_1^n)) \subseteq \mathfrak{P}$. Then $supp(g(\chi_{\{a_i\}}, \ldots, \chi_{\{a_n\}})) \subseteq \mathfrak{P}$. Since $\mathfrak{P}$ is a prime F-hyperideal of $\mathfrak{R}$, we have $supp(\chi_{\{a_i\}})) \subseteq \mathfrak{P}$ for some $1 \leq i \leq n$. This implies that $\{a_i\} \subseteq \mathfrak{P}$ and so $a_i \in \mathfrak{P}$, as needed.
$\Longleftarrow$Let $supp(g(\mu_1^n)) \subseteq \mathfrak{P}$ for some $\mu_1^n \in L_*^{\mathfrak{R}}$. Let for all $1 \leq i \leq n$, $supp(\mu_i) \nsubseteq \mathfrak{P}$. Then for each $1 \leq i \leq n$, there exists an element $a_i \in supp(\mu_i)$ such that $a_i \notin \mathfrak{P}$. By Proposition 3.2 in [24], $supp(g(a_1^n)) \subseteq supp(g(\mu_1^n))$. Therefore we have $supp(g(a_1^n)) \subseteq \mathfrak{P}$. By the hypothesis, there exists $1 \leq i \leq n$ such that $a_i \in \mathfrak{P}$ and this is a contradiction. Thus $\mathfrak{P}$ is a prime F-hyperideal of $\mathfrak{R}$. □

Example 3.3. *Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers and $t_1, t_2 \in (0, 1]$. We define an F^m-operation f and an F^n- operation g on $\mathbb{Z}$ as follows:*

$$f(a_1, \cdots, a_m) = (a_1 + \cdots + a_m)_{t_1} \quad \text{for all } a_1, \cdots, a_m \in \mathbb{Z}$$

$$g(a_1, \cdots, a_n) = (a_1. \cdots .a_n)_{t_2} \qquad \text{for all } a_1, \cdots, a_n \in \mathbb{Z}.$$

It is easy to verify that $(\mathbb{Z}, f, g)$ is a Krasner $F^{(m,n)}$-hyperring. All F-hyperideals

$p\mathbb{Z}$, where p is a prime natural number are prime F-hyperideals of Krasner $F^{(m,n)}$-hyperring $(\mathbb{Z}, f, g)$.

Example 3.4. *Consider the Krasner $F^{(m,n)}$-hyperring (G, f, g), given in Example 3.3 in [24]. Let $a \in G$. Then $\{e, a\}$ is a hyperideal of G but it is not a prime F-hyperideal of G.*

Let $\mathfrak{I}$ be a normal F-hyperideal of a Krasner $F^{(m,n)}$-hyperring $(\mathfrak{R}, f, g)$. The set $[\mathfrak{R} : \mathfrak{I}^*] = \{\mathfrak{I}^*[x] \mid x \in \mathfrak{R}\}$ is a Krasner (m, n)-hyperring with m-hyperoperation $f_{|I^*}$ and n-operation $g_{|I^*}$ as follows:

$$f_{|I^*}(I^*{}^{[x_n]}_{[x_1]}) = \{I^*[x]\}, \quad \forall x \in supp(f(x_1^m))$$

$$g_{|I^*}(I^*{}^{[x_n]}_{[x_1]}) = \{I^*[supp(g(x_1^n))]\}.$$

Thus $[\mathfrak{R} : \mathfrak{I}^*]_i$ is a Krasner $F^{(m,n)}$-hyperring (for more details refer to [24]). Now, we determine when the F-hyperideal $[\mathfrak{J} : \mathfrak{I}^*]_i$ of $[\mathfrak{R} : \mathfrak{I}^*]_i$ is prime.

Theorem 3.5. *Let $\mathfrak{I}$ be a normal F-hyperideal of a Krasner $F^{(m,n)}$-hyperring $(\mathfrak{R}, f, g)$ and let $\mathfrak{J}$ be an F-hyperideal of $(\mathfrak{R}, f, g)$. If $\mathfrak{J}$ is a prime F-hyperideal of $\mathfrak{R}$, then $[\mathfrak{J} : \mathfrak{I}^*]_i$ is a prime F-hyperideal of $[\mathfrak{R} : \mathfrak{I}^*]_i$.*

Proof. By Theorem 5.10 in [24], $[\mathfrak{J} : \mathfrak{I}^*]_i$ is an F-hyperideal of $[\mathfrak{R} : \mathfrak{I}^*]_i$. Let for $I^*{}^{[x_n]}_{[x_1]} \in [\mathfrak{R} : \mathfrak{I}^*]_i$, $g_{|I^*}(I^*{}^{[x_n]}_{[x_1]}) \subseteq [\mathfrak{J} : \mathfrak{I}^*]_i$. Then $I^*[supp(g(x_1^n))] \subseteq [\mathfrak{J} : \mathfrak{I}^*]_i$. This implies that $supp(g(x_1^n)) \subseteq \mathfrak{J}$. Since $\mathfrak{J}$ is a prime F-hyperideal of $\mathfrak{R}$, then there exists $1 \leq i \leq n$ such that $x_i \in \mathfrak{J}$. This means that $I^*[x_i] \in [\mathfrak{J} : \mathfrak{I}^*]_i$. Thus $[\mathfrak{J} : \mathfrak{I}^*]_i$ is a prime F-hyperideal of $[\mathfrak{R} : \mathfrak{I}^*]_i$. □

The following result investigates the stability of prime F-hyperideals property under a transfer.

Theorem 3.6. *Let $(\mathfrak{R}_1, f_1, g_1)$ and $(\mathfrak{R}_2, f_2, g_2)$ be two Krasner $F^{(m,n)}$-hyperrings and $h : \mathfrak{R}_1 \longrightarrow \mathfrak{R}_1$ be a homomorphism. If $\mathfrak{P}$ is a prime F-hyperideal of $\mathfrak{R}_2$, then $h^{-1}(\mathfrak{P})$ is a prime F-hyperideal of $\mathfrak{R}_2$.*

Proof. Suppose that $(\mathfrak{R}_1, f_1, g_1)$ and $(\mathfrak{R}_2, f_2, g_2)$ are two Krasner $F^{(m,n)}$-hyperrings and $h : \mathfrak{R}_1 \longrightarrow \mathfrak{R}_1$ is a homomorphism. Let $supp(g_1(a_1^n)) \subseteq h^{-1}(\mathfrak{P})$ for some $a_1^n \in \mathfrak{R}_1$. Then we obtain $h(supp(g_1(a_1^n))) \subseteq \mathfrak{P}$ which implies $supp(g_2(h(a_1), \cdots, h(a_n)) \subseteq \mathfrak{P}$. Since $\mathfrak{P}$ is a prime F-hyperideal of $\mathfrak{R}_2$, there exists some $1 \leq i \leq n$ such that $h(a_i) \in \mathfrak{P}$. Therefore $a_i \in h^{-1}(\mathfrak{P})$, as needed. □

Let $(\mathfrak{R}_1, f_1, g_1)$ and $(\mathfrak{R}_2, f_2, g_2)$ be two Krasner $F^{(m,n)}$-hyperrings and $\mathfrak{R}_1 \times \mathfrak{R}_2 = \{(a, b) \mid a \in \mathfrak{R}_1, b \in \mathfrak{R}_2\}$. Then, by Proposition 5.6 in [24], $(\mathfrak{R}_1 \times \mathfrak{R}_2, f_\otimes, g_\otimes)$ is a Krasner $F^{(m,n)}$-hyperring, where

$$f_\otimes((a_1, b_1), \cdots, (a_m, b_m))(a, b) = min\{f_1(a_1^m)(a), f_2(b_1^m)(b)\}$$

$$g_\otimes((a_1, b_1), \cdots, (a_n, b_n))(a, b) = min\{g_1(a_1^n)(a), g_2(b_1^n)(b)\}$$

Now, we establish the following result.

Theorem 3.7. *Let $(\mathfrak{R}_1, f_1, g_1)$ and $(\mathfrak{R}_2, f_2, g_2)$ be two Krasner $F^{(m,n)}$-hyperrings. If $\mathfrak{P}_1$ is a prime F-hyperideal of $\mathfrak{R}_1$, then $\mathfrak{P}_1 \times \mathfrak{R}_2$ is a prime F-hyperideal of $\mathfrak{R}_1 \times \mathfrak{R}_2$.*

Proof. Let $supp(g_{\otimes}((a_1, b_1), \cdots, (a_n, b_n))) \subseteq \mathfrak{P}_1 \times \mathfrak{R}_2$ for $(a_1, b_1), \cdots, (a_n, b_n) \in \mathfrak{R}_1 \times \mathfrak{R}_2$. Therefore we have

$$\begin{aligned}
&supp(g_{\otimes}((a_1, b_1), \cdots, (a_n, b_n)))\\
&\quad = \{(a, b) \mid g_{\otimes}((a_1, b_1), \cdots, (a_n, b_n))(a, b) \neq 0\}\\
&\quad = \{(a, b) \mid min\{g_1(a_1^n)(a), g_2(b_1^n)(b)\} \neq 0\}\\
&\quad \subseteq \mathfrak{P}_1 \times \mathfrak{R}_2.
\end{aligned}$$

This implies that $supp(g_1(a_1^n)) \subseteq \mathfrak{P}_1$. Since $\mathfrak{P}_1$ is a prime F-hyperideal of $\mathfrak{R}_1$, we get $a_i \in \mathfrak{P}_1$ for some $1 \leq i \leq n$. Thus we have $(a_i, b_i) \in \mathfrak{P}_1 \times \mathfrak{R}_2$. Consequently, $\mathfrak{P}_1 \times \mathfrak{R}_2$ is a prime F-hyperideal of $\mathfrak{R}_1 \times \mathfrak{R}_2$. □

Let $\mathfrak{I}$ be a F-hyperideal of a Krasner $F^{(m,n)}$-hyperring $(\mathfrak{R}, f, g)$. Then the set

$$\mathfrak{R}/\mathfrak{I} = \{supp(f(a_1^{i-1}, \mathfrak{I}, a_{i+1}^m)) \mid a_1^{i-1}, a_{i+1}^n \in \mathfrak{R}\}$$

endowed with F^m-hyperoperation f which for all $a_{11}^{1m}, \ldots, a_{m1}^{mm} \in \mathfrak{R}$

$$\begin{aligned}
&f(supp(f(a_{11}^{1(i-1)}, \mathfrak{I}, a_{1(i+1)}^{1m})), \ldots, supp(f(a_{m1}^{m(i-1)}, \mathfrak{I}, a_{m(i+1)}^{mm})))\\
&= supp(f(supp(f(a_{11}^{m1})), \ldots, supp(f(a_{1(i-1)}^{m(i-1)})), \mathfrak{I}, supp(f(a_{1(i+1)}^{m(i+1)})), \ldots, supp(f(a_{1m}^{mm})))
\end{aligned}$$

and with F^n-operation g which for all $a_{11}^{1m}, \ldots, a_{n1}^{nm} \in \mathfrak{R}$

$$\begin{aligned}
&g(supp(f(a_{11}^{1(i-1)}, \mathfrak{I}, a_{1(i+1)}^{1m})), \ldots, supp(f(a_{n1}^{n(i-1)}, \mathfrak{I}, a_{n(i+1)}^{nm})))\\
&= supp(f(supp(g(a_{11}^{n1})), \ldots, supp(g(a_{1(i-1)}^{n(i-1)})), \mathfrak{I}, supp(g(a_{1(i+1)}^{n(i+1)})), \ldots, supp(g(a_{1m}^{nm})))
\end{aligned}$$

construct a Krasner $F^{(m,n)}$-hyperring, and $(\mathfrak{R}/\mathfrak{I}, f, g)$ is called the quotient Krasner $F^{(m,n)}$-hyperring of R by $\mathfrak{I}$.

Theorem 3.8. *Let $\mathfrak{I}$ be an F-hyperideal of a Krasner $F^{(m,n)}$-hyperring $(\mathfrak{R}, f, g)$. Then the natural map $\pi : \mathfrak{R} \longrightarrow \mathfrak{R}/\mathfrak{I}$, by $\pi(a) = supp(f(a, \mathfrak{I}, e^{(m-2)}))$ is an epimorphism.*

Proof. Let $\mathfrak{I}$ be an F-hyperideal of a Krasner $F^{(m,n)}$-hyperring $(\mathfrak{R}, f, g)$. It is clear that π is a projection map. We have to show that π is a homomorphism. $\pi(e_R) = supp(f(e_R, \mathfrak{I}, e_R^{(m-2)})) = \mathfrak{I} = e_{R/\mathfrak{I}}$, by Lemma 3.14 in [24]. For all $a_1^m \in \mathfrak{R}$,

$$\begin{aligned}
\pi(supp(f(a_1^m)) &= supp(f(supp(f(a_1^m)), \mathfrak{I}, \{e\}^{(m-2)})\\
&= supp(f(supp(f(a_1^m)), \mathfrak{I}, (supp(f(e^m))^{(m-2)}))\\
&= supp(f(supp(f(a_1, \mathfrak{I}, e^{(m-2)})), \ldots, supp(f(a_m, \mathfrak{I}, e^{(m-2)}))\\
&= supp(f(\pi(a_1), \ldots, \pi(a_m)).
\end{aligned}$$

Furthermore, for all $b_1^n \in \mathfrak{R}$ we have

$$\begin{aligned}
\pi(supp(g(b_1^n)) &= supp(f(supp(g(b_1^n)), \mathfrak{I}, \{e\}^{(m-2)})\\
&= supp(f(supp(g(b_1^n)), \mathfrak{I}, (supp(g(e^n))^{(m-2)}))\\
&= supp(g(supp(f(b_1, \mathfrak{I}, e^{(m-2)})), \ldots, supp(f(b_n, \mathfrak{I}, e^{(m-2)})))\\
&= supp(g(\pi(b_1), \ldots, \pi(b_n)).
\end{aligned}$$

Hence, π is a homomorphism. □

Definition 3.9. A Krasner $F^{(m,n)}$-hyperring $(\mathfrak{R}, f, g)$ is called a hyperintegral F-domain, if for all $a_1^n \in \mathfrak{R}$, $supp(g(a_1^n)) = \{e\}$ implies that $a_i = e$ for some $1 \leq i \leq n$.

Example 3.10. *Let $(H, +, \cdot)$ be a Krasner hyperintegral domain. We define an F^m-hyperoperation f and an F^n-operation g on H as follows:*

$$f(x_1, \cdots, x_m) = (x_1 + \cdots + x_m)_{0.25} \quad \text{for all } x_1, \cdots, x_m \in H$$

$$g(x_1, \cdots, x_n) = (x_1 . \cdots . x_n)_{0.75} \quad \text{for all } x_1, \cdots, x_n \in H.$$

It is easy to verify that (H, f, g) is a hyperintegral F-domain.

The next theorem characterizes prime F-hyperideals in the sense of quotient Krasner $F^{(m,n)}$-hyperrings .

Theorem 3.11. *Let $\mathfrak{P}$ be an F-hyperideal of a Krasner $F^{(m,n)}$-hyperring $(\mathfrak{R}, f, g)$. Then $\mathfrak{P}$ is prime if and only if $\mathfrak{R}/\mathfrak{P}$ is a hyperintegral F-domain.*

Proof. $\Longrightarrow$ Let $\mathfrak{P}$ be a F-hyperideal of $\mathfrak{R}$ and for all $a_{11}^{1m}, \ldots, a_{n1}^{nm}$,

$supp(f(a_{11}^{1(i-1)}, \mathfrak{P}, a_{1(i+1)}^{1m}), \ldots, supp(f(a_{n1}^{n(i-1)}, \mathfrak{P}, a_{n(i+1)}^{nm}) \in \mathfrak{R}/\mathfrak{P}$

such that

$g(supp(f(a_{11}^{1(i-1)}, \mathfrak{I}, a_{1(i+1)}^{1m})), \ldots, supp(f(a_{n1}^{n(i-1)}, \mathfrak{I}, a_{n(i+1)}^{nm}))) = \mathfrak{P} = \{e_{\mathfrak{R}/\mathfrak{I}}\}$.

Then we get

$supp(f(supp(g(a_{11}^{n1})), \ldots, supp(g(a_{1(i-1)}^{n(i-1)})), \mathfrak{I}, supp(g(a_{1(i+1)}^{n(i+1)})), \ldots, supp(g(a_{1m}^{nm}))) = \mathfrak{P}$

and so

$supp(f(supp(g(a_{11}^{n1})), \ldots, supp(g(a_{1(i-1)}^{n(i-1)})), e, supp(g(a_{1(i+1)}^{n(i+1)})), \ldots, supp(g(a_{1m}^{nm}))) \subseteq \mathfrak{P}$.

This means that

$g(supp(f(a_{11}^{1(i-1)}, e, a_{1(i+1)}^{1m})), \ldots, supp(f(a_{n1}^{n(i-1)}, e, a_{n(i+1)}^{nm}))) \subseteq \mathfrak{P}$

which imples

$g(\chi_{supp(f(a_{11}^{1(i-1)}, e, a_{1(i+1)}^{1m}))}, \ldots, \chi_{supp(f(a_{n1}^{n(i-1)}, e, a_{n(i+1)}^{nm}))}) \subseteq \mathfrak{P}$.

Since $\mathfrak{P}$ is a prime F-hyperideal of $\mathfrak{R}$, we obtain $supp(\chi_{supp(f(a_{j1}^{j(i-1)}, e, a_{j(i+1)}^{jm}))}) \subseteq \mathfrak{P}$ for some $1 \leq j \leq n$. Therefore $supp(f(a_{j1}^{j(i-1)}, e, a_{j(i+1)}^{jm})) \subseteq \mathfrak{P}$. Then we get $supp(f(a_{j1}^{j(i-1)}, \mathfrak{P}, a_{j(i+1)}^{jm})) = \mathfrak{P}$, by Lemma 3.14 (1) in [24]. Consequently, $\mathfrak{R}/\mathfrak{P}$ is a hyperintegral F-domain.

$\Longleftarrow$ Let $\mathfrak{R}/\mathfrak{P}$ be a hyperintegral F-domain. Suppose that $supp(g(a_1^n)) \subseteq \mathfrak{P}$ for all $a_1^n \in \mathfrak{R}$. Then we have $supp(f(supp(g(a_1^n)), \mathfrak{P}, e^{(m-2)})) = \mathfrak{P}$ by Lemma 3.14 (1) in [24]. Therefore

$supp(f(supp(g(a_1^n)), \mathfrak{P}, (supp(g(e^n))^{(m-2)})) = \mathfrak{P}$.

Then by the definition of the quotient Krasner $F^{(m,n)}$-hyperring we have

$g(supp(f(a_1, \mathfrak{P}, e^{(m-2)}), \ldots, supp(f(a_n, \mathfrak{P}, e^{(m-2)}))) = \mathfrak{P} = \{e_{\mathfrak{R}/\mathfrak{I}}\}$.

Since $\mathfrak{R}/\mathfrak{P}$ is a hyperintegral F-domain, then we get $supp(f(a_i, \mathfrak{P}, e^{(m-2)})) = \mathfrak{P}$ for some $1 \leq i \leq n$ which implies $a_i \in \mathfrak{P}$. consequently, $\mathfrak{P}$ is a prime F-hyperideal of $\mathfrak{R}$. $\square$

Theorem 3.12. *Let $(\mathfrak{R}, f, g)$ be a Krasner $F^{(m,n)}$-hyperring with the scalar identity e'. Then for all $x \in \mathfrak{R}$, the set $\{a \in supp(g(r, x, e'^{(n-2)})) \mid r \in \mathfrak{R}\}$ is an F-hyperideal of $\mathfrak{R}$. We say that the F-hyperideal is the hyperideal generated by x and it is denoted by $< x >_F$.*

Proof. We first show $< x >_F$ is support closed under f and g. Let $a_1^m \in < x >_F$. Then for each $1 \leq i \leq m$ there exists $r_i \in \mathfrak{R}$ such that $a_i \in supp(g(r_i, x, e'^{(n-2)}))$. By Proposition 3.2 in [24], we get

$$\begin{aligned} supp(f(a_1^m)) &= supp(f(g(r_1, x, e'^{(n-2)}), \ldots, g(r_m, x, e'^{(n-2)}))) \\ &= supp(g(f(r_1, \ldots, r_m), x, e'^{(n-2)}) \\ &= \bigcup_{r \in supp(f(r_1,\ldots,r_m))} supp(g(r, x, e'^{(n-2)})). \end{aligned}$$

Since $r \in supp(f(r_1, \ldots, r_m)) \subseteq \mathfrak{R}$, then $supp(f(a_1^m)) \subseteq < x >_F$. Let $b_1^n \in < x >_F$. Then for each $1 \leq i \leq n$ there exists $r_i \in \mathfrak{R}$ such that $b_i \in supp(g(r_i, x, e'^{(n-2)}))$. Therefore we have

$$\begin{aligned} supp(g(b_1^n)) &= supp(g(g(r_1, x, e'^{(n-2)}), \ldots, g(r_n, x, e'^{(n-2)}))) \\ &= supp(g(g(r_1, \ldots, r_n), x, e'^{(n-2)})) \\ &= supp(g(r, x, e'^{(n-2)})) \end{aligned}$$

such that $r \in supp(g(r_1, \ldots, r_n)) \subseteq \mathfrak{R}$. Hence $supp(g(b_1^n)) \subseteq < x >_F$. It is easy to see that $(< x >_F, f, g)$ is a F-subhyperring of $\mathfrak{R}$. Now we show that for all $r_1^n \in \mathfrak{R}$ and $1 \leq i \leq n$, $supp(g(r_1^{i-1}, < x >, r_{i+1}^n)) \subseteq < x >_F$. Let $r_1^n \in \mathfrak{R}$. Then

$$\begin{aligned} supp(g(r_1^{i-1}, < x >_F, r_{i+1}^n)) &= supp(g(r_1^{i-1}, \chi_{<x>_F}, r_{i+1}^n)) \\ &= \bigcup_{a \in supp(\chi_{<x>_F})} supp(g(r_1^{i-1}, a, r_{i+1}^n)) \\ &= \bigcup_{a \in <x>_F} supp(g(r_1^{i-1}, a, r_{i+1}^n)) \\ &\subseteq \bigcup_{r \in \mathfrak{R}} supp(g(r_1^{i-1}, g(r, x, e'^{(n-2)}), r_{i+1}^n)) \\ &= \bigcup_{r \in \mathfrak{R}} supp(g(g(r_1^{i-1}, r, r_{i+1}^n), x, e'^{(n-2)})) \\ &= \bigcup_{r \in \mathfrak{R}} \bigcup_{r' \in supp(g(r_1^{i-1}, r, r_{i+1}^n))} supp(g(r', x, e'^{(n-2)})) \\ &\subseteq < x >_F. \end{aligned}$$

Thus F-subhyperring $< x >_F = \{a \in supp(g(r, x, e'^{(n-2)})) \mid r \in \mathfrak{R}\}$ of $\mathfrak{R}$ is an F-hyperideal of $\mathfrak{R}$. □

Definition 3.13. An F-hyperideal $\mathfrak{M}$ of a Krasner $F^{(m,n)}$-hyperring is called maximal if for every F-hyperideal $\mathfrak{N}$ of $\mathfrak{R}$, $\mathfrak{M} \subseteq \mathfrak{N} \subseteq \mathfrak{R}$ implies that $\mathfrak{N} = \mathfrak{M}$ or $\mathfrak{N} = \mathfrak{R}$.

The intersection of all maximal F-hyperideals of $\mathfrak{R}$ is called Jacobson radical of $\mathfrak{R}$ and it is denoted by $J(\mathfrak{R})$. If $\mathfrak{R}$ does not have any maximal F-hyperideal, we let $J(\mathfrak{R}) = \mathfrak{R}$.

Example 3.14. *Let us consider the Krasner $F^{(m,2)}$-hypering (G, f, g), given in Example 3.4 in [24]. Then $\{e\}$ is a maximal F-hyperideal of G.*

The following theorem ensures that there is always a sufficient supply of the maximal F-hyperideals.

Theorem 3.15. *Every Krasner $F^{(m,2)}$-hypering $\mathfrak{R}$ with scalar F-identity e', has at least one maximal F-hyperideal.*

Proof. The proof is the same as the proof in the classical context of maximal ideals of rings. $\square$

We say that an element $x \in \mathfrak{R}$ is F-invertible if there exists $y \in \mathfrak{R}$ such that $supp(g(x, y, {e'}^{(n-2)})) = \{e'\}$. Also, the subset U of $\mathfrak{R}$ is F-invertible if and only if every element of U is F-invertible. The Jacobson radical of a Krasner $F^{(m,n)}$-hyperring $\mathfrak{R}$ can be characterized as follows:

Theorem 3.16. *Let $\mathfrak{I}$ be an F-hyperideal of a Krasner $F^{(m,n)}$-hyperring with the scalar F-identity e'. Then every element of $supp(f(e', \mathfrak{I}, e^{(m-2)}))$ is F-invertible if and only if $\mathfrak{I} \subseteq J(\mathfrak{R})$.*

Proof. $\Longrightarrow$ Suppose that every element of $supp(f(e', \mathfrak{I}, e^{(n-2)}))$ is F-invertible. Let $\mathfrak{I} \nsubseteq J(\mathfrak{R})$. Then there exists a maximal F-hyperideal $\mathfrak{M}$ of $\mathfrak{R}$ such that $\mathfrak{I} \nsubseteq \mathfrak{M}$. Let $x \in \mathfrak{I}$ but $x \notin \mathfrak{M}$. By Lemma 3.12 in [24], $supp(f(\mathfrak{M}, < x >_F , e^{(m-2)}))$ is an F-hyperideal of $\mathfrak{R}$. Since $\mathfrak{M} \subseteq supp(f(\mathfrak{M}, < x >_F, e^{(m-2)}))$ and $\mathfrak{M}$ is a maximal F-hyperideal of $\mathfrak{R}$, we have $supp(f(\mathfrak{M}, < x >_F, e^{(m-2)})) = \mathfrak{R}$ and so $e' \in supp(f(\mathfrak{M}, < x >_F, e^{(m-2)}))$. This means that there exist $m \in supp(\chi_{\mathfrak{M}}) = \mathfrak{M}$ and $a \in supp(\chi_{<x>_F}) =< x >_F$ such that $e' \in supp(f(m, a, e^{(m-2)}))$. Since $(\mathfrak{R}, f)$ is a canonical F^m-hypergroup, then $m \in supp(f(e', -a, e^{(m-2)}))$. Since $supp(f(e', -a, e^{(m-2)})) \subseteq supp(f(e', -g(r, x, e'), e^{(m-2)}))$ for some $r \in \mathfrak{R}$, we have $m \in supp(f(e', g(r, x, e'), e^{(m-2)}))$ which implies $m \in supp(f(e', \mathfrak{I}, e^{(m-2)}))$. This m is F-invertible, a contradiction.
$\Longleftarrow$ Suppose that $\mathfrak{I} \subseteq J(\mathfrak{R})$. Assume that $a \in supp(f(e', \mathfrak{I}, e^{(m-2)}))$ is not F-invertible. Then there exists $x \in \mathfrak{I}$ such that $a \in supp(f(e', x, e^{(m-2)}))$. We have $a \in \mathfrak{M}$ for some maximal F-hyperideal $\mathfrak{M}$, because a is not F-invertible. From $a \in supp(f(e', x, e^{(m-2)}))$, it follows that $e' \in supp(f(a, -x, e^{(m-2)})) \subseteq \mathfrak{M}$, a contradiction. Thus every element of $supp(f(e', \mathfrak{I}, e^{(m-2)}))$ is F-invertible. $\square$

In view of Theorem 3.16, we have the following result.

Corollary 3.17. *Let $\mathfrak{M}$ be a maximal F-hyperideal of a Krasner $F^{(m,n)}$-hyperring with the scalar F-identity e'. If every element of $supp(f(e', \mathfrak{M}, e^{(m-2)}))$ is F-invertible, then $\mathfrak{M}$ is the only maximal hyperideal of $\mathfrak{R}$.*

Theorem 3.18. *Suppose that T is a non-empty subset of a Krasner $F^{(m,n)}$-hyperring $(\mathfrak{R}, f, g)$ that is support closed under g and $\mathfrak{I}$ is an F-hyperideal of $\mathfrak{R}$ such that $\mathfrak{I} \cap T = \varnothing$. Then there exists an F-hyperideal $\mathfrak{P}$ which is maximal in the set of all hyperideals of $\mathfrak{R}$ disjoint from T containing $\mathfrak{I}$. Furthermore any such F-hyperideal is prime.*

Proof. Let Σ be the set of all hyperideals of $\mathfrak{R}$ disjoint from T containing $\mathfrak{I}$. Since $\mathfrak{I} \in \Sigma$, then $\Sigma \neq \varnothing$. Thus Σ is a partially ordered set with respect to set inclusion relation. Then there is an F-hyperideal $\mathfrak{P}$ which is maximal in Σ, by Zorn's lemma. Our task now is to show that $\mathfrak{P}$ is a prime F-hyperideal of $\mathfrak{R}$. Let $supp(g(a_1^n)) \subseteq \mathfrak{P}$ for some $a_1^n \in \mathfrak{R}$ such that for all $1 \leq i \leq n$, $a_i \notin \mathfrak{P}$. Then for each $1 \leq i \leq n$, $\mathfrak{P} \subseteq supp(f(\mathfrak{P}, < a_i >, e^{(m-2)}))$. By maximality of $\mathfrak{P}$, we conclude that $supp(f(\mathfrak{P}, < a_i >, e^{(m-2)})) \cap T \neq \varnothing$. Hence there exist $p_1^n \in \mathfrak{P}$ and $x_i \in < a_i >$ such that $supp(f(p_i, x_i, e^{(m-2)})) \cap T \neq \varnothing$ for each $1 \leq i \leq n$. Since T is support closed under g, then there exists $r_i \in supp(f(p_i, x_i, e^{(m-2)}))$ for each $1 \leq i \leq n$ such that $supp(g(r_1^n)) \cap T \neq \varnothing$. Therefore

$$\begin{aligned} supp(g(r_1^n)) &\subseteq supp(g(f(p_1, x_1, e^{(m-2)})), \ldots, f(p_n, x_n, e^{(m-2)}))) \\ &= supp(f(g(p_1^n), g(p_1^{n-1}, x_n), \cdots, g(p_1, x_2^n), \cdots, g(x_1^n), e^{(m-2^n)})) \\ &\subseteq \mathfrak{P}. \end{aligned}$$

This means that $\mathfrak{P} \cap T \neq \varnothing$ which is contradiction with $\mathfrak{P} \in \Sigma$. Thus, by Theorem 3.2, $\mathfrak{P}$ is a prime F-hyperideal of $\mathfrak{R}$. □

Definition 3.19. Let $\mathfrak{I}$ be an F-hyperideal of a Krasner $F^{(m,n)}$-hyperring $(\mathfrak{R}, f, g)$ with scalar F-identity e'. The intersection of all prime F-hyperideals of $\mathfrak{R}$ containing $\mathfrak{I}$ is called F-radical of $\mathfrak{I}$, being denoted by $\sqrt{\mathfrak{I}}^F$. If $\mathfrak{R}$ does not have any prime F-hyperideal containing $\mathfrak{I}$, we define $\sqrt{\mathfrak{I}}^F = \mathfrak{R}$.

Example 3.20. *In the Krasner $F^{(m,n)}$-hyperring defined in Example 3.3, F-radical of F-hyperideal $4\mathbb{Z}$ is $2\mathbb{Z}$.*

The following theorem gives an alternative definition of $\sqrt{\mathfrak{I}}^F$.

Theorem 3.21. *$\mathfrak{I}$ be an F-hyperideal of a Krasner $F^{(m,n)}$-hyperring $(\mathfrak{R}, f, g)$ with scalar F-identity e'. Then*

$$\sqrt{\mathfrak{I}}^F = \left\{ a \in \mathfrak{R} \mid \left\{ \begin{array}{ll} supp(g(a^{(s)}, e'^{(n-s)})) \subseteq \mathfrak{I}, & s \leq n \\ supp(g_{(l)}(a^{(s)})) \subseteq \mathfrak{I} & s > n, s = l(n-1)+1 \end{array} \right\} \right\}$$

.

Proof. Let $a \in \sqrt{\mathfrak{I}}^F$ and let $\mathfrak{P}$ be a prime F-hyperideal of $\mathfrak{R}$ with $\mathfrak{I} \subseteq \mathfrak{P}$. Thus there exists $s \in \mathbb{N}$ with $supp(g(a^{(s)}, e'^{(n-s)})) \subseteq \mathfrak{I}$ for $s \leq n$, or $supp(g_{(l)}(a^{(s)})) \subseteq \mathfrak{I}$ for $s = l(n-1)+1$. In the first case, we have $supp(g(a^{(s)}, e'^{(n-s)})) \subseteq \mathfrak{P}$ and so $supp(g(a, g(a^{(s-1)}, e'^{(n-s+1)}), e'^{(n-2)})) \subseteq \mathfrak{P}$. Since $\mathfrak{P}$ is a prime F-hyperideal of $\mathfrak{R}$, we get $a \in \mathfrak{P}$ or $supp(g(a^{(s-1)}, e'^{(n-s+1)})) \subseteq \mathfrak{P}$.From $supp(g(a^{(s-1)}, e'^{(n-s+1)})) \subseteq \mathfrak{P}$, it follows that $supp(g(a, g(a^{(s-2)}, e'^{(n-s+2)}), e'^{(n-2)}) \subseteq \mathfrak{P}$ which implies $a \in \mathfrak{P}$ or $supp(g(a^{(s-2)}, e'^{(n-s+2)})) \subseteq \mathfrak{P}$. By continuing this process, we obtain $a \in \mathfrak{P}$. Hence we have $a \in \mathfrak{P}$ for all $\mathfrak{I} \subseteq \mathfrak{P}$ and so $a \in \bigcap_{\mathfrak{I} \subseteq \mathfrak{P}} \mathfrak{P}$. This means $\sqrt{\mathfrak{I}}^F \subseteq \bigcap_{\mathfrak{I} \subseteq \mathfrak{P}} \mathfrak{P}$. In the second case, we get $supp(g(g(\ldots g(g(a^{(n)}), a^{(n-1)}), \ldots), a^{(n-1)})) \subseteq \mathfrak{P}$. By using a similar argument , we get $\sqrt{\mathfrak{I}}^F \subseteq \bigcap_{\mathfrak{I} \subseteq \mathfrak{P}} \mathfrak{P}$. Now, suppose that $a \in \bigcap_{\mathfrak{I} \subseteq \mathfrak{P}} \mathfrak{P}$ but $a \notin \sqrt{\mathfrak{I}}^F$. Hence we conclude that for every $s \in \mathbb{N}$, $supp(g(a^{(s)}, e'^{(n-s)})) \nsubseteq \mathfrak{I}$. Let $T = \{e', a\} \cup \{r \in supp(g(a^{(t)}, e'^{(n-t)})) \mid 2 \leq t\}$. Clearly, T is a subset of

$\mathfrak{R}$ that is support closed under g and $T \cap \mathfrak{I} = \varnothing$. By Theorem 3.18, there exists a prime F-hyperideal $\mathfrak{P}$ with $\mathfrak{I} \subseteq \mathfrak{P}$ and $T \cap \mathfrak{P} = \varnothing$. This means $a \notin \mathfrak{P}$. This is contradiction as $a \in \bigcap_{\mathfrak{I} \subseteq \mathfrak{P}} \mathfrak{P}$. Thus $a \in \sqrt{\mathfrak{I}}^F$. Consequently, $\sqrt{\mathfrak{I}}^F = \bigcap_{\mathfrak{I} \subseteq \mathfrak{P}} \mathfrak{P}$. □

4. Primary F-hyperideals

In this section, we aim to present the definition of primary F-hyperideals in a Krasner $F^{(m,n)}$-hyperring $(\mathfrak{R}, f, g)$ and give some basic properties of them.

Definition 4.1. An F-hyperideal $\mathfrak{Q}$ of a Krasner $F^{(m,n)}$-hyperring $(\mathfrak{R}, f, g)$ (with scalar F-identity e') is called a primary F-hyperideal if for all $\mu_1^n \in L_*^{\mathfrak{R}}$, $supp(g(\mu_1^n)) \subseteq \mathfrak{Q}$ implies that $supp(\mu_i) \subseteq \mathfrak{Q}$ or $supp(g(\mu_1^{i-1}, \chi_{\{e'\}}, \mu_{i+1}^n)) \subseteq \sqrt{\mathfrak{Q}}^F$ for some $1 \leq i \leq n$.

Theorem 4.2. *Let $\mathfrak{Q}$ be an F-hyperideal of a Krasner $F^{(m,2)}$-hyperring $(\mathfrak{R}, f, g)$ (with scalar F-identity e'). Then $\mathfrak{Q}$ is primary if and only if for all $a_1^2 \in \mathfrak{R}$, $supp(g(a_1^2)) \subseteq \mathfrak{Q}$ implies that $a_1 \in \mathfrak{Q}$ or $a_2 \in \sqrt{\mathfrak{Q}}^F$.*

Proof. $\Longrightarrow$ Let $\mathfrak{Q}$ be a primary F-hyperideal of $\mathfrak{R}$. Suppose that $supp(g(a_1^2)) \subseteq \mathfrak{Q}$ for some $a_1^2 \in \mathfrak{R}$. Since $supp(g(a_1^2)) = supp(g(\chi_{\{a_1\}}, \chi_{\{a_2\}}))$, then we have $supp(g(\chi_{\{a_1\}}, \chi_{\{a_2\}})) \subseteq \mathfrak{Q}$. Since $\mathfrak{Q}$ is a primary F-hyperideal of $\mathfrak{R}$, we get $supp(\chi_{\{a_1\}}) \subseteq \mathfrak{Q}$ or $supp(\chi_{\{a_2\}}) \subseteq \sqrt{\mathfrak{Q}}^F$. This means $a_1 \in \mathfrak{Q}$ or $a_2 \in \sqrt{\mathfrak{Q}}^F$.
$\Longrightarrow$ Let $supp(g(\mu_1^2)) \subseteq \mathfrak{Q}$ for some $\mu_1^2 \in L_*^{\mathfrak{R}}$ such that neither $supp(\mu_1) \subseteq \mathfrak{Q}$ nor $supp(\mu_2) \subseteq \sqrt{\mathfrak{Q}}^F$. Suppose that $x_1 \in supp(\mu_1) - \mathfrak{Q}$ and $x_2 \in supp(\mu_2) - \sqrt{\mathfrak{Q}}^F$. Clearly, $supp(g(x_1^2)) \subseteq supp(g(\mu_1^2)) \subseteq \mathfrak{Q}$. By the hypothesis, we get $x_1 \in \mathfrak{Q}$ or $x_2 \in \sqrt{\mathfrak{Q}}^F$, a contradiction. □

Corollary 4.3. *Let $\mathfrak{Q}$ be an F-hyperideal of a Krasner $F^{(m,n)}$-hyperring $(\mathfrak{R}, f, g)$ (with scalar F-identity e'). Then $\mathfrak{Q}$ is primary if and only if for all $a_1^n \in \mathfrak{R}$, $supp(g(a_1^n)) \subseteq \mathfrak{Q}$ implies that $a_i \in \mathfrak{Q}$ or $supp(g(a_1^{i-1}, e', a_{i+1}^n)) \subseteq \sqrt{\mathfrak{Q}}^F$ for some $1 \leq i \leq n$.*

Example 4.4. *Suppose that $R = [0,1]$ and $t \in (0,1]$. Then (R, f, g) is a Krasner $F^{(2,3)}$-hyperring, where f and g defined by*

$$f(a,b) = \begin{cases} \chi_{max\{a,b\}} & \text{if } a \neq b \\ \chi_{[0,a]} & \text{if } a = b \end{cases}$$

and

$$g(a,b,c) = (a.b.c)_t$$

for all $a, b, c \in R$. The F-hyperideal $I = [0, 0.5]$ is a primary F-hyperideal of R.

The following is a direct consequence and can be proved easily and so the proof is omited.

Theorem 4.5. *If $\mathfrak{P}$ is a prime F-hyperideal of $\mathfrak{R}$, then $\mathfrak{P}$ is a primary F-hyperideal of $\mathfrak{R}$.*

The next example shows that the inverse of Theorem 4.5 is not true, in general.

Example 4.6. *In Example 4.4, the primary F-hyperideal I of R is not prime as the fact that $supp(g(0.8, 0.7, .0.6)) \subseteq I$ but $0.8, 0.7, 0.6 \notin I$.*

In next theorem, we establish a relationship between prime F-hyperideals and primary F-hyperideals of a Krasner $F^{(m,n)}$-hyperring $(\mathfrak{R}, f, g)$.

Theorem 4.7. *Let $\mathfrak{Q}$ be an F-hyperideal of a Krasner $F^{(m,n)}$-hyperring $(\mathfrak{R}, f, g)$ (with scalar F-identity e'). If $\mathfrak{Q}$ is primary, then $\sqrt{\mathfrak{Q}}^F$ is a prime F-hyperideal of $\mathfrak{R}$.*

Proof. Let $supp(g(x_1^n)) \subseteq \sqrt{\mathfrak{Q}}^F$ for some $x_1^n \in \mathfrak{R}$ such that $x_1^{i-1}, x_{i+1}^n \notin \sqrt{\mathfrak{Q}}^F$. Our task now is to show that $a_i \in \sqrt{\mathfrak{Q}}^F$. Let $x \in supp(g(x_1^n)$. From $supp(g(x_1^n)) \subseteq \sqrt{\mathfrak{Q}}^F$, it follows that there exists $s \in \mathbb{N}$ such that if $s \leq n$, then $supp(g(x^{(s)}, {e'}^{(n-s)})) \subseteq \mathfrak{Q}$. Therefore we have $supp(g(g(x_1^n)^{(s)}, {e'}^{(n-s)})) \subseteq \mathfrak{Q}$, because $supp(g(x^{(s)}, {e'}^{(n-s)})) = supp(g(g(x_1^n)^{(s)}, {e'}^{(n-s)}))$.
Thus $supp(g(x_i^{(s)}, g(x_1^{i-1}, e', x_{i+1}^n)^{(s)}, {e'}^{(n-2s)})) \subseteq \mathfrak{Q}$ and so $supp(g(g(x_i^{(s)}, {e'}^{(n-s)}), g(g(x_1^{i-1}, e', x_{i+1}^n)^{(s)}, {e'}^{(n-s)}), {e'}^{(n-2)})) \subseteq \mathfrak{Q}$. Since $\mathfrak{Q}$ is a primary F-hyperideal of $\mathfrak{R}$, we get $supp(g(g(x_1^{i-1}, e', x_{i+1}^n)^{(s)}, {e'}^{(n-s)})) \subseteq \mathfrak{Q}$ or $supp(g(x_i^{(s)}, {e'}^{(n-s)})) \subseteq \sqrt{\mathfrak{Q}}^F$. Suppose that $supp(g(g(x_1^{i-1}, e', x_{i+1}^n)^{(s)}, {e'}^{(n-s)})) \subseteq \mathfrak{Q}$. Then we get $supp(g(g(x_1^{(s)}, {e'}^{(n-s)}), g(x_2^{i-1}, {e'}^2, x_{i+1}^n)^{(s)}, {e'}^{(n-s-1)})) \subseteq \mathfrak{Q}$ which means $supp(g(x_1^{(s)}, {e'}^{(n-s)})) \subseteq \mathfrak{Q}$ or $supp(g(g(x_2^{i-1}, {e'}^2, x_{i+1}^n)^{(s)}, {e'}^{(n-s)})) \subseteq \sqrt{\mathfrak{Q}}^F$. Since $x_1 \notin \sqrt{\mathfrak{Q}}^F$, then $supp(g(g(x_2^{i-1}, {e'}^2, x_{i+1}^n)^{(s)}, {e'}^{(n-s)})) \subseteq \sqrt{\mathfrak{Q}}^F$. By continuing this process, since $x_1^{i-1}, x_{i+1}^n \notin \sqrt{\mathfrak{Q}}^F$, we have $supp(g(x_i^{(s)}, {e'}^{(n-s)})) \subseteq \sqrt{\mathfrak{Q}}^F$. By using the definition of F-radical of $\mathfrak{Q}$, we conclude that $a_i \in \sqrt{\mathfrak{Q}}^F$. If $s = l(n-1)+1$, then by using a similar argument , one can easily complete the proof. □

The next example shows that the inverse of Theorem 4.7 is not true, in general.

Example 4.8. *Let $R = \mathbb{Z}_3[x, y, z]$ and $I = \langle xy - z^2 \rangle$. Then $H = R/I$ is a Krasner $(2, 2)$-hyperring with ordinary addition and ordinary multiplication. Note that (H, f, g) is a Krasner $F^{(2,2)}$-hyperring, where*

$$f(\alpha, \beta) = \chi_{(\alpha+\beta)} \qquad \textit{for all } \alpha, \beta \in H$$

$$g(\alpha, \beta) = \chi_{(\alpha\cdot\beta)} \qquad \textit{for all } \alpha, \beta \in H.$$

Then $K = \langle \bar{x}, \bar{z} \rangle$ is a prime F-hyperideal of H where $\bar{x}, \bar{z}$ denote the images of x, z, respectively, in H. K^2 is not a primary F-hyperideal of H, while its F-radical is a prime F-hyperideal of H.

5. Conclusion

In this paper, our purpose is to extend the study initiated in [24] about Krasner $F^{(m,n)}$-hyperrings by Farshi and Davvaz. We defined prime F-hyperideals, maximal F-hyperideals and primary F-hyperideals of a Krasner $F^{(m,n)}$-hyperring $\mathfrak{R}$. We obtain many specific results explaining the structures. Moreover, their connection with other concepts such as the quotient structure, Jacobson radical and F-radical was investigated. The stability of prime F-hyperideals property was examined under a transfer. An alternative definition of F-radical of an F-hyperideal was given.

The future work can be on defining the concept of δ-primary F-hyperideals unifing the notions of the prime and primary F-hyperideal in a frame.

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7. Conflict of interest

The authors declare no conflict of interest.

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