Journal of Mahani Mathematical Research
Shahid Bahonar University of Kerman
Print ISSN: 2251-7952 Online ISSN: 2645-4505

PERIODIC OSCILLATION FOR A CLASS OF DELAYED ECONOMIC MODELS

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Article type: Research Article
(Received: 14 August 2022, Received in revised form 20 December 2022)
(Accepted: 21 January 2023, Published Online: 23 February 2022)

Abstract. A five coupled Kaldor-Kalecki economic model with one delay appeared in the literature, in which the periodic solution of the model was verified by numerical analysis. The periodic solution is an important characteristic of the mutual interactions of economic systems. Also, different investment functions may have different delays. The present paper extends the five coupled Kaldor-Kalecki economic model with one delay to a multiple delay system and discusses the existence of periodic oscillation of this multiple delay model. By linearizing the investment functions at the positive equilibrium and analyzing the instability of the positive equilibrium together with the boundedness of the solutions, some sufficient conditions to guarantee the existence of periodic oscillatory solutions for this model are established. Computer simulations are given to illustrate the validity of the theoretical results. The present result is new.

Keywords: Economic model, Delay, Periodic oscillation.
2020 MSC: Primary 34K13.

1. Mathematical model

Recently, many researchers have studied various economic models with or without discrete and distributed delays. For example, Grasman and Wentzel have investigated the Kaldor-Kalecki business cycle model as follows [1]:

$$\begin{cases} y'(t) = \alpha(I(y(t), k(t)) - \gamma y(t)), \\ k'(t) = I(y(t), k(t)) - qk(t), \end{cases} \tag{1}$$

where $y(t)$ is the gross product, $k(t)$ is the capital stock at time, α is the adjustment coefficient in the goods market, q is the depreciation rate of the capital stock, γ represents the propensity to save, and $I(y(t), k(t))$ is the investment. The existence of a limit cycle with an equilibrium point was considered for the mode (1). Concerned delay τ in the form of investment and saving functions as the following:

$$\begin{cases} y'(t) = \alpha(I(y(t), k(t)) - \gamma y(t)), \\ k'(t) = I(y(t-\tau), k(t-\tau)) - qk(t). \end{cases} \tag{2}$$

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DOI: 10.22103/jmmr.2023.20085.1321
Publisher: Shahid Bahonar University of Kerman
How to cite: C. Feng, *Periodic Oscillation for a Class of Delayed Economic Models*, J. Mahani Math. Res. 2023; 12(2): 513-527.

The dynamic behaviors such as stability, Hopf bifurcation, codimension-two bifurcation, and chaos were established in [2-10]. For a Kaldor-Kalecki model of the business cycle with two discrete time delays as follows:

$$(3)\qquad \begin{cases} y'(t) = \alpha(I(y(t), k(t))) - S(y(t), k(t)), \\ k'(t) = I(y(t-\tau_1), k(t-\tau_2)) - qk(t), \end{cases}$$

where $I(y, k)$ and $S(y, k)$ are the investment function and the saving function respectively, τ_1 is the time delay for the investment due to the past gross product, τ_2 is the time delay for capital stock in the past. By analyzing the corresponding characteristic equations, the local stability of the positive equilibrium was discussed. Choosing the adjustment coefficient α in the goods market as a bifurcation parameter, the existence of Hopf bifurcation was investigated in detail [11, 12]. Due to the importance of anticipation for making decisions and organizational transformations, the Kaldor-Kalecki model of the business cycle was studied in view of showing its anticipatory capabilities [13]. The dynamics behaviors of Kaldor-Kalecki business cycle model with diffusion effect and time delay under the Neumann boundary conditions were investigated [14, 15]. Mircea et al., considered a Kaldor-Kalecki stochastic model of the business cycle [16]. Caraballo and Silva investigated the stability of a delay differential in Kaldor's model with government policies [17].

On the other hand, when local economies are subject to various economic and various unions, macroeconomic models cannot be treated as isolated systems. Especially, the influence of large and dominating economies on the Gross Domestic Product of local economies cannot be neglected. Therefore, Zduniak et al., have investigated a coupled Kaldor-Kalecki model with delay [18]:

$$(4)\qquad \begin{cases} y_1'(t) = \alpha_1(F_1(y_1(t)) - \delta_1 y_2(t) - \gamma_1 y_1(t)), \\ y_2'(t) = F_1(y_1(t-\tau)) - \delta_1 y_2(t-\tau) - \delta y_2(t), \\ y_3'(t) = \alpha_2(F_2(y_2(t)) - \delta_2 y_4(t) - \gamma_2 y_3(t)), \\ y_4'(t) = F_2(y_2(t-\tau)) - \delta_2 y_4(t-\tau) - \delta y_4(t) + s(y_1(t) - y_3(t)), \end{cases}$$

where $F_i(y_i)$ are investment functions, and it is taken from the published literature as $\frac{e^{y_i(t)}}{1+e^{y_i(t)}}$. s is a coupling coefficient, α_1 and α_2 are the adjustment coefficients (correction factors), $\delta \in (0, 1)$ is the depreciation rate of capital stock, $\gamma_1, \gamma_2, \delta_1$ and δ_2 are constants, and τ denotes the time delay. The authors considered two types of investment functions that lead to the different behavior of the system. The model with unidirectional coupling to investigate the influence of a global economy on a local economy was also considered. Zduniak and Orlowsereki also extended two coupled Kaldor-Kalecki model to a five coupled Kaldor-Kalecki model with one delay, in which the numerical simulation for the parameter values was provided [19]. However, different investment functions may have different time delays as in model (2). A multiple time delay system may be more in keeping with practical situations. Therefore, in this paper, we extend the five coupled Kaldor-Kalecki model to the following system with ten delays:

$$
(5)\quad \begin{cases}
y_1'(t) = \alpha_1(F_1(y_1(t)) - \delta_1 y_2(t) - \gamma_1 y_1(t)), \\
y_2'(t) = F_1(y_1(t-\tau_1)) - \delta_1 y_2(t-\tau_2) - \delta y_2(t) - s_4(y_9(t) - y_1(t)) \\
\qquad +s_5(y_7(t) - y_1(t)), \\
y_3'(t) = \alpha_2(F_3(y_3(t)) - \delta_2 y_4(t) - \gamma_2 y_3(t)), \\
y_4'(t) = F_3(y_3(t-\tau_3)) - \delta_2 y_4(t-\tau_4) - \delta y_4(t) - s_1(y_1(t) - y_3(t)) \\
\qquad +s_9(y_7(t) - y_3(t)) - s_{11}(y_5(t) - y_3(t)), \\
y_5'(t) = \alpha_3(F_5(y_5(t)) - \delta_3 y_6(t) - \gamma_3 y_5(t)), \\
y_6'(t) = F_5(y_5(t-\tau_5)) - \delta_3 y_6(t-\tau_6) - \delta y_6(t) - s_2(y_1(t) - y_5(t)) \\
\qquad +s_8(y_7(t) - y_5(t)) - s_{12}(y_3(t) - y_5(t)), \\
y_7'(t) = \alpha_4(F_7(y_7(t)) - \delta_4 y_8(t) - \gamma_4 y_7(t)), \\
y_8'(t) = F_7(y_7(t-\tau_7)) - \delta_4 y_8(t-\tau_8) - \delta y_8(t) - s_7(y_9(t) - y_7(t)) \\
\qquad -s_{10}(y_1(t) - y_7(t)), \\
y_9'(t) = \alpha_5(F_9(y_9(t)) - \delta_5 y_{10}(t) - \gamma_5 y_9(t)), \\
y_{10}'(t) = F_9(y_9(t-\tau_9)) - \delta_5 y_{10}(t-\tau_{10}) - \delta y_{10}(t) - s_3(y_1(t) - y_9(t)) \\
\qquad +s_6(y_7(t) - y_9(t)),
\end{cases}
$$

where $F_i(y_i(t)) = \frac{e^{y_i(t)}}{1+e^{y_i(t)}}, i = 1, 3, 5, 7, 9$, all parameter values $\delta, \alpha_i, \delta_i, s_i$ are positive real numbers. Our goal is to investigate the periodic oscillatory solutions of the model (5) not only by numerical simulation, but also by means of the theoretical analysis.

2. Preliminaries

It is known that bifurcation can give rise to a periodic solution. However, the bifurcation method is hard to deal with in model (5) due to the complexity of the bifurcating equation since there are ten time delays. In this paper, we will use the method of mathematical analysis to discuss the existence of periodic solutions of the model (5). First, we provide the following definition and two lemmas.

Definition 2.1. The trivial solution of system (5) is unstable, if there exists at least one component of the trivial solution which is unstable.

For selecting parameter values, set up a square matrix M as follows:

$$
M = \begin{pmatrix}
m_{11} & m_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
m_{21} & m_{22} & 0 & 0 & 0 & 0 & m_{27} & 0 & m_{29} & 0 \\
0 & 0 & m_{33} & m_{34} & 0 & 0 & 0 & 0 & 0 & 0 \\
m_{41} & 0 & m_{43} & m_{44} & m_{45} & 0 & m_{47} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & m_{55} & m_{56} & 0 & 0 & 0 & 0 \\
m_{61} & 0 & m_{63} & 0 & m_{65} & m_{66} & m_{67} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & m_{77} & m_{78} & 0 & 0 \\
m_{81} & 0 & 0 & 0 & 0 & 0 & m_{87} & m_{88} & m_{89} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_{99} & m_{910} \\
m_{101} & 0 & 0 & 0 & 0 & 0 & m_{107} & 0 & m_{109} & m_{1010}
\end{pmatrix},
$$

where $m_{11} = \alpha_1 c_1 - \alpha_1\gamma_1, m_{12} = -\alpha_1\delta_1, m_{21} = c_1 + s_4 - s_5, m_{22} = -\delta_1 - \delta, m_{27} = s_5, m_{29} = -s_4, m_{33} = \alpha_2 c_3 - \alpha_2\gamma_2, m_{34} = -\alpha_2\delta_2, m_{41} = -s_1, m_{43} = c_3 + s_1 + s_9 - s_{11}, m_{44} = -\delta_2 - \delta, m_{45} = -s_{11}, m_{47} = s_9, m_{55} = \alpha_3 c_5 - \alpha_3\gamma_3, m_{56} = -\alpha_3\delta_5, m_{61} = -s_2, m_{63} = -s_{12}, m_{65} = c_5 + s_2 - s_8 - s_{12}, m_{66} = -\delta_3 - \delta, m_{67} = s_8, m_{77} = \alpha_4 c_7 - \alpha_4\gamma_4, m_{78} = -\alpha_4\delta_4, m_{81} = -s_{10}, m_{87} = c_7 + s_7 + s_{10}, m_{88} = -\delta_4 - \delta, m_{89} = -s_7, m_{99} = \alpha_5 c_9 - \alpha_5\gamma_5, m_{910} = -\alpha_5\delta_5, m_{101} = -s_3, m_{107} = s_6, m_{109} = c_9 + s_3 - s_6, m_{1010} = -\delta_5 - \delta$, where $0 < c_i < 1, i = 1, 3, 5, 7, 9$. Then we have

Lemma 2.2. *Assuming that M is a nonsingular matrix, then system (5) has a unique positive equilibrium point.*

Proof. An equilibrium point $y^* = [y_1^*, y_2^*, \ldots, y_{10}^*]^T$ of the system (5) is a constant solution of the following algebraic equation

$$(6)\quad \begin{cases} \alpha_1 F_1(y_1^*) - \alpha_1\delta_1 y_2^* - \alpha_1\gamma_1 y_1^* = 0, \\ F_1(y_1^*) - \delta_1 y_2^* - \delta y_2^* - s_4(y_9^* - y_1^*) + s_5(y_7^* - y_1^*) = 0, \\ \alpha_2 F_3(y_3^*) - \alpha_2\delta_2 y_4^* - \gamma_2 y_3^* = 0, \\ F_3(y_3^*) - \delta_2 y_4^* - \delta y_4^* - s_1(y_1^* - y_3^*) + s_9(y_7^* - y_3^*) - s_{11}(y_5^* - y_3^*) = 0, \\ \alpha_3 F_5(y_5^*) - \alpha_3\delta_3 y_6^* - \gamma_3 y_5^* = 0, \\ F_5(y_5^*) - \delta_3 y_6^* - \delta y_6^* - s_2(y_1^* - y_5^*) + s_8(y_7^* - y_5^*) - s_{12}(y_3^* - y_5^*) = 0, \\ \alpha_4 F_7(y_7^*) - \alpha_4\delta_4 y_8^* - \alpha_4\gamma_4 y_7^* = 0, \\ F_7(y_7^*) - \delta_4 y_8^* - \delta y_8^* - s_7(y_9^* - y_7^*) - s_{10}(y_1^* - y_7^*) = 0, \\ \alpha_5 F_9(y_9^*) - \alpha_5\delta_5 y_{10}^* - \alpha_5\gamma_5 y_9^* = 0, \\ F_9(y_9^*) - \delta_5 y_{10}^* - \delta y_{10}^* - s_3(y_1^* - y_9^*) + s_6(y_7^* - y_9^*) = 0. \end{cases}$$

If $\bar{y}^* = [\bar{y}_1^*, \bar{y}_2^*, \ldots, \bar{y}_{10}^*]^T$ is another set of the equilibrium point of system (5), then we have

$$(7)\quad \begin{cases} \alpha_1[F_1(y_1^*) - F_1(\bar{y}_1^*)] - \alpha_1\gamma_1(y_1^* - \bar{y}_1^*) - \alpha_1\delta_1(y_2^* - \bar{y}_2^*) = 0, \\ F_1(y_1^*) - F_1(\bar{y}_1^*) + (s_4 - s_5)(y_1^* - \bar{y}_1^*) - (\delta_1 + \delta)(y_2^* - \bar{y}_2^*) \\ \qquad + s_5(y_7^* - \bar{y}_7^*) - s_4(y_9^* - \bar{y}_9^*) = 0, \\ \alpha_2 F_3(y_3^* - F_3(\bar{y}_3^*)] - \gamma_2 y_3^*(y_3^* - \bar{y}_3^*) - \alpha_2\delta_2(y_4^* - \bar{y}_4^*) = 0, \\ -s_1(y_1^* - \bar{y}_1^*) + F_3(y_3^*) - F_3(\bar{y}_3^*) + (s_1 - s_9 + s_{11})(y_3^* - \bar{y}_3^*) \\ \qquad -(\delta_2 + \delta)(y_4^* - \bar{y}_4^*) - s_{11}(y_5^* - \bar{y}_5^*) + s_9(y_7^* - \bar{y}_7^*) = 0, \\ \alpha_3[F_5(y_5^*) - F_5(\bar{y}_5^*)] - \gamma_3(y_5^* - \bar{y}_5^*) - \alpha_3\delta_3(y_6^* - \bar{y}_6^*) = 0, \\ -s_2(y_1^* - \bar{y}_1^*) - s_{12}(y_3^* - \bar{y}_3^*) + F_5(y_5^*) - F_5(\bar{y}_5^*) + (s_2 - s_8 + s_{12})(y_5^* \\ \qquad -\bar{y}_5^*) - (\delta_3 + \delta)(y_6^* - \bar{y}_6^*) + s_8(y_7^* - \bar{y}_7^*) = 0, \\ \alpha_4[F_7(y_7^*) - F_7(\bar{y}_7^*)] - \alpha_4\gamma_4(y_7^* - \bar{y}_7^*) - \alpha_4\delta_4(y_8^* - \bar{y}_8^*) = 0, \\ -s_{10}(y_1^* - \bar{y}_1^*) + F_7(y_7^*) - F_7(\bar{y}_7^*) + (s_7 + s_{10})(y_7^* - \bar{y}_7^*) - (\delta_4 + \delta)(y_8^* \\ \qquad -\bar{y}_8^*) - s_7(y_9^* - \bar{y}_9^*) = 0, \\ \alpha_5[F_9(y_9^*) - F_9(\bar{y}_9^*)] - \alpha_5\gamma_5(y_9^* - \bar{y}_9^*) - \alpha_5\delta_5(y_{10}^* - \bar{y}_{10}^*) = 0, \\ -s_3(y_1^* - \bar{y}_1^*) + s_6(y_7^* - \bar{y}_7^*) + F_9(y_9^*) - F_9(\bar{y}_9^*) + (s_3 - s_6)(y_9^* - \bar{y}_9^*) \\ \qquad -(\delta_5 + \delta)(y_{10}^* - \bar{y}_{10}^*) = 0. \end{cases}$$

Noting that $F_i(y_i(t)) = \frac{e^{y_i(t)}}{1+e^{y_i(t)}} (i = 1, 3, 5, 7, 9)$, then $F_i'(y_i(t)) = \frac{e^{y_i(t)}}{(1+e^{y_i(t)})^2}$. Therefore, $0 < F_i(y_i(t)) < 1$, and $F_i(y_i)$ are monotone increasing functions. By the mean value theorem, $F_i(y_i^*) - F_i(\bar{y}_i^*) = F_i'(\eta_i)(y_i^* - \bar{y}_i^*)$, where $\eta_i \in (y_i^*, \bar{y}_i^*)$. Let $c_i = F_i'(\eta_i)$, then $0 < c_i < 1 (i = 1, 3, 5, 7, 9)$. Thus, from the system (7) we have

$$M(y^* - \bar{y}^*) = \mathbf{0}. \tag{8}$$

System (8) is a matrix equation about variables $(y_i^* - \bar{y}_i^*)$. Since M is a nonsingular matrix, based on the basic algebraic knowledge, system (8) has a unique trivial solution, namely, $y_i^* - \bar{y}_i^* = 0 (i = 1, 2, \ldots, 10)$, implying that system (5) has a unique equilibrium point $y_1^*, y_2^*, \ldots, y_{10}^*$. □

Lemma 2.3. *Assume that all parameters are positive constants, then all solutions of system (5) are bounded.*

Proof. To prove the boundedness of the solutions in the system (5), we construct a Lyapunov function $V(t) = \sum_{i=1}^{10} \frac{1}{2} y_i^2$. Noting that $0 < F_i(y_i(t)) < 1$, calculating the derivative of $V(t)$ through system (5) we get

$$\begin{aligned} V'(t)|_{(5)} = \textstyle\sum_{i=1}^{10} y_i y_i' &= y_1(t)[\alpha_1(F_1(y_1(t)) - \delta_1 y_2(t) - \gamma_1 y_1(t))] \\ &\quad + y_2(t)[F_1(y_1(t-\tau_1)) - \delta_1 y_2(t-\tau_2) - \delta y_2(t) - s_4(y_9(t) - y_1(t)) \\ &\quad + s_5(y_7(t) - y_1(t))] + \cdots + y_9(t)[\alpha_5 F_9(y_9(t-\tau_9)) - \delta_5 y_{10}(t-\tau_{10}) \\ &\quad - \gamma_5 y_9(t)] + y_{10}(t)[F_9(y_9(t-\tau_9)) - \delta_5 y_{10}(t-\tau_{10}) - \delta y_{10}(t) \\ &\quad - s_3(y_1(t) - y_9(t)) + s_6(y_7(t) - y_9(t))] \\ &\leq -\alpha_1\gamma_1 y_1^2 - (\delta_1 + \delta) y_2^2 - \alpha_2\gamma_2 y_3^2 - (\delta_2 + \delta) y_4^2 - \alpha_3\gamma_3 y_5^2 \\ &\quad - (\delta_3 + \delta) y_6^2 - \alpha_4\gamma_4 y_7^2 - (\delta_4 + \delta) y_8^2 - \alpha_5\gamma_5 y_9^2 \\ &\quad - (\delta_5 + \delta) y_{10}^2 - \alpha_1\delta_1 y_1 y_2 - s_4 y_2 y_9 + (s_4 - s_5) y_1 y_2 \\ &\quad - \alpha_5\delta_5 y_9 y_{10} - s_3 y_1 y_{10} + (s_3 - s_6) y_9 y_{10} + s_6 y_7 y_{10} \\ &\quad + \alpha_1 y_1 + \alpha_2 y_3 + \alpha_3 y_5 + \alpha_4 y_7 + \alpha_5 y_9. \end{aligned} \tag{9}$$

Since all parameters are positive real numbers, obviously, there exists a positive number L such that $V'(t)|_{(5)} < 0$ when $y_i > L$. This means that all solutions of system (5) are bounded. □

If $y_1^*, y_2^*, \ldots, y_{10}^*$ is a positive equilibrium point of system (5), and make the change of $x_i(t) = y_i(t) - y_i^*$, then by the linearizing system (5) around

$(0, 0, \ldots, 0)$ we have

$$
(10)\qquad \begin{cases}
x_1'(t) = \alpha_1 F_1'(y_1^*)x_1(t) - \alpha_1\delta_1 x_2(t) - \alpha_1\gamma_1 x_1(t)),\\
x_2'(t) = F_1'(y_1^*)x_1(t-\tau_1) - \delta_1 x_2(t-\tau_2) - \delta x_2(t) - s_4(x_9(t)\\
\qquad -x_1(t)) + s_5(x_7(t) - x_1(t)),\\
x_3'(t) = \alpha_2 F_3'(y_3^*)x_3(t) - \alpha_2\delta_2 x_4(t) - \alpha_2\gamma_2 x_3(t)),\\
x_4'(t) = F_3'(y_3^*)x_3(t-\tau_3) - \delta_2 x_4(t-\tau_4) - \delta x_4(t) - s_1(x_1(t)\\
\qquad -x_3(t)) + s_9(x_7(t) - x_3(t)) - s_{11}(x_5(t) - x_3(t)),\\
x_5'(t) = \alpha_3 F_5'(y_5^*)x_5(t) - \alpha_3\delta_3 x_6(t) - \alpha_3\gamma_3 x_5(t)),\\
x_6'(t) = F_5'(y_5^*)x_5(t-\tau_5)) - \delta_3 x_6(t-\tau_6) - \delta x_6(t) - s_2(x_1(t)\\
\qquad -x_5(t)) + s_8(x_7(t) - x_5(t)) - s_{12}(x_3(t) - x_5(t)),\\
x_7'(t) = \alpha_4 F_7'(y_7^*)x_7(t) - \alpha_4\delta_4 x_8(t) - \alpha_4\gamma_4 y_7(t)),\\
x_8'(t) = F_7'(y_7^*)x_7(t-\tau_7)) - \delta_4 x_8(t-\tau_8) - \delta x_8(t) - s_7(x_9(t)\\
\qquad -x_7(t)) - s_{10}(x_1(t) - x_7(t)),\\
x_9'(t) = \alpha_5 F_9'(y_9^*)x_9(t) - \alpha_5\delta_5 x_{10}(t) - \alpha_5\gamma_5 x_9(t)),\\
x_{10}'(t) = F_9'(y_9^*)x_9(t-\tau_9)) - \delta_5 x_{10}(t-\tau_{10}) - \delta x_{10}(t)\\
\qquad -s_3(x_1(t) - x_9(t)) + s_6(x_7(t) - x_9(t)).
\end{cases}
$$

The matrix form of system (10) is the following:

$$
(11)\qquad x'(t) = Ax(t) + Bx(t-\tau),
$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_{10}(t)]^T$, $x(t-\tau) = [x_1(t-\tau_1), x_2(t-\tau_2), \ldots, x_{10}(t-\tau_{10})]^T$. Both $A = (a_{ij})_{10\times 10}$ and $B = (b_{ij})_{10\times 10}$ are 10×10 matrices as follows:

$$
A = (a_{ij})_{10\times 10} = \begin{pmatrix}
a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
a_{21} & a_{22} & 0 & 0 & 0 & 0 & a_{27} & 0 & a_{29} & 0\\
0 & 0 & a_{33} & a_{34} & 0 & 0 & 0 & 0 & 0 & 0\\
a_{41} & 0 & a_{43} & a_{44} & a_{45} & 0 & a_{47} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & a_{55} & a_{56} & 0 & 0 & 0 & 0\\
a_{61} & 0 & a_{63} & 0 & a_{65} & a_{66} & a_{67} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & a_{77} & a_{78} & 0 & 0\\
a_{81} & 0 & 0 & 0 & 0 & 0 & a_{87} & a_{88} & a_{89} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{99} & a_{910}\\
a_{101} & 0 & 0 & 0 & 0 & 0 & a_{107} & 0 & a_{109} & a_{1010}
\end{pmatrix},
$$

where $a_{11} = \alpha_1 F_1'(y_1^*) - \alpha_1\gamma_1, a_{12} = -\alpha_1\delta_1, a_{21} = s_4 - s_5, a_{22} = -\delta, a_{27} = s_5, a_{29} = -s_4, a_{33} = \alpha_2 F_3'(y_3^*) - \alpha_2\gamma_2, a_{34} = -\alpha_2\delta_2, a_{41} = -s_1, a_{43} = s_1 + s_9 - s_{11}, a_{44} = -\delta, a_{45} = -s_{11}, a_{47} = s_9, a_{55} = \alpha_3 F_5'(y_5^*) - \alpha_3\gamma_3, a_{56} = -\alpha_3\delta_3, a_{61} = -s_2, a_{63} = -s_{12}, a_{65} = -s_2 + s_8 - s_{12}, a_{66} = -\delta, a_{67} = s_8, a_{77} = \alpha_4 F_7'(y_7^*) - \alpha_4\gamma_4, a_{78} = -\alpha_4\delta_4, a_{81} = -s_{10}, a_{87} = s_7 + s_{10}, a_{88} = -\delta, a_{89} = -s_7, a_{99} = \alpha_5 F_9'(y_9^*) - \alpha_5\gamma_5, a_{910} = -\alpha_5\delta_5, a_{101} = -s_3, a_{107} = s_6, a_{109} = s_3 - s_6, a_{1010} =$

$-\delta.$

$$B = (b_{ij})_{10\times 10} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{43} & b_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{65} & b_{66} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{87} & b_{88} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{109} & b_{1010} \end{pmatrix},$$

where $b_{21} = F_1'(y_1^*), b_{22} = -\delta_1, b_{43} = F_3'(y_3^*), b_{44} = -\delta_2, b_{65} = F_5'(y_5^*), b_{66} = -\delta_3, b_{87} = F_7'(y_7^*), b_{88} = -\delta_4, b_{109} = F_9'(y_9^*), b_{1010} = -\delta_5$.
It is known that if the trivial solution of the linearized system (10) (or (11)) is unstable, then the positive equilibrium point of the original system (5) is unstable. Therefore, for proving the instability of the unique positive equilibrium point of system (5) we only need to prove the instability of the trivial solution of system (10) (or (11)).

3. Existence of oscillatory solutions

Based on the definition and the above two lemmas, we have the following main result.

Theorem 3.1. *Assume that the conditions of Lemmas 1 and 2 hold. Let $\alpha_1, \alpha_2, \ldots, \alpha_{10}$ represent the eigenvalues of matrix A, and $\beta_1, \beta_2, \ldots, \beta_{10}$ the eigenvalues of matrix B. If there exists one eigenvalue, say α_1 which is a positive real number, or α_1 is a complex number that has a positive real part, then the trivial solution of system (10) (or (11)) is unstable, implying that the unique equilibrium point $y_1^*, y_2^*, \ldots, y_{10}^*$ of system (5) is unstable, and system (5) generates a limit cycle, namely, a periodic solution.*

Proof. Since the eigenvalues of matrix A are $\alpha_1, \alpha_2, \ldots, \alpha_{10}$, and the eigenvalues of matrix B are $\beta_1, \beta_2, \ldots, \beta_{10}$, system (11) has the following characteristic equation:

$$\prod_{i=1}^{10} (\lambda - \alpha_i - \beta_i e^{-\lambda \tau_i}) = 0. \tag{12}$$

Noting that there exist five-row entries of matrix B which are zeros, so there is a characteristic value, say $\beta_1 = 0$. then we have

$$\lambda - \alpha_1 - \beta_1 e^{-\lambda \tau_1} = \lambda - \alpha_1 = 0. \tag{13}$$

This means that there exists an eigenvalue that is a positive number or is a complex number that has a positive real part, implying that the trivial solution of system (10) (or (11)) is unstable. This suggests that the unique positive equilibrium point of system (5) is unstable. The instability of the unique positive equilibrium point with the boundedness of the solution will force system (5) to generate a limit cycle, namely, a periodic solution [22, 23]. □

Now let $\mu = \max_{1\leq j\leq 10}(a_{jj} + \sum_{i=1,i\neq j}^{10} |a_{ij}|)$, and $b = \max_{1\leq i,j\leq 10} |b_{ij}|$ [20], then we have

Theorem 3.2. *Assume that the conditions of Lemmas 1 and 2 hold. If the following condition holds:*

$$\mu + b > 0. \tag{14}$$

Then the trivial solution of system (10) (or (11)) is unstable, implying that the unique positive equilibrium point of system (5) is unstable, and system (5) generates a limit cycle, namely, a periodic solution.

Proof. System (10) (or (11)) can be rewritten as

$$\left\{\begin{array}{l} \frac{d|x_1(t)|}{dt} \leq a_{11}|x_1(t)| + |a_{12}||x_2(t)|, \\ \frac{d|x_2(t)|}{dt} \leq a_{22}|x_2(t)| + |a_{21}||x_1(t)| + |a_{27}||x_7(t)| + |a_{29}||x_9(t)| \\ \qquad +|b_{21}||x_1(t-\tau_1)| + |b_{22}||x_2(t-\tau_2)|, \\ \frac{d|x_3(t)|}{dt} \leq a_{33}|x_3(t)| + |a_{34}||x_4(t)|, \\ \frac{d|x_4(t)|}{dt} \leq a_{44}|x_4(t)| + |a_{41}||x_1(t)| + |a_{43}||x_3(t)| + |a_{45}||x_5(t)| \\ \qquad +|a_{47}||x_7(t)| + |b_{43}||x_3(t-\tau_3)| + |b_{44}||x_4(t-\tau_4)|, \\ \frac{d|x_5(t)|}{dt} \leq a_{55}|x_5(t)| + |a_{56}||x_6(t)|, \\ \frac{d|x_6(t)|}{dt} \leq a_{66}|x_6(t)| + |a_{61}||x_1(t)| + |a_{63}||x_3(t)| + |a_{65}||x_5(t)| \\ \qquad +|a_{67}||x_7(t)| + |b_{65}||x_5(t-\tau_5)| + |b_{66}||x_6(t-\tau_6)|, \\ \frac{d|x_7(t)|}{dt} \leq a_{77}|x_7(t)| + |a_{78}||x_8(t)|, \\ \frac{d|x_8(t)|}{dt} \leq a_{88}|x_8(t)| + |a_{81}||x_1(t)| + |a_{87}||x_7(t)| + |a_{89}||x_9(t)| \\ \qquad +|b_{87}||x_7(t-\tau_7)| + |b_{88}||x_8(t-\tau_8)|, \\ \frac{d|x_9(t)|}{dt} \leq a_{99}|x_9|(t) + |a_{910}||x_{10}(t)|, \\ \frac{d|x_{10}(t)|}{dt} \leq a_{1010}|x_{10}(t)| + |a_{101}||x_1(t)| + |a_{107}||x_7(t)| + |a_{109}||x_9(t)| \\ \qquad +|b_{109}||x_9(t-\tau_9)| + |b_{1010}||x_{10}(t-\tau_{10})|. \end{array}\right. \tag{15}$$

Let $u(t) = \sum_{i=1}^{10} |x_i(t)|, u(t-\tau) = \sum_{i=1}^{10} |x_i(t-\tau_i)|$. Then we get

$$\frac{du(t)}{dt} \leq \mu u(t) + bu(t-\tau). \tag{16}$$

Consider the scalar delay differential equation

$$\frac{dv(t)}{dt} = \mu v(t) + bv(t-\tau). \tag{17}$$

By the property of the delay differential equation [21] we have

$$u(t) \leq v(t). \tag{18}$$

We must prove that the trivial solution of system (17) is unstable. The characteristic equation associated with the system (17) is the following:

$$\lambda = \mu + be^{-\lambda\tau}. \tag{19}$$

Equation (19) is a transcendental equation, it is hard to find all solutions of this equation. However, we claim that there exists a positive characteristic value or a positive real part of complex characteristic value under the restrictive condition (14). Indeed, we define a function $h(\lambda) = \lambda - \mu - be^{-\lambda\tau}$, then $h(\lambda)$ is a continuous function of λ. When $\lambda = 0$ we have $h(0) = -\mu - b = -(\mu + b) < 0$. Obviously, there exists a suitably large λ, say $\lambda_1 > 0$ such that $h(\lambda_1) = \lambda_1 - \mu - be^{-\lambda_1\tau} > 0$. Based on the Intermediate Value Theorem, there exists a value of λ, say $\lambda_0 \in (0, \lambda_1)$ such that $h(\lambda_0) = \lambda_0 - \mu - be^{-\lambda_0\tau} = 0$. In other words, equation (19) has a positive eigenvalue, implying that the trivial solution of system (17) (also system (16)) is unstable, this means that the unique positive equilibrium point of system (5) is unstable. The instability of the unique positive equilibrium point of system (5) with the boundedness of the solution will force system (5) to generate a limit cycle, namely, a periodic solution. The proof is completed. □

4. Numerical results

These simulations are performed based on system (5). Firstly we select $\delta = 0.055$, the other parameter values as table 1 for figure 1 and figure 2. The characteristic values of matrix M are $0.7354, -1.2370, -0.2545 \pm 1.1222i, -0.1542 \pm 0.7271i, -0.1307 \pm 0.6972i, -0.0593 \pm 0.1672i$. Therefore, M is a nonsingular matrix. The characteristic values of matrix A are $0.8355, -1.1837$, $0.0168 \pm 0.0158i, -0.1593 \pm 1.1154i, -0.0909 \pm 0.7178i, -0.0570 \pm 0.6865i$. Since 0.8355 is a positive characteristic value of matrix A, the conditions of Theorem 1 are satisfied. We see that there are periodic oscillatory solutions of system (5) (see Fig. 1 and Fig. 2). Then we only set $\delta = 0.0085, \alpha_1 = 1.98, \alpha_2 = 1.25, \alpha_3 = 2.12, \alpha_4 = 1.30, \alpha_5 = 2.16$, the other parameters are the same as in Fig. 1. When time delays are selected as $\tau_1 = 0.52, \tau_2 = 0.55, \tau_3 = 0.58, \tau_4 = 0.56, \tau_5 = 0.50, \tau_6 = 0.44, \tau_7 = 0.42, \tau_8 = 0.43, \tau_9 = 0.46, \tau_{10} = 0.48$, and $\tau_1 = 0.96, \tau_2 = 0.95, \tau_3 = 0.98, \tau_4 = 0.94, \tau_5 = 0.90, \tau_6 = 0.92, \tau_7 = 0.88, \tau_8 = 0.86, \tau_9 = 0.85, \tau_{10} = 0.84$, respectively, there are periodic oscillatory solutions of system (5) (see Fig. 3 and Fig. 4). However, we see that the oscillatory frequency and amplitude both are changed comparison with figure 1 and figure 2. Finally, we select $\delta = 0.015$, the other parameter values are as table 2, we see that $\mu + b = 6.95 + 0.2396 > 0$. The conditions of Theorem 2 are satisfied. There are periodic oscillatory solutions of system (5) (see Fig. 5 and Fig. 6). Recalling in [19], the parameters are $\delta = 0.1, \alpha_1 = 4, \alpha_2 = 2, \alpha_3 = 3, \alpha_4 = 3.5, \alpha_5 = 3.8, \gamma_i = \delta_i = 0.2, s_1 = s_4 = s_8 = 0.1, s_2 = s_3 = s_5 = 0.05, s_5 = s_7 = s_{10} = s_{12} = 0.2, s_6 = 0.3, s_{11} = 0.25,$

Table 1. Parameter values of figure 1 and figure 2

α_1	α_2	α_3	α_4	α_5	δ_1	δ_2	δ_3	δ_4	δ_5
1.25	1.18	2.16	2.14	2.15	0.15	0.12	0.16	0.18	0.16
γ_1	γ_2	γ_3	γ_4	γ_5	s_1	s_2	s_3	s_4	s_5
0.18	0.15	0.18	0.15	0.17	2.34	2.26	2.24	2.25	2.27
s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}			
2.32	2.28	2.22	2.25	2.29	2.24	2.25			
y_1^*	y_2^*	y_3^*	y_4^*	y_5^*	y_6^*	y_7^*	y_8^*	y_9^*	y_{10}^*
0.2848	3.2646	3.4633	0.3319	4.4374	0.3266	3.2637	0.2850	2.9337	0.2847
c_1	c_3	c_5	c_7	c_9					
0.2450	0.0295	0.0116	0.0355	0.0480					
τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	τ_9	τ_{10}
1.30	1.25	1.32	1.36	1.22	1.25	1.38	1.28	1.24	1.26
τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	τ_9	τ_{10}
1.92	1.85	1.88	1.86	1.90	1.84	1.82	1.94	1.96	1.87

Table 2. Parameter values of figure 5 and figure 6

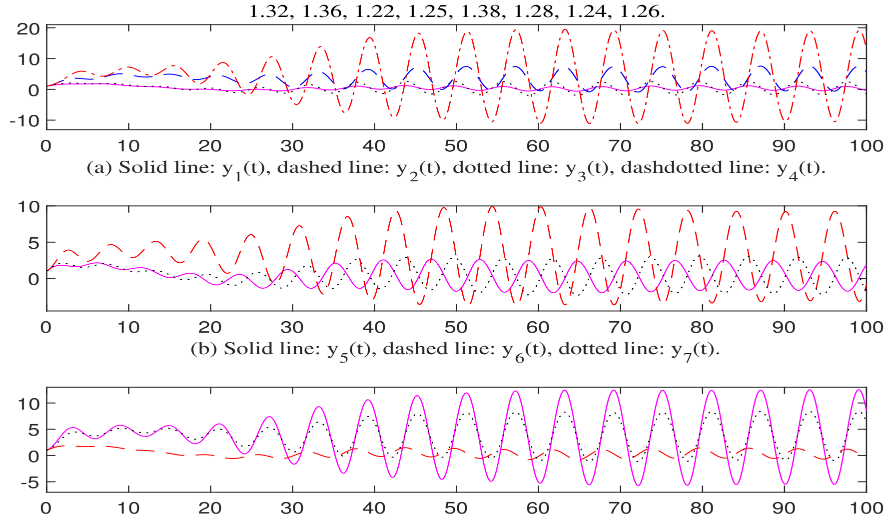
α_1	α_2	α_3	α_4	α_5	δ_1	δ_2	δ_3	δ_4	δ_5
3.65	3.45	2.92	1.95	3.76	0.065	0.064	0.055	0.068	0.054
γ_1	γ_2	γ_3	γ_4	γ_5	s_1	s_2	s_3	s_4	s_5
0.28	0.25	0.24	0.22	0.26	3.25	3.86	3.78	3.75	3.95
s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}			
3.15	4.94	3.92	3.45	3.75	3.35	3.65			
y_1^*	y_2^*	y_3^*	y_4^*	y_5^*	y_6^*	y_7^*	y_8^*	y_9^*	y_{10}^*
0.4129	7.4794	0.4760	7.7780	0.4803	9.1370	0.4201	7.5158	0.4213	9.1527
c_1	c_3	c_5	c_7	c_9					
0.2396	0.2364	0.2361	0.2393	0.2391					
τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	τ_9	τ_{10}
0.85	0.82	0.88	0.84	0.80	0.78	0.86	0.76	0.75	0.74
τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	τ_9	τ_{10}
1.45	1.42	1.43	1.44	1.38	1.36	1.35	1.40	1.32	1.34

only one delay τ was fixed at $\tau = 3$. Obviously, the present work is an extension of the result in [19] not only in time delays, but also in the adjustment and coupling coefficients.

5. Conclusion

In this paper, we have discussed the existence of periodic solutions for a financial model with time delays. The original work in [19] has been extended

Fig. 1 Periodic oscillation of the solutions, delays: 1.30, 1.25, 1.32, 1.36, 1.22, 1.25, 1.38, 1.28, 1.24, 1.26.

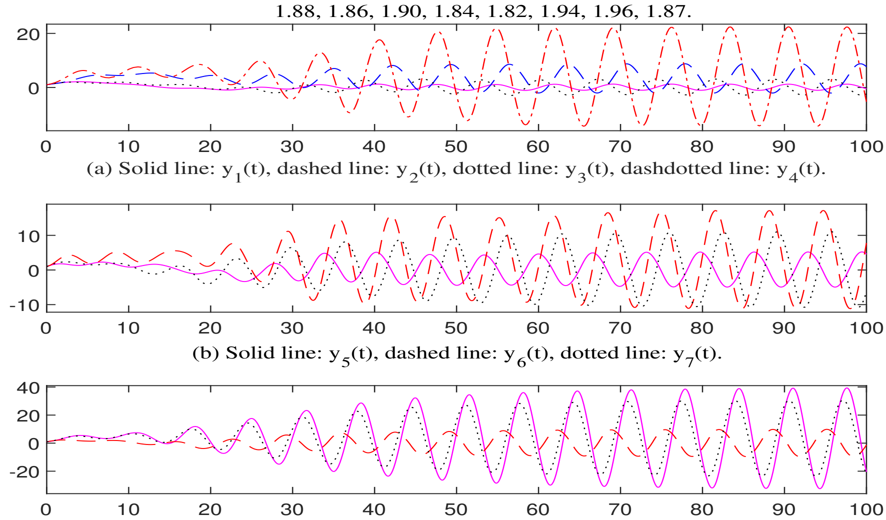


(a) Solid line: $y_1(t)$, dashed line: $y_2(t)$, dotted line: $y_3(t)$, dashdotted line: $y_4(t)$.

(b) Solid line: $y_5(t)$, dashed line: $y_6(t)$, dotted line: $y_7(t)$.

(c) Solid line: $y_8(t)$, dashed line: $y_9(t)$, dotted line: $y_{10}(t)$.

Fig. 2 Periodic oscillation of the solutions, delays: 1.92, 1.85, 1.88, 1.86, 1.90, 1.84, 1.82, 1.94, 1.96, 1.87.



(a) Solid line: $y_1(t)$, dashed line: $y_2(t)$, dotted line: $y_3(t)$, dashdotted line: $y_4(t)$.

(b) Solid line: $y_5(t)$, dashed line: $y_6(t)$, dotted line: $y_7(t)$.

(c) Solid line: $y_8(t)$, dashed line: $y_9(t)$, dotted line: $y_{10}(t)$.

theoretically. The existence of a limit circle which is easy to check, as compared to the general bifurcation method. Some simulations are provided to indicate

Fig. 3 Periodic oscillation of the solutions, delays: 0.52, 0.55, 0.58, 0.56, 0.50, 0.44, 0.42, 0.43, 0.46, 0.48.

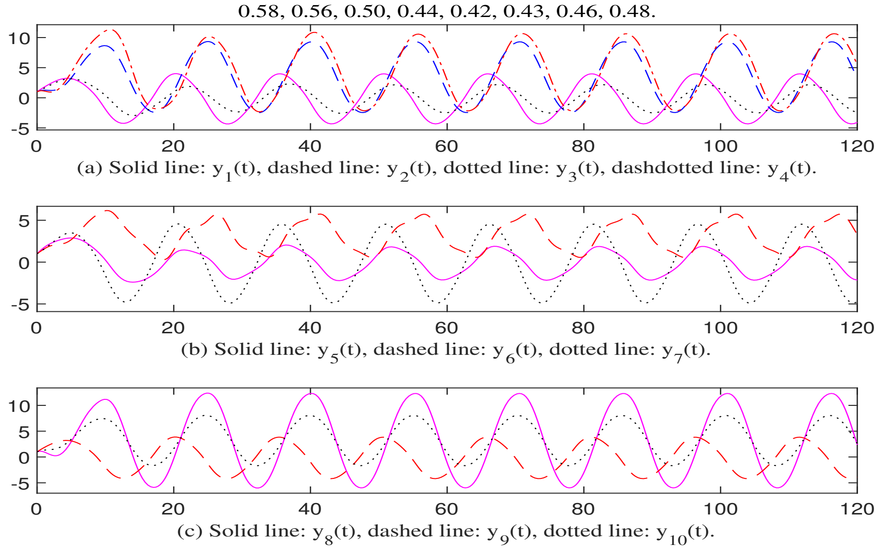
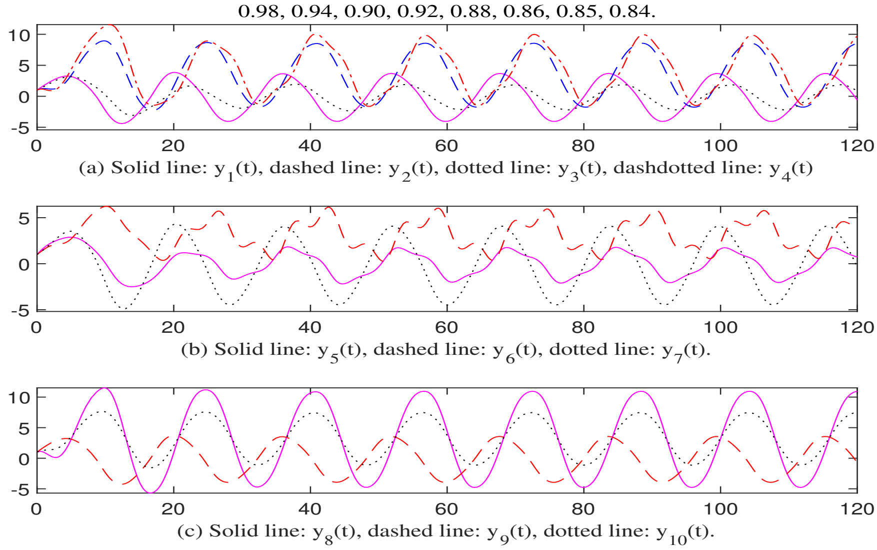
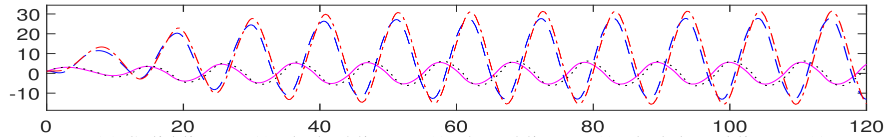


Fig. 4 periodic oscillation of the solutions, delays: 0.96, 0.95, 0.98, 0.94, 0.90, 0.92, 0.88, 0.86, 0.85, 0.84.

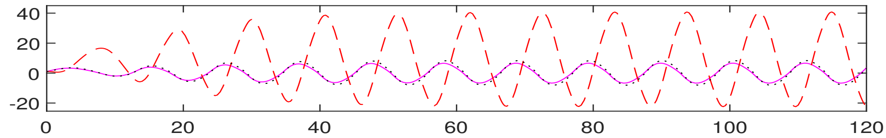


the result of the criterion. Time delays affect the oscillatory frequency and amplitude when there exists a limit cycle of the system. The simulations also

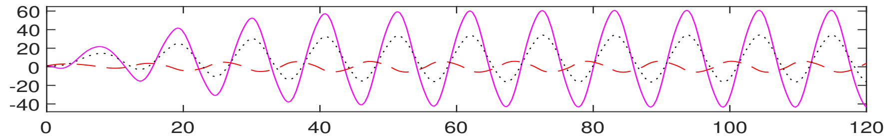
Fig. 5 Periodic oscillation of the solutions, delays: 0.85, 0.82, 0.88, 0.84, 0.80, 0.78, 0.86, 0.76, 0.75, 0.74.



(a) Solid line: $y_1(t)$, dashed line: $y_2(t)$, dotted line: $y_3(t)$, dashdotted line: $y_4(t)$.

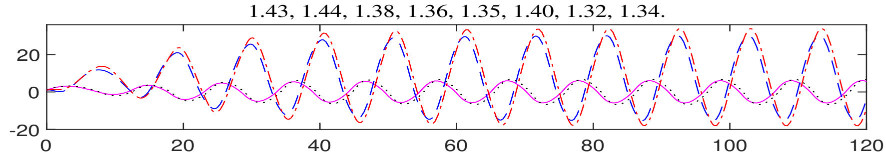


(b) Solid line: $y_5(t)$, dashed line: $y_6(t)$, dotted line: $y_7(t)$.

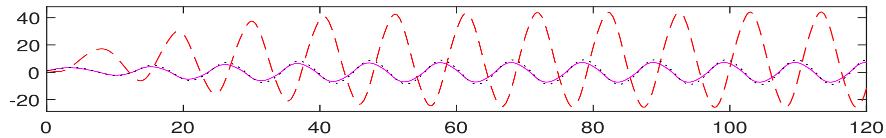


(c) Solid line: $y_8(t)$, dashed line: $y_9(t)$, dotted line: $y_{10}(t)$.

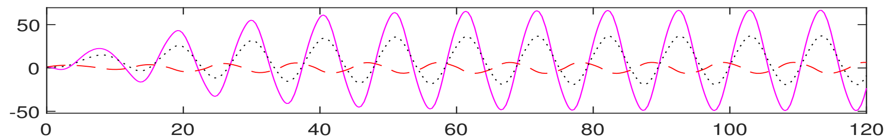
Fig. 6 Periodic oscillation of the solutions, delays: 1.45, 1.42, 1.43, 1.44, 1.38, 1.36, 1.35, 1.40, 1.32, 1.34.



(a) Solid line: $y_1(t)$, dashed line: $y_2(t)$, dotted line: $y_3(t)$, dashdotted line: $y_4(t)$.



(b) Solid line: $y_5(t)$, dashed line: $y_6(t)$, dotted line: $y_7(t)$.



(c) Solid line: $y_8(t)$, dashed line: $y_0(t)$, dotted line: $y_{10}(t)$.

indicate that our theorems are only sufficient conditions. The present method can be used to deal with any n coupled Kaldor-Kalecki economic models. This is our research work for the future.

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