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AUTOMATIC CONTINUITY OF MULTIPLICATIVE POLYNOMIAL OPERATORS

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Abstract. This paper deals with the automatic continuity of multiplicative polynomial operators on a class of topological algebras. Several results are derived in this direction. We also support our results by some examples.

Keywords: Topological algebra, Multiplicative polynomial functional, Automatic continuity, Spectral radius.
2020 MSC: 46H05, 46H20, 46H40.

1. Introduction

Automatic continuity of homomorphisms, in particular, multiplicative linear functionals are very significant in functional analysis and topological algebras. The starting point for continuity of homomorphisms is the theorem that every multiplicative linear functionals on a Banach algebra is automatically continuous [3, 14]. The continuity of multiplicative linear functionals implies that every homomorphism from a Banach algebra onto a commutative semi-simple Banach algebra is continuous. A celebrated theorem due to Johnson [12] generalizes this result to semi-simple Banach algebras. Several results for automatic continuity of homomorphisms in Banach and Frechet algebras have also been studied by Aupetit [3] and Husain [11]. However, it remains an interesting open question, known as Michael's problem: is every multiplicative linear functionals on Frechet algebras automatically continuous? Automatic continuity of multiplicative polynomial operators has been investigated widely by many authors and researchers. The most important results in this field are due to Bochnak, Siciak [6], Gajda [9] and Zagorodnyuk [16], who have extended this topic on a class of topological algebras.

Polynomial operators are very crucial in functional analysis and have numerous applications in many parts of real life. They are the simplest analytic functions which are widely used in mathematics, engineering, management, medicine and even in farming. The motivation for the study of polynomial operators is to investigate the most important facts about multiplicative polynomial operators which are basic for the further development of the automatic

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continuity. Also, several conditions are given for multiplicative polynomial operators to be continuous. We hope that some of the results of this note may be interesting for functional analysis and topological algebras.

In this paper, we derive some results on automatic continuity of multiplicative polynomial operators on a class of topological algebras. Some examples are investigated in this direction as well.

Throughout this article, all algebras are considered unital and the units are denoted by e.

2. Definitions

In this section, we present a collection of definitions.

Definition 2.1. [10] Let A be an algebra. We denote the group of invertible elements of A by $Inv(A)$.

Definition 2.2. [4] Let A be an algebra. For each $x \in A$, the set

$$sp_A(x) = \{\lambda \in \mathbb{C} : \ x - \lambda e \notin Inv(A)\},$$

is called the spectrum of x. The number

$$r_A(x) = \sup\{|\lambda| : \lambda \in sp_A(x)\},$$

is called the spectral radius of x.

Definition 2.3. [4] By a topological algebra we mean an algebra over $\mathbb{C}$ endowed with a topology that makes the multiplication separately continuous.

Definition 2.4. [10] A topological algebra A is called a Q-algebra if $Inv(A)$ is open.

Definition 2.5. [11] Let A be a Hausdorff topological algebra. A is said to be strongly sequential if there exists a neighborhood U of 0 such that for all $x \in U$ we have $x^n \to 0$ as $n \to \infty$.

Definition 2.6. [2, 2.13] Let A be a Hausdorff topological algebra and $x \in A$. The radius of boundedness $\beta_A(x)$ of x is defined by

$$\beta_A(x) = \inf\left\{r > 0 : \left(\frac{x^n}{r^n}\right) \to 0\right\},$$

with the convention: $\inf \emptyset = +\infty$.

Definition 2.7. [15, 16] Let X and Y be two topological algebras. A mapping $p_n : X \to Y$ is an n-homogeneous polynomial if there is a necessarily unique symmetric n-linear mapping $\overline{p_n} : X \times \cdots \times X \to Y$ such that $p_n(x) = \overline{p_n}(x, \ldots, x)$ for all $x \in X$. A polynomial operator $p : X \to Y$ is a finite sum of homogeneous polynomials. We say that a polynomial operator $p : X \to Y$ is multiplicative if $p(x_1x_2) = p(x_1)p(x_2)$ for each $x_1, x_2 \in X$. It is clear that $p(0) = 0$. It is known that every multiplicative polynomial functional is homogeneous.

Definition 2.8. [11] An F-algebra A is a topological algebra which is an F-space, i.e., a complete metrizable topological vector space. The topology of an F-space X can be given by means of F-norm, i.e., a map $x \mapsto \|x\|_F$ from X to non-negative real numbers such that

(1) $\|x\|_F \geq 0$ and $\|x\|_F = 0$ if and only if $x = 0$,
(2) $\|x + y\|_F \leq \|x\|_F + \|y\|_F$,
(3) $\|\lambda x\|_F \leq \|x\|_F$ for $|\lambda| \leq 1$.

A norm in an F-space X is given by means of $\|x - y\|_F$ $(x, y \in X)$; we shall also write $x_n \to x_0$ if $\lim_n \|x_n - x_0\|_F = 0$. If the F-norm is submultiplicative, i.e.,

$$\|xy\|_F \leq \|x\|_F \|y\|_F \quad \text{for all} \quad x, y \in A,$$

then A is called a submultiplicative F-algebra.

Definition 2.9. [5] A locally convex algebra is a complex algebra A with a locally convex Hausdorff topology for which the multiplication is separately continuous.

Definition 2.10. [5] A locally m-convex algebra is a locally convex algebra in which the topology is defined by a family of submultiplicative seminorms.

Definition 2.11. [5] A complete metrizable locally m-convex algebra is called a Frechet algebra.

Definition 2.12. [10] A Frechet algebra $(A, (p_k))$ is called uniform if $p_k(a^2) = (p_k(a))^2$ for each $k \in \mathbb{N}$ and for all $a \in A$.

3. continuity of multiplicative polynomial operators

In this section, we study the automatic continuity of multiplicative polynomial operators (in particular, multiplicative polynomial functionals) on certain topological algebras.

Theorem 3.1. *Let $(A, \| \cdot \|_F)$ be a submultiplicative F-algebra. If $p : A \to \mathbb{C}$ is a multiplicative polynomial functional, then p is continuous.*

Proof. Suppose that p is not continuous. By [7, Theorem 1], $\ker p$ is a dense set in A. Hence, there exists a sequence $(x_n)_n$ of $\ker p$ such that $x_n \to e$ as $n \to \infty$. Since $\|x_n - e\|_F < 1$ for sufficiently large n, $x_n \in Inv(A)$ by [4, 3.3.20 (ii)]. Therefore,

$$p(e) = p(x_n x_n^{-1}) = p(x_n)p(x_n^{-1}) = 0.$$

So

$$p(x) = p(e)p(x) = 0, \quad \text{for all} \quad x \in A.$$

Therefore, $p = 0$ which is a contradiction. □

Corollary 3.2. *If $(A, (p_k))$ is a Frechet algebra, then each multiplicative polynomial functional is continuous on A.*

Theorem 3.3. *Let $(A, \|\cdot\|_F)$ be a submultiplicative F-algebra and $(B, (p_k))$ be a commutative Frechet algebra such that B is uniform. If $f : A \to B$ is a multiplicative polynomial operator, then f is continuous.*

Proof. Let $\varphi : B \to \mathbb{C}$ be a multiplicative linear functional on B. Then $\varphi \circ f$ is a multiplicative polynomial functional. From Theorem 3.1, it follows that $\varphi \circ f$ is continuous. Suppose that f is a discontinuous polynomial operator. So condition (i) of [6, Theorem 1], is not satisfied. Then there exists a seminorm $q \in (p_k)_{k\in\mathbb{N}}$ such that for every $m \in \mathbb{N}$, there exists a point $x_m \in A$ such that $q(f(x_m)) > m$ and $\|x_m\|_F < \dfrac{1}{m}$. By [11, Proposition 1.11 (d)], we have

$$r_B(f(x_m)) = \sup_k \lim_{n\to\infty} [p_k(f(x_m))^n]^{\frac{1}{n}} = \sup_k p_k(f(x_m)) > m.$$

On the other hand,

$$r_B(f(x_m)) = \sup_{\varphi\in M(B)} |\varphi(f(x_m))| \to 0, \text{ as } m \to \infty,$$

where $M(B)$ is the set of multiplicative linear functionals on B. Hence, $r_B(f(x_m)) < 1$, for sufficiently large m. This contradiction implies that f is continuous. □

Theorem 3.4. *Let A be a submultiplicative F-algebra and $(f_n)_n$ be the sequence of multiplicative polynomial functionals on A such that*

$$\lim_{n\to\infty} f_n(x) = p(x) \quad \textit{for} \quad x \in A.$$

Then p is a continuous multiplicative polynomial functional on A.

Proof. By Theorem 3.1, each f_n, $n = 1, 2, \ldots$, is continuous. Also, it follows directly from the definition of polynomial operator, p is a polynomial functional. On the other hand,

$$p(xy) = \lim_{n\to\infty} f_n(xy) = \lim_{n\to\infty} f_n(x) \lim_{n\to\infty} f_n(y) = p(x)p(y)$$

for all $x, y \in A$. Therefore, p is continuous by Theorem 3.1, again. □

Theorem 3.5. *Let A be a Hausdorff Q-algebra. Then each multiplicative polynomial functional $p : A \to \mathbb{C}$ is continuous.*

Proof. Suppose that $p : A \to \mathbb{C}$ is a discontinuous polynomial functional. By [7, Theorem 1], $\ker p$ is dense in A. So, there exists a sequence $(x_n)_n$ of $\ker p$ such that $x_n \to e$ as $n \to \infty$. Since $Inv(A)$ is a neighborhood of e, $x_n \in Inv(A)$ for sufficiently large n. Hence,

$$p(e) = p(x_n x_n^{-1}) = p(x_n)p(x_n^{-1}) = 0.$$

Therefore, as in Theorem 3.1, $p = 0$ which is a contradiction. □

Corollary 3.6. *If A is a Banach algebra, then each multiplicative polynomial functional is continuous on A.*

Theorem 3.7. *Let A be a strongly sequential Hausdorff topological algebra such that $r_A \leq \beta_A$. Then every multiplicative polynomial functional is continuous on A.*

Proof. Since A is strongly sequential, β_A is continuous at zero by [1, Lemma 2]. Since $r_A \leq \beta_A$, r_A is continuous at zero as well. Hence, $\{x \in A : r_A(x) \leq 1\}$ is a neighborhood of zero in A. Therefore, A is a Q-algebra by [13, 4.2]. The result follows from Theorem 3.5. □

Now, we disprove the converse of Theorem 3.7 with the following example.

Example 3.8. *Let $A = C(\mathbb{R})$ denote the algebra of all continuous complex-valued functions on $\mathbb{R}$. Then A with the sequence $\{p_n\}$ of seminorms defined by $p_n(f) = \sup_{|x|\leq n} |f(x)|$ is a Frechet algebra which is not a Q-algebra. So, the spectral radius function is not continuous at zero. Also, we have $r_A \leq \beta_A$ by [8, Proposition II.3]. So, β_A is not continuous at zero as well. It follows from [8, Proposition III.1] that A is not strongly sequential. However, Corollary 3.2 implies that each multiplicative polynomial functional is continuous on A.*

Corollary 3.9. *Let A be a strongly sequential locally convex algebra. If A is pseudo-complete, then every multiplicative polynomial functional is continuous on A.*

Proof. By [8, Proposition II.3], we have $r_A \leq \beta_A$. Now the result follows from Theorem 3.7. □

Example 3.10. *Consider the projective mapping $p_i : \mathbb{R}^n \to \mathbb{R}$ by*

$$p_i(x_1, \ldots, x_n) = x_i.$$

It is easy to check that each p_i is a multiplicative linear functional. So, the finite product of projective mappings is a multiplicative polynomial functional.

Example 3.11. *Let $A = \mathbb{C}[x]$ be the algebra of polynomials in one indeterminate x. Endow A with the topology of uniform convergence on $[0, 1]$. For every $\lambda \in \mathbb{C}$, let $f_\lambda : A \to \mathbb{C}$ be the linear functional assigning to x^n the scalar λ^n, $n \in \mathbb{N}$. It is a continuous multiplicative linear functional on A. Clearly, the finite product of multiplicative linear functionals $f_{\lambda_1} \cdot f_{\lambda_2} \cdot \ldots f_{\lambda_n}$, $f_{\lambda_k} \in M(A)$, $k = 1, 2, \ldots, n$ is a continuous multiplicative polynomial functional.*

Example 3.12. *Let A be the field $\mathbb{C}(x)$ of rational functions of the indeterminate x on $\mathbb{C}$ endowed with the strongest locally convex topology. Then A is a complete locally convex Q-algebra which is not strongly sequential. By Theorem 3.5, every multiplicative polynomial functional is continuous.*

4. Conclusion

In this article, we obtained some results concerning the automatic continuity of multiplicative polynomial operators on a class of topological algebras.

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