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ON REES FACTOR S-POSETS SATISFYING CONDITIONS $(\mathbf{PWP_E})$ OR $(\mathbf{PWP_E})_\mathbf{w}$

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Abstract. Golchin and Rezaei introduced conditions (PWP) and $(PWP)_w$ in (Subpullbacks and flatness properties of S-posets). In this paper, we introduce conditions (PWP_E) and $(PWP_E)_w$ as generalizations of these conditions, respectively, and show that the relevant implications are strict. In general, we observe that condition $(PWP_E)_w$ follows from condition (PWP_E), but not conversely. Also, we prove that principal weak po-flatness follows from condition $(PWP_E)_w$, but not conversely. Then, we obtain some general properties of conditions (PWP_E) and $(PWP_E)_w$, and find sufficient and necessary conditions for the S-poset $A(I)$ to satisfy these conditions. Finally, we find conditions on a pomonoid S under which a cyclic or Rees factor S-poset satisfies condition (PWP_E) or condition $(PWP_E)_w$. Thereby, we present some homological classifications of pomonoids over which each of the conditions (PWP_E) and $(PWP_E)_w$ implies a specific property, and vice versa, for Rees factor S-posets.

Keywords: pomonoid, S-posets, Conditions (PWP_E) and $(PWP_E)_w$, Rees factor S-posets.
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1. Introduction

By a *pomonoid*, we mean a monoid S on which a partial ordering compatible with the binary operation is defined. Also, we call a non-empty subset I of a pomonoid S a *right ideal* if $IS \subseteq I$. A *right S-poset* is a non-empty poset A, commonly denoted by A_S (or simply A), on which S acts from the right. This means that a mapping $A \times S \to A$, defined by $(a, s) \mapsto as$, exists for which the following conditions are satisfied.

(i) The action is monotonic, with respect to each variable.
(ii) $a1 = a$ and $(as)t = a(st)$ for any $a \in A$ and every $s, t \in S$.

Left S-posets are defined similarly. Also, we denote by $\Theta_S = \{\theta\}$ a one-element right S-poset. Moreover, we refer to order-preserving maps which preserve the action of S as *S-poset morphisms*.

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Given a right S-poset A, let θ be a right S-act congruence for which the S-act A/θ can be considered as an S-poset and the natural mapping $A \to A/\theta$ is an S-morphism. We call θ an *S-poset congruence*. Now, let A be an S-poset and H be an arbitrary subset of $A \times A$. Then, an S-poset congruence $\vartheta(H)$ can be defined on A for which the following statements are true (see [1]).

(1) For $(h, h') \in H$, the relation $[h]_{\vartheta(H)} \leq [h']_{\vartheta(H)}$ holds.
(2) Assume that for an S-poset congruence θ on A, $(h, h') \in H$ implies $[h]_\theta \leq [h']_\theta$. Then, $\vartheta(H) \subseteq \theta$.

A *convex subpomonoid* of a pomonoid S is a subpomonoid K satisfying $K = [K]$, in which $[K] = \{x \in S | \exists k, k' \in K, k \leq x \leq k'\}$. For a convex, proper right ideal K of S, we write S/K to denote $S/\vartheta(K \times K)$.

For a right S-poset A and a left S-poset B, we consider the componentwise order on the Cartesian product $A \times B$ to define the *tensor product* $A \otimes B$. If ρ denotes the order congruence on the right S-poset $A \times B$ generated by $H = \{\big((as, b), (a, sb)\big) | a \in A, b \in B, s \in S\}$, then we let $A \otimes_S B = (A \times B)/\rho$. Moreover, we denote by $a \otimes b$ the equivalence class of (a, b) in $A \otimes_S B$. We can define an order relation on $A \otimes_S B$ by writing $a \otimes b \leq a' \otimes b'$ in $A \otimes_S B$ if and only if $a_1, ..., a_n \in A$, $b_1, ..., b_n \in B$, and $s_1, t_1, ..., s_n, t_n \in S$ can be found such that

$$\begin{aligned} a &\leq a_1 s_1 & & \\ a_1 t_1 &\leq a_2 s_2 & s_1 b &\leq t_1 b_2 \\ a_2 t_2 &\leq a_3 s_3 & s_2 b_2 &\leq t_2 b_3 \\ &\vdots & &\vdots \\ a_n t_n &\leq a' & s_n b_n &\leq t_n b'. \end{aligned}$$

Let S be a monoid. In [7], the pullback diagram of homomorphisms $f : {}_S M \to {}_S Q$ and $g : {}_S N \to {}_S Q$ is denoted by $P(M, N, f, g, Q)$ in the category of left S-acts. If we tensor this diagram by A_S, we obtain a diagram in the category of sets. It may or may not be a pullback diagram, depending on whether or not the mapping φ, obtained from the universal property of pullbacks in the category of sets, is bijective.

It was shown that, by requiring either bijectivity or surjectivity of φ for certain pullback diagrams, we not only recover most of the well-known forms of flatness, but also obtain conditions (WP) and (PWP). Based on this observation, some concepts were introduced in the category of S-posets.

As defined in [5], an S-poset A satisfies *condition* (WP) if for every subpullback diagram $P(I, I, f, f, S)$, the corresponding φ is surjective. Here, I denotes a left ideal of S. Also, we say that A satisfies *condition* $(WP)_w$ if $af(s) \leq a'f(t)$ implies the existence of $a'' \in A$, $u, v \in S$, and $s', t' \in \{s, t\}$ such that $f(us') \leq f(vt')$ and $a \otimes s \leq a'' \otimes us'$, $a'' \otimes vt' \leq a' \otimes t$ in $A \otimes_S (Ss \cup St)$, for every $s, t \in S$, any homomorphism $f : {}_S(Ss \cup St) \to {}_S S$, and every $a, a' \in A$. Furthermore, the S-poset A satisfies *condition* (PWP) if for every subpullback diagram $P(Ss, Ss, f, f, S)$, $s \in S$, the corresponding φ is surjective. Also, when

$as \leq a's$, for $a, a' \in A$ and $s \in S$, implies the existence of $a'' \in A$ and $u, v \in S$ such that $a \leq a''u$, $a''v \leq a'$ and $us \leq vs$, we say that the S-poset A satisfies *condition* $(PWP)_w$.

If $as \leq a's'$, for $a, a' \in A$ and $s, s' \in S$, implies $a = a''u$ and $a' = a''v$, for some $a'' \in A$ and $u, v \in S$ with $us \leq vs'$, then we say that the S-poset A satisfies *condition* (P). If $as \leq at$, for $a \in A$ and $s, t \in S$, implies $a = a'u$ for some $a' \in A$ and $u \in S$ with $us \leq ut$, then we say that the S-poset A satisfies *condition* (E). If A satisfies conditions (P) and (E), we call it *strongly flat.*

We define projectivity using the standard, categorical approach. By *left* PP we mean a pomonoid S whose all principal left ideals are projective. Moreover, condition (P_w) was introduced in [14]. If $as \leq a's'$, for $a, a' \in A$ and $s, s' \in S$, implies $a \leq a''u$ and $a''u' \leq a'$, for some $a'' \in A$ and $u, u' \in S$ with $us \leq u's'$, then we say that the S-poset A satisfies *condition* (P_w).

Recently, numerous studies have been conducted on flatness properties of S-posets, including strong flatness, projectivity and, conditions (P), (P_w) and $(PWP)_w$. Flatness properties of S-posets were first studied in the 1980s, by Fakhruddin (see [2,3]). In [11], flatness properties of the amalgamated coproduct $A(I)$ were discussed.

In Section 2, we introduce conditions (PWP_E) and $(PWP_E)_w$ and obtain some of their general properties. We determine conditions under which the amalgamated coproduct $A(I)$ and cyclic S-posets satisfy conditions (PWP_E) and $(PWP_E)_w$. In Section 3, we consider Rees factor S-posets that satisfy condition (PWP_E) or condition $(PWP_E)_w$. Also, we present some homological classifications of pomonoids over which all Rees factor S-posets satisfying condition (PWP_E) or condition $(PWP_E)_w$ also satisfy some other conditions, including conditions (P) and (PWP), the conditions of being weakly subpullback flat or strongly flat, and vice versa.

An S-poset A is said to be *flat* if for every $a, a' \in A$ and $b, b' \in B$, $a \otimes b = a' \otimes b'$ in $A \otimes_S B$ implies $a \otimes b = a' \otimes b'$ in $A \otimes_S (Sb \cup Sb')$. It is (*principally*) *weakly flat* ((p.) w. flat) if for any (principal) left ideal I of S, and every $s, s' \in I$, $a, a' \in A$, $a \otimes s = a' \otimes s'$ in $A \otimes_S S$ implies $a \otimes s = a' \otimes s'$ in $A \otimes_S I$. Replacing $=$ with $\leq$, the conditions of being po-flat and (principally) weakly po-flat can be defined similarly.

A right *po-cancellable* element of a pomonoid S is an element c for which $sc \leq s'c$, for $s, s' \in S$, implies $s \leq s'$. An S-poset A is said to be *po-torsion free* (po-t.f, for short) if $ac \leq a'c$, for $a, a' \in A$ and a right po-cancellable element c of S, implies $a \leq a'$ (see [1]).

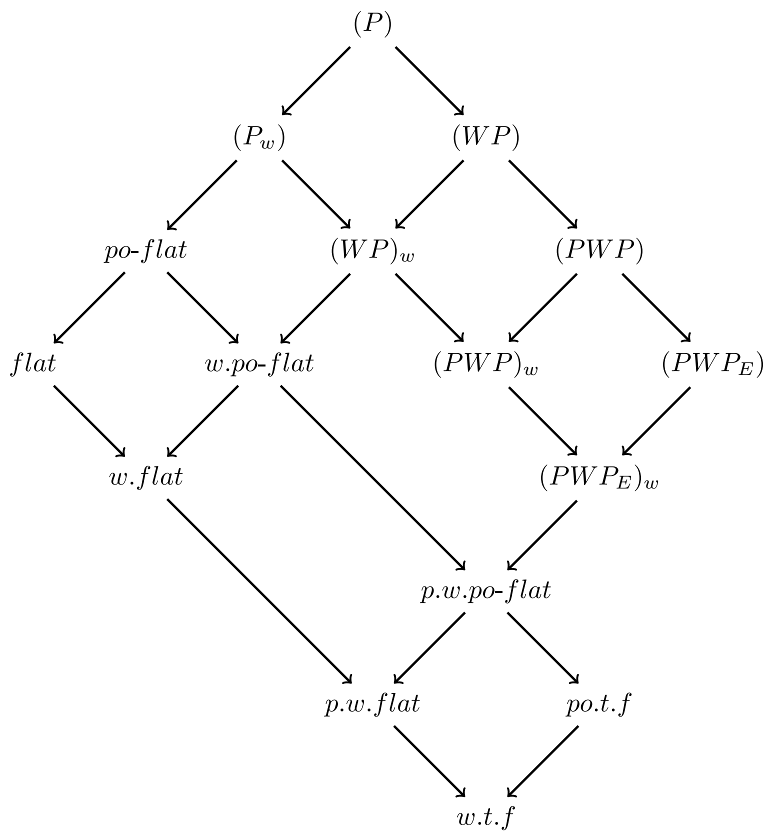
If for every $a, a' \in A$ and any right po-cancellable element c of S, $a = a'$ follows from $ac = a'c$, then we say that the S-poset A is *weakly torsion free* (w.t.f) (see [9]).

The required preliminaries of the theory of S-posets can be found in [1] and the references therein. Throughout this paper, S always will stand for a pomonoid. Also, by an ideal we mean a convex, proper right ideal, unless otherwise stated.

2. S-posets satisfying conditions $(\mathbf{PWP_E})$ and $(\mathbf{PWP_E})_\mathbf{w}$

In this section, we introduce conditions (PWP_E) and $(PWP_E)_w$. Moreover, we show that condition (PWP_E) implies condition $(PWP_E)_w$, and also, condition $(PWP_E)_w$ implies principal weak po-flatness. Then, we find necessary and sufficient conditions for the S-poset $A(I)$ and cyclic S-posets to satisfy conditions (PWP_E) and $(PWP_E)_w$.

The following diagram illustrates the way the conditions are related to the properties already studied.



Definition 2.1. Let A be a right S-poset. Suppose that for every $a, a' \in A$ and $s \in S$, $as \leq a's$ implies the existence of $a'' \in A$, $u, v \in S$ and $e, f \in E(S)$ such that $ae = a''u$, $a''v = a'f$, $es = s = fs$ and $us \leq vs$. Then, we say that A satisfies condition (PWP_E).

Definition 2.2. Let A be a right S-poset. Suppose that for every $a, a' \in A$ and $s \in S$, $as \leq a's$ implies the existence of $a'' \in A$, $u, v \in S$ and $e, f \in E(S)$ such that $ae \leq a''u$, $a''v \leq a'f$, $es = s = fs$ and $us \leq vs$. Then, we say that A satisfies condition $(PWP_E)_w$.

In the final section, we will show that all newly obtained implications are strict.

Remark 2.3. If S is left PP, then every principally weakly po-flat right S-poset A satisfies condition $(PWP_E)_w$, by the duality of [14, Corollary 3.15]. Because, for $a, a' \in A$ and $s \in S$, $as \leq a's$ implies the existence of $e \in E(S)$ such that $es = s$ and $ae \leq a'e$. Therefore, we can take $a'' = a$ and $u = v = f = e$.

Proposition 2.4. *Let A be a right S-poset. Then the following statements are true.*

1. *S_S and Θ_S satisfy condition $(PWP_E)_w\big((PWP_E)\big)$.*
2. *For any family $\{A_i\}_{i\in I}$, of right S-posets, if $A = \prod\limits_{i\in I} A_i$ satisfies condition $(PWP_E)_w$ $\big((PWP_E)\big)$, then A_i satisfies condition $(PWP_E)_w$ $\big((PWP_E)\big)$ for every $i \in I$.*
3. *For any family $\{A_i\}_{i\in I}$, of right S-posets, $A = \coprod\limits_{i\in I} A_i$ satisfies condition $(PWP_E)_w\big((PWP_E)\big)$ if and only if each A_i satisfies condition $(PWP_E)_w\big((PWP_E)\big)$.*
4. *If $\{A_i | i \in I\}$ is a chain of subposets of A, and every A_i satisfies condition $(PWP_E)_w\big((PWP_E)\big)$, then so does $\bigcup\limits_{i\in I} A_i$.*
5. *If A satisfies condition $(PWP_E)_w\big((PWP_E)\big)$, then every retract of A satisfies condition $(PWP_E)_w\big((PWP_E)\big)$.*

Proof. The proofs are straightforward. □

As defined in [8], an S-poset A is said to be *GP-po-flat* if for every $a, a' \in A$ and $s \in S$, $a \otimes s \leq a' \otimes s$ in $A \otimes_S S$, implies the existence of $n \in \mathbb{N}$ such that $a \otimes s^n \leq a' \otimes s^n$ in $A \otimes_S Ss^n$.

Theorem 2.5. *The following statements are true for the right S-poset A.*

1. *Condition (PWP_E) $\Rightarrow$ condition $(PWP_E)_w$ $\Rightarrow$ principally weakly po-flat.*
2. *If S is right po-cancellative, then*

condition $(PWP)_w \Leftrightarrow$ condition $(PWP_E)_w \Leftrightarrow$ principally weakly po-flat $\Leftrightarrow$ GP-po-flat $\Leftrightarrow$ po-torsion free.

Proof. (1). Obviously, condition (PWP_E) implies condition $(PWP_E)_w$. Now, let $as \leq a's$, for $a, a' \in A$ and $s \in S$. Then, by the assumption, there exist $a'' \in A$, $u, v \in S$ and $e, f \in E(S)$ such that $ae \leq a''u$, $a''v \leq a'f$, $es = s = fs$ and $us \leq vs$. Thus,

$$a \otimes s = a \otimes es = ae \otimes s \leq a''u \otimes s = a'' \otimes us \leq a'' \otimes vs = a''v \otimes s \leq a'f \otimes s = a' \otimes fs = a' \otimes s$$

in $A \otimes_S Ss$, which implies that A is principally weakly po-flat.
(2). This is obvious, by [8, Corollary 2.6]. □

Now, we provide an alternative description for condition $(PWP_E)_w$.

Proposition 2.6. *The right S-poset A satisfies condition $(PWP_E)_w$ if and only if $af(s) \leq a'f(s)$, for $a, a' \in A$, $s \in S$ and a homomorphism $f : {}_SS \to {}_SS$, implies the existence of $a'' \in A$, $u, v \in S$ and $e_1, e_2 \in E(S)$ such that $ase_1 \leq a''u$, $a''v \leq a'se_2$, $f(u) \leq f(v)$, and $f(e_1) = f(1) = f(e_2)$.*

Proof. **Necessity**. Let $af(s) \leq a'f(s)$, for a homomorphism $f : {}_SS \to {}_SS$, $a, a' \in A$ and $s \in S$. Then $asf(1) \leq a'sf(1)$ and so, by the assumption, there exist $a'' \in A$, $u, v \in S$ and $e_1, e_2 \in E(S)$ such that $ase_1 \leq a''u$, $a''v \leq a'se_2$, $uf(1) \leq vf(1)$ and $e_1f(1) = f(1) = e_2f(1)$. Thus, $f(u) \leq f(v)$ and $f(e_1) = f(1) = f(e_2)$, as required.
Sufficiency. Suppose that $as \leq a's$, for $a, a' \in A$ and $s \in S$, and let $f = \rho_s : {}_SS \to {}_SS$ be defined by $f(x) = xs$, for $x \in S$. It is obvious that f is a homomorphism satisfying $af(1) \leq a'f(1)$. By the assumption, there exist $a'' \in A$, $u, v \in S$ and $e_1, e_2 \in E(S)$ such that $ae_1 \leq a''u$, $a''v \leq a'e_2$, $f(u) \leq f(v)$, and $f(e_1) = f(1) = f(e_2)$, which imply $us \leq vs$ and $e_1s = s = e_2s$. Therefore, A satisfies condition $(PWP_E)_w$. $\square$

Letting $e_1 = e_2 = 1$ in the above proposition, we obtain the following corollary.

Corollary 2.7. *The right S-poset A satisfies condition $(PWP)_w$ if and only if $af(s) \leq a'f(s)$, for $a, a' \in A$, $s \in S$, and a homomorphism $f : {}_SS \to {}_SS$, implies the existence of $a'' \in A$ and $u, v \in S$ such that $as \leq a''u$, $a''v \leq a's$ and $f(u) \leq f(v)$.*

By an argument similar to the proof of Proposition 2.6, we obtain the following result.

Proposition 2.8. *The right S-poset A satisfies condition (PWP_E) if and only if $af(s) \leq a'f(s)$, for $a, a' \in A$, $s \in S$, and a homomorphism $f : {}_SS \to {}_SS$, implies the existence of $a'' \in A$, $u, v \in S$ and $e_1, e_2 \in E(S)$ such that $ase_1 = a''u$, $a''v = a'se_2$, $f(u) \leq f(v)$, and $f(e_1) = f(1) = f(e_2)$.*

Letting $e_1 = e_2 = 1$ in the above proposition, we obtain the following corollary.

Corollary 2.9. *The right S-poset A satisfies condition (PWP) if and only if $af(s) \leq a'f(s)$, for $a, a' \in A$, $s \in S$, and a homomorphism $f : {}_SS \to {}_SS$, implies the existence of $a'' \in A$ and $u, v \in S$ such that $as = a''u$, $a''v = a's$ and $f(u) \leq f(v)$.*

For an ideal I (not necessarily convex) of S and any $\alpha, \beta, \gamma \notin S$, set $A(I) := (\{\alpha, \beta\} \times (S \setminus I)) \cup (\{\gamma\} \times I)$ and define a right S-action on $A(I)$ by

$$(w, v)t = \begin{cases} (w, vt) & if \ vt \notin I \\ \\ (\gamma, vt) & if \ vt \in I \end{cases},$$

for every $w \in \{\alpha, \beta\}$, $v \in S \setminus I$ and $t \in S$, and

$$(\gamma, u)t = (\gamma, ut),$$

for every $u \in I$ and $t \in S$. The order of $A(I)$ is defined by

$$(w, u) \leq (w', v) \Leftrightarrow (w = w', u \leq v) \text{ or } (w \neq w', u \leq i \leq v, \text{ for some } i \in I).$$

As is proved in [12], $A(I)$ is a right S-poset.

Theorem 2.10. *For an ideal I (not necessarily convex) of S, the right S-poset $A(I)$ satisfies condition $(PWP_E)_w$ if and only if for every $u, v, s \in S$ and $i \in I$,*

$$us \leq i \leq vs \Rightarrow \big(\exists e, f \in E(S)\big)(\exists j \in I)\big((es = s = fs) \wedge \\ \big((us \leq js \wedge j \leq vf) \vee (js \leq vs \wedge ue \leq j)\big)\big).$$

Proof. **Necessity**. Let $us \leq i \leq vs$, for $u, v, s \in S$ and $i \in I$. Then, $(\alpha, 1)us \leq (\beta, 1)vs$. There are four cases that we should consider.

Case 1. $u, v \notin I$. Then $(\alpha, u)s \leq (\beta, v)s$ and so, by the assumption, there exist $(w, p) \in A(I)$, $u', v' \in S$ and $e, f \in E(S)$ such that

(1) $$(\alpha, u)e \leq (w, p)u', (w, p)v' \leq (\beta, v)f, es = s = fs \text{ and } u's \leq v's.$$

Now, three subcases arise.

1.1. $w = \alpha$. If $pv' \notin I$, then $(\alpha, pv') \leq (\beta, v)f$ implies the existence of $j \in I$ such that $pv' \leq j \leq vf$. Since $ue \leq pu'$, $us = ues \leq pu's \leq pv's \leq js$. If $pv' \in I$, then $(\alpha, p)v' = (\gamma, pv')$. Also, $(\beta, v)f = (\beta, vf)$ or $(\beta, v)f = (\gamma, vf)$. If $(\beta, v)f = (\beta, vf)$, then $(\gamma, pv') \leq (\beta, vf)$ implies the existence of $j \in I$ such that $pv' \leq j \leq vf$. Since $ue \leq pu'$, $us = ues \leq pu's \leq pv's \leq js$. If $(\beta, v)f = (\gamma, vf)$, then $(\gamma, pv') \leq (\gamma, vf)$ implies $pv' \leq vf$. We can take $j = pv'$ and so, $j \leq vf$. Since $ue \leq pu'$, $us = ues \leq pu's \leq pv's = js$.

1.2. $w = \beta$. If $pu' \notin I$, then $(\alpha, u)e \leq (\beta, pu')$ implies the existence of $j \in I$ such that $ue \leq j \leq pu'$. Since $pv' \leq vf$, $js \leq pu's \leq pv's \leq vfs = vs$. If $pu' \in I$, then $(\beta, p)u' = (\gamma, pu')$. Also, $(\alpha, u)e = (\alpha, ue)$ or $(\alpha, u)e = (\gamma, ue)$. If $(\alpha, u)e = (\alpha, ue)$, then $(\alpha, ue) \leq (\gamma, pu')$ implies the existence of $j \in I$ such that $ue \leq j \leq pu'$. Since $pv' \leq vf$, $js \leq pu's \leq pv's \leq vfs = vs$. If $(\alpha, u)e = (\gamma, ue)$, then $(\gamma, ue) \leq (\gamma, pu')$ implies $ue \leq pu'$. We can take $j = pu'$ and so, $ue \leq j$. Since $pv' \leq vf$, $js = pu's \leq pv's \leq vfs = vs$.

1.3. $w = \gamma$. If $ue \notin I$, then $(\alpha, ue) \leq (\gamma, pu')$ implies the existence of $j \in I$ such that $ue \leq j \leq pu'$. Since $(\gamma, p)v' \leq (\beta, v)f$, $pv' \leq vf$ or there exists $j' \in I$ such that $pv' \leq j \leq vf$. In any case, we obtain $js \leq pu's \leq pv's \leq vfs = vs$. If $ue \in I$, then by letting $j = ue$, the result follows.

Case 2. $u \notin I$, $v \in I$. This is similar to Case 1.

Case 3. $u \in I$, $v \notin I$. This is similar to Case 1.

Case 4. $u, v \in I$. Then, $(\gamma, u)s \leq (\gamma, v)s$. Since $A(I)$ satisfies condition $(PWP_E)_w$, there exist $(w,p) \in A(I)$, $u', v' \in S$ and $e, f \in E(S)$ such that $(\gamma, u)e \leq (w,p)u'$, $(w,p)v' \leq (\gamma, v)f$, $es = s = fs$ and $u's \leq v's$. Since $us \leq vs$, $ues \leq ves$. Then, we can take $j = ue$ or $j = vf$. If $j = ue$, then $ue \leq j$ and $js = ues \leq ves = vs$. If $j = vf$, then $j \leq vf$ and $us = ues \leq ves = vfs = js$.

Sufficiency. Let $(w_1, z)s \leq (w_2, z')s$, for $(w_1, z), (w_2, z') \in A(I)$ and $s \in S$. There are four cases that we should consider.

Case 1. If $w_1 = w_2 = \alpha$, then $(\alpha, z)s \leq (\alpha, z')s$. Hence $(\alpha, z) \leq (\alpha, 1)z$, $(\alpha, 1)z' \leq (\alpha, z')$ and $zs \leq z's$. Thus, we can take $e = f = 1$, $a'' = (\alpha, 1)$, $u = z$, $v = z'$ and so, $A(I)$ satisfies condition $(PWP_E)_w$.
Case 2. $w_1 = w_2 = \beta$ or $w_1 = w_2 = \gamma$. This is similar to Case 1.
Case 3. $w_1 = \alpha$ and $w_2 = \beta$. If at least one of zs or $z's$, say zs, is in I, then $(\gamma, zs) \leq (w, z's)$ $(w \in \{\beta, \gamma\})$ and so, there exists $i \in I$ such that $zs \leq i \leq z's$. Also, if $zs, z's \notin I$, then $(\alpha, zs) \leq (\beta, z's)$ implies the existence of i in I such that $zs \leq i \leq z's$. In each case, by the assumption, there exist $e, f \in E(S)$ and $j \in I$ such that $es = s = fs$, $zs \leq js$ and $j \leq z'f$, or $js \leq z's$ and $ze \leq j$. If $zs \leq js$ and $j \leq z'f$, then by letting $a'' = (\alpha, 1)$, $u = ze$ and $v = j$ we obtain $(\alpha, z)e = (\alpha, 1)ze = a''u$, $a''v = (\alpha, 1)j = (\gamma, j) \leq (\beta, z')f$, and $us = zes = zs \leq js = vs$. If $js \leq z's$ and $ze \leq j$, then by letting $a'' = (\beta, 1)$, $u = j$ and $v = z'f$ we obtain $(\alpha, z)e = (\alpha, 1)ze \leq (\beta, 1)j = a''u$, $a''v = (\beta, 1)z'f = (\beta, z')f$, and $us = js \leq z's = z'fs = vs$.
Case 4. $(w_1 = \alpha \wedge w_2 = \gamma)$ or $(w_1 = \beta \wedge w_2 = \gamma)$. This is similar to Case 3.

□

Similarly to the argument of Theorem 2.10, we obtain the following result.

Theorem 2.11. *For an ideal I (not necessarily convex) of S, the right S-poset $A(I)$ satisfies condition (PWP_E) if and only if for every $u, v, s \in S$ and $i \in I$,*

$$us \leq i \leq vs \Rightarrow \big(\exists e, f \in E(S)\big)(\exists j \in I)\big((es = s = fs) \wedge \big((us = js \wedge j = vf) \vee (js = vs \wedge ue = j)\big)\big).$$

We conclude this section by considering cyclic S-posets satisfying conditions (PWP_E) or $(PWP_E)_w$.

Theorem 2.12. *For a right order congruence ρ on S, the cyclic right S-poset S/ρ satisfies condition $(PWP_E)_w$ if and only if $[x]_\rho t \leq [y]_\rho t$, for $x, y, t \in S$, implies the existence of $u, v \in S$ and $e, f \in E(S)$ such that $[x]_\rho e \leq [u]_\rho$, $[v]_\rho \leq [y]_\rho f$, $et = t = ft$ and $ut \leq vt$.*

Proof. **Necessity**. Let $[x]_\rho t \leq [y]_\rho t$, for $x, y, t \in S$. By the assumption, there exist $[z]_\rho \in S/\rho$, $u', v' \in S$ and $e, f \in E(S)$ such that $[x]_\rho e \leq [z]_\rho u'$, $[z]_\rho v' \leq [y]_\rho f$, $et = t = ft$ and $u't \leq v't$. If $zu' = u$ and $zv' = v$, then

$[x]_\rho e \leq [u]_\rho$, $[v]_\rho \leq [y]_\rho f$, $et = t = ft$ and $ut = zu't \leq zv't = vt$.

Sufficiency. Let $[x]_\rho t \leq [y]_\rho t$, for $x, y, t \in S$. By the assumption, there exist $u, v \in S$ and $e, f \in E(S)$ such that $[x]_\rho e \leq [u]_\rho$, $[v]_\rho \leq [y]_\rho f$, $et = t = ft$ and $ut \leq vt$. Therefore, $[x]_\rho e \leq [1]_\rho u$, $[1]_\rho v \leq [y]_\rho f$, $et = t = ft$ and $ut \leq vt$. Hence, S/ρ satisfies condition $(PWP_E)_w$. □

Similarly to the argument of Theorem 2.12, we obtain the following theorem.

Theorem 2.13. *For a right order congruence ρ on S, the cyclic right S-poset S/ρ satisfies condition (PWP_E) if and only if $[x]_\rho t \leq [y]_\rho t$, for $x, y, t \in S$, implies the existence of $u, v \in S$ and $e, f \in E(S)$ such that $[x]_\rho e = [u]_\rho$, $[v]_\rho = [y]_\rho f$, $et = t = ft$ and $ut \leq vt$.*

3. Rees factor S-posets satisfying conditions $(\mathbf{PWP_E})$ or $(\mathbf{PWP_E})_\mathbf{w}$

In this section, we find conditions on S under which a Rees factor S-poset satisfies conditions (PWP_E) or $(PWP_E)_w$. Then, we show that condition $(PWP_E)_w \not\Rightarrow$ condition $(PWP)_w \not\Rightarrow$ condition (PWP) and principal weak po-flatness $\not\Rightarrow$ condition $(PWP_E)_w \not\Rightarrow$ condition (PWP_E). Finally, we present some classifications of pomonoids over which each of the conditions (PWP_E) and $(PWP_E)_w$ implies a specific property, and vice versa, for Rees factor S-posets.

As defined in [10, 13], for an ideal K of S, if for every $s \in S$ and $u, v \in S\backslash K$, $[u]_{\rho_K} s \leq [v]_{\rho_K} s$ implies $us \leq vs$, then we say that K is *strongly left annihilating* (briefly, SLA), and if for every $u, v \in S\backslash K$ and a homomorphism $f : {}_S(Su \cup Sv) \to {}_S S$, $[f(u)]_{\rho_K} \leq [f(v)]_{\rho_K}$ implies $f(u) \leq f(v)$, then we call it *double-strongly left annihilating* (briefly, D-SLA).

Note that for any ideal of S, SLA follows from D-SLA, but not conversely (see [10, Example 3.6]).

An ideal K of S is called *w-strongly left annihilating* (briefly, w-SLA), if $[u]_{\rho_K} s \leq [v]_{\rho_K} s$, for $u, v \in S \setminus K$ and $s \in S$, implies the existence of $t, t' \in S$ and $k_1, k_2, l_1, l_2 \in K$ such that one of the following four conditions is satisfied.

(a) $u \leq t, t' \leq v$, and $ts \leq t's$.
(b) $u \leq t, t' \leq l_1, l_2 \leq v$, and $ts \leq t's$.
(c) $u \leq k_1, k_2 \leq t, t' \leq v$, and $ts \leq t's$.
(d) $u \leq k_1, k_2 \leq t, t' \leq l_1, l_2 \leq v$, and $ts \leq t's$.

Also, for any ideal of S, w-SLA follows from SLA, but not conversely (see [10, Example 3.11]).

For an ideal K of S, if for every $k \in K$ and $s \in S$,

$$\big(k \leq s \Rightarrow (\exists l \in K)(ls \leq s)\big), \text{ and } \big(s \leq k \Rightarrow (\exists l \in K)(s \leq ls)\big),$$

then we say that K is *strongly left stabilizing.*

The following definitions are descriptions for Rees factor S-posets satisfying conditions (PWP_E) and $(PWP_E)_w$.

Definition 3.1. We say that an ideal K of S is *E-strongly left annihilating* (briefly, E-SLA), if $[u]_{\rho_K} s \leq [v]_{\rho_K} s$, for $u, v \in S \setminus K$ and $s \in S$, implies the existence of $t, t' \in S$, $e, f \in E(S)$ and $k_1, k_2, l_1, l_2 \in K$ such that one of the following four conditions is satisfied.

(a) $ue \leq t, t' \leq vf$, $es = s = fs$, and $ts \leq t's$.
(b) $ue \leq t, t' \leq l_1, l_2 \leq vf$, $es = s = fs$, and $ts \leq t's$.
(c) $ue \leq k_1, k_2 \leq t, t' \leq vf$, $es = s = fs$, and $ts \leq t's$.
(d) $ue \leq k_1, k_2 \leq t, t' \leq l_1, l_2 \leq vf$, $es = s = fs$, and $ts \leq t's$.

Definition 3.2. We say that an ideal K of S is *SE-strongly left annihilating* (briefly, SE-SLA), if $[u]_{\rho_K} s \leq [v]_{\rho_K} s$, for $u, v \in S \setminus K$ and $s \in S$, implies the existence of $e, f \in E(S)$ and $k, l \in K$ such that one of the following four conditions is satisfied.

(a) $es = s = fs$ and $us \leq vs$.
(b) $vf \in K$, $es = s = fs$ and $us \leq ls$.
(c) $ue \in K$, $es = s = fs$ and $ks \leq vs$.
(d) $ue, vf \in K$, $es = s = fs$ and $ks \leq ls$.

It is clear that for any ideal of S, E-SLA follows from w-SLA, but the converse is not true by Example 3.5. Also, E-SLA follows from SE-SLA, but the converse is not true by Example 3.8. Moreover, E-SLA and SE-SLA follow from SLA, but the converses are not true by Example 3.6.

Now, we provide an example for Definitions 3.1 and 3.2.

Example 3.3. *Consider the semilattice $S = \{1, 0, e, f\}$ with $ef = 0$ and the trivial order relation on S. Then $K = \{0, e, f\}$ is a convex proper right ideal of S. Obviously, K is SE-SLA and so, it is E-SLA.*

Theorem 3.4. *For an ideal K of S, S/K satisfies condition $(PWP_E)_w$ $\big((PWP_E)\big)$ if and only if*

(1) *K is strongly left stabilizing, and*
(2) *K is E-SLA (SE-SLA).*

Proof. **Necessity**. Since by part (1) of Theorem 2.5, S/K is principally weakly po-flat, K is strongly left stabilizing, by [1, Proposition 10]. Now, let $[u]_{\rho_K} s \leq [v]_{\rho_K} s$ for $u, v \in S \setminus K$ and $s \in S$. Then, by Theorem 2.12, there exist $t, t' \in S$ and $e, f \in E(S)$ such that $[ue]_{\rho_K} \leq [t]_{\rho_K}$, $[t']_{\rho_K} \leq [vf]_{\rho_K}$, $es = s = fs$ and $ts \leq t's$. By [1, Lemma 3], $[ue]_{\rho_K} \leq [t]_{\rho_K}$ implies $ue \leq t$ or $ue \leq k_1$ and $k_2 \leq t$, for $k_1, k_2 \in K$. Similarly, $[t']_{\rho_K} \leq [vf]_{\rho_K}$ implies $t' \leq vf$ or $t' \leq l_1$ and $l_2 \leq vf$, for $l_1, l_2 \in K$. Hence, we get the four possible conditions of Definition 3.1, and so K is E-SLA.
Sufficiency. Let $[z]_{\rho_K} s \leq [w]_{\rho_K} s$, for $z, w, s \in S$. We show that there exist $u, v \in S$ and $e, f \in E(S)$ such that $[z]_{\rho_K} e \leq [u]_{\rho_K}$, $[v]_{\rho_K} \leq [w]_{\rho_K} f$, $es = s = fs$ and $us \leq vs$. Since $[zs]_{\rho_K} \leq [ws]_{\rho_K}$, by [1, Lemma 3], $zs \leq ws$, or $zs \leq k_1$

and $k_2 \leq ws$ for $k_1, k_2 \in K$. If $zs \leq ws$, then by letting $e = f = 1$ and $u = z$, $v = w$, the desired result follows. Otherwise, there are the following four cases.

Case 1. $z, w \in K$. We can take $e = f = 1$ and $u = v = z$.

Case 2. $z \in K$, $w \notin K$. Since $k_2 \leq ws$, by part (1) there exists $k_3 \in K$ such that $k_3ws \leq ws$ and so, we can take $e = f = 1$, $u = k_3w$ and $v = w$.

Case 3. $z \notin K$, $w \in K$. This is similar to Case 2.

Case 4. $z, w \notin K$. By part (2), there exist $u, v \in S$, $e, f \in E(S)$ and $k_1, k_2, l_1, l_2 \in K$ such that one of the conditions of Definition 3.1 holds. Hence, in any condition, we get $[ze]_{\rho_K} \leq [u]_{\rho_K}$, $[v]_{\rho_K} \leq [wf]_{\rho_K}$, $es = s = fs$ and $us \leq vs$.

The case condition(PWP_E) could be proved similarly. □

Example 3.5. $\big((PWP_E)_w \nRightarrow (PWP)_w\big)$
Consider the semilattice $S = \{1, 0, e, f\}$ with $ef = 0$ and the trivial order relation on S. Obviously, S is regular and so, by [12, Theorem 2.3], all right S-posets are principally weakly po-flat. Since S is left PP, all right S-posets satisfy condition $(PWP_E)_w$, by Remark 2.3. If $K = eS = \{e, 0\}$, then it is a convex proper right ideal of S and $[1]e \leq [f]e$, but no elements $a'' \in S/K$ and $u, v \in S$ exist such that $[1] \leq a''u$, $a''v \leq [f]$ and $ue \leq ve$. Thus, S/K does not satisfy condition $(PWP)_w$.

From the above example, we deduce that condition $(PWP_E)_w$ does not imply conditions (P), (P_w), (WP), $(WP)_w$ and (PWP).

Example 3.6. $\big((PWP_E) \nRightarrow (PWP)\big)$
Consider the semilattice $S = \{1, 0, e, f\}$ with $ef = 0$ and the trivial order relation on S. Since S is regular, all right Rees factor acts of S satisfy condition (PWP_E), by [4, Theorem 3.1]. Now, let $K = \{e, f\}$. Since $1e \neq fe$ and $1, f \notin 1eS \cup fes = K$, by [7, Corollary 3.8], S/K does not satisfy condition (PWP).

The following examples show that the implications of part (1) of Theorem 2.5 are strict.

Example 3.7. $\big($*principal weak po-flatness* $\nRightarrow (PWP_E)_w\big)$
Let $S = \{0, 1, r, s, t\}$ denote the monoid with the following table.

	0	1	r	s	t
0	0	0	0	0	0
1	0	1	r	s	t
r	0	r	r	0	r
s	0	s	0	s	0
t	0	t	r	0	r

Suppose that the only nontrivial order relations are $t < 0$ and $r < 0$. Let $K = \{r, 0\}$. Then, $(S, \leq)$ is a pomonoid and by [5, Example 6.3], S/K is principally weakly po-flat. Nevertheless, S/K does not satisfy condition $(PWP_E)_w$, since

$[1]t \leq [t]t$, and $e = 1$ is the only idempotent such that $et = t$, but no elements $u, v \in S$ exist such that $[1] \leq [u]$, $[v] \leq [t]$ and $ut \leq vt$.

Since principally weakly flat $\nRightarrow$ principally weakly po-flat, it is obvious that the property of being principally weakly flat does not imply condition $(PWP_E)_w$.

Example 3.8. $\big((PWP_E)_w \nRightarrow (PWP_E)\big)$
Let G be an ordered group. Then, all G-posets satisfy condition (P_w), by [14, Theorem 3.7], and so, all G-posets satisfy condition $(PWP_E)_w$. Now, let $A = \{a, a'\}$ be a two-element chain with $a \leq a'$ and, $as = a$ and $a's = a'$ for every $s \in S$. Then, A is an S-poset which fails to satisfy condition (PWP_E).

Recall from [1] that S is called *weakly right reversible* if for all $s, t \in S$, there exist $u, v \in S$ such that $us \leq vt$.

Theorem 3.9. *The following statements are equivalent.*

(1) *Every Rees factor S-poset satisfying condition $(PWP_E)_w\big((PWP_E)\big)$ is weakly po-flat.*
(2) *Every Rees factor S-poset satisfying condition $(PWP_E)_w\big((PWP_E)\big)$ is weakly flat.*
(3) *S is weakly right reversible.*

Proof. Implication (1) $\Rightarrow$ (2) is obvious.
(2) $\Rightarrow$ (3). This is obvious, by [1, Proposition 14].
(3) $\Rightarrow$ (1). Every Rees factor S-poset satisfying condition $(PWP_E)_w\big((PWP_E)\big)$ is principally weakly po-flat, by part (1) of Theorem 2.5. So, every Rees factor S-poset satisfying condition $(PWP_E)_w\big((PWP_E)\big)$ is weakly po-flat, by [1, Proposition 13]. □

Theorem 3.10. *The following statements are equivalent.*

(1) *Every Rees factor S-poset satisfying condition $(PWP_E)_w$ satisfies condition $(PWP)_w$ $\big((PWP_E)\big)$.*
(2) *Every strongly left stabilizing and E-SLA ideal K of S is w-SLA (SE-SLA).*

Proof. (1) $\Rightarrow$ (2). Let the ideal K of S be strongly left stabilizing and E-SLA. Then, by Theorem 3.4, S/K satisfies condition $(PWP_E)_w$ and so, by the assumption, it satisfies condition $(PWP)_w$. Thus, K is w-SLA by [10, Theorem 3.10].
(2) $\Rightarrow$ (1). For an ideal K (not necessarily proper) of S, suppose that S/K satisfies condition $(PWP_E)_w$. If K is proper, then by Theorem 3.4, K is strongly left stabilizing and E-SLA. So, by the assumption, K is w-SLA. Thus, S/K satisfies condition $(PWP)_w$, by [10, Theorem 3.10]. If $K = S$, then $S/K = \Theta_S$ satisfies conditions $(PWP_E)_w$ and $(PWP)_w$.
One can prove the case condition (PWP_E) similarly. □

By an argument similar to the proof of Theorem 3.10, we obtain the following theorem.

Theorem 3.11. *The following statements are equivalent.*

(1) *Every Rees factor S-poset satisfying condition $(PWP_E)_w$ $\big((PWP_E)\big)$ satisfies condition (PWP).*

(2) *Every strongly left stabilizing and E-SLA (SE-SLA) ideal K of S is SLA.*

As defined in [6], if an ideal K of S satisfies one of the following conditions, then we say that it has *property* (X).

(a) $p < 1$, for some $p \in K$, and $K = (Kk]$ for every $k \in K$.

(b) $p > 1$, for some $p \in K$, and $K = [Kk)$ for every $k \in K$.

Theorem 3.12. *The following statements are equivalent.*

(1) *Every Rees factor S-poset satisfying condition $(PWP_E)_w$ satisfies condition (P_w).*

(2) *S is weakly right reversible, and if S has a strongly left stabilizing and E-SLA ideal K, then $|K| = 1$ or K satisfies property (X).*

Proof. (1) ⇒ (2). Since Θ_S satisfies condition $(PWP_E)_w$, by the assumption, it satisfies condition (P_w) and so, by [1, Theorem 1], S is weakly right reversible. Now, for a strongly left stabilizing and E-SLA ideal K of S, S/K satisfies condition $(PWP_E)_w$, by Theorem 3.4, and so, by the assumption, it satisfies condition (P_w). Thus, by [6, Lemma 2.3], $|K| = 1$ or K satisfies property (X).
(2) ⇒ (1). For an ideal K (not necessarily proper) of S, suppose that S/K satisfies condition $(PWP_E)_w$. If K is proper, then by Theorem 3.4, it is strongly left stabilizing and E-SLA. Then, by the assumption, $|K| = 1$ or K satisfies property (X). So, by [6, Lemma 2.3], S/K satisfies condition (P_w). If $K = S$, then $\Theta_S = S/K$ satisfies condition (P_w), by [1, Theorem 1]. □

Theorem 3.13. *The following statements are equivalent.*

(1) *Every Rees factor S-poset satisfying condition $(PWP_E)_w$ $\big((PWP_E)\big)$ satisfies condition (P).*

(2) *S is weakly right reversible, and if S has a strongly left stabilizing and E-SLA (SE-SLA) ideal K, then $|K| = 1$.*

Proof. (1) ⇒ (2). Since Θ_S satisfies condition $(PWP_E)_w$, by the assumption, it satisfies condition (P) and so, by [1, Theorem 1], S is weakly right reversible. Now, for a strongly left stabilizing and E-SLA ideal K of S, S/K satisfies condition $(PWP_E)_w$, by Theorem 3.4, and so, by the assumption, it satisfies condition (P). Thus, by [13, Lemma 1.8], $|K| = 1$.
(2) ⇒ (1). For an ideal K (not necessarily proper) of S, suppose that S/K satisfies condition $(PWP_E)_w$. If K is proper, then by Theorem 3.4, it is strongly left stabilizing and E-SLA and so, by the assumption, $|K| = 1$. Thus, by [13, Lemma 1.8], S/K satisfies condition (P). If $K = S$, then $\Theta_S = S/K$ satisfies

condition (P), by [1, Theorem 1].
The case condition (PWP_E) could be proved similarly. □

As defined in [10], if for any $t, t' \in S$ there exists $v \in S$ such that $vt = vt'$, then we say that S is *left collapsible*, and if $tw = t'w$, for $t, t', w \in S$, implies the existence of $v \in S$ such that $vt = vt'$, then we say that it is *weakly left collapsible*

If for every $a \in A$, $t, t', w \in S$, $at \leq at'$ and $tw = t'w$ imply the existence of $a' \in A$ and $v \in S$ such that $a = a'v$ and $vt \leq vt'$, then we say that the S-poset A satisfies condition (E'). If the S-poset A satisfies conditions (P) and (E'), then we say that it is *weakly subpullback flat.*

Theorem 3.14. *The following statements are equivalent.*

(1) *Every Rees factor S-poset satisfying condition* $(PWP_E)_w$ $\big((PWP_E)\big)$ *is weakly subpullback flat.*
(2) *S is weakly right reversible and weakly left collapsible, and S has no strongly left stabilizing and E-SLA (SE-SLA) ideal K with* $|K| > 1$.

Proof. (1) ⇒ (2). Since Θ_S satisfies condition $(PWP_E)_w$, by the assumption, it is weakly subpullback flat and so, by [10, Theorem 3.19], S is weakly right reversible and weakly left collapsible. Now, let K be a strongly left stabilizing and E-SLA ideal of S with $|K| > 1$. Then, by Theorem 3.4, S/K satisfies condition $(PWP_E)_w$ and so, by the assumption, it is weakly subpullback flat. Since $K \neq S$, [10, Theorem 3.19] shows that $|K| = 1$, which is a contradiction.
(2) ⇒ (1). For an ideal K (not necessarily proper) of S, suppose that S/K satisfies condition $(PWP_E)_w$. If K is proper, then by Theorem 3.4, it is strongly left stabilizing and E-SLA. So, by the assumption, $|K| = 1$, which implies that S/K is weakly subpullback flat, by [10, Theorem 3.19]. If $K = S$, then by [1, Theorem 1], $\Theta_S = S/K$ satisfies conditions (P) and (E'). So, Θ_S is weakly subpullback flat.
The case condition (PWP_E) could be proved similarly. □

Theorem 3.15. *The following statements are equivalent.*

(1) *Every Rees factor S-poset satisfying condition* $(PWP_E)_w$ $\big((PWP_E)\big)$ *is strongly flat.*
(2) *S is left collapsible and S has no strongly left stabilizing and E-SLA (SE-SLA) ideal K with* $|K| > 1$.

Proof. (1) ⇒ (2). Since Θ_S satisfies condition $(PWP_E)_w$, by the assumption, it is strongly flat and so, by [1, Theorem 1], S is left collapsible. Now, let K be a strongly left stabilizing and E-SLA ideal of S with $|K| > 1$. By Theorem 3.4, S/K satisfies condition $(PWP_E)_w$, and so by the assumption, it is strongly flat. Thus, [13, Lemma 1.8] shows that $|K| = 1$, which is a contradiction.
(2) ⇒ (1). For an ideal K (not necessarily proper) of S, suppose that S/K satisfies condition $(PWP_E)_w$. If K is proper, then by Theorem 3.4, it is strongly left stabilizing and E-SLA. So, by the assumption, $|K| = 1$. Thus, by [13,

Lemma 1.8], S/K is strongly flat. If $K = S$, then $\Theta_S = S/K$ is strongly flat, by the note before [6, Proposition 3.3].
The case condition (PWP_E) could be proved similarly. □

Theorem 3.16. *The following statements are equivalent.*

(1) *Every Rees factor S-poset satisfying condition $(PWP_E)_w$ $\big((PWP_E)\big)$ is projective.*
(2) *S has a left zero element, and S has no strongly left stabilizing and E-SLA (SE-SLA) ideal K with $|K| > 1$.*

Proof. (1) ⇒ (2). Since Θ_S satisfies condition $(PWP_E)_w$, by the assumption, it is projective and so, by [1, Theorem 1], S has a left zero element. Now, let K be a strongly left stabilizing and E-SLA ideal of S with $|K| > 1$. By Theorem 3.4, S/K satisfies condition $(PWP_E)_w$ and so, by the assumption, it is projective. Thus, [13, Lemma 1.8] shows that $|K| = 1$, which is a contradiction.
(2) ⇒ (1). For an ideal K (not necessarily proper) of S, suppose that S/K satisfies condition $(PWP_E)_w$. If K is proper, then by Theorem 3.4, it is strongly left stabilizing and E-SLA. So, by the assumption, $|K| = 1$. Thus, by [13, Lemma 1.8], S/K is projective. If $K = S$, then $\Theta_S = S/K$ is projective, by [1, Theorem 1].
The case condition (PWP_E) could be proved similarly. □

Theorem 3.17. *The following statements are equivalent.*

(1) *Every Rees factor S-poset satisfying condition $(PWP_E)_w\big((PWP_E)\big)$ is free.*
(2) $|S| = 1$.

Proof. (1) ⇒ (2). Since Θ_S satisfies condition $(PWP_E)_w\big((PWP_E)\big)$, by the assumption, it is free and so, $|S| = 1$ by [1, Theorem 1].
(2) ⇒ (1). This is obvious, by [13, Theorem 2.13]. □

Theorem 3.18. *For any ideal K of S, the following statements are equivalent.*

(1) *All principally weakly po-flat right Rees factor S-posets satisfy condition $(PWP_E)_w$ $\big((PWP_E)\big)$.*
(2) *If K is strongly left stabilizing, then it is E-SLA (SE-SLA).*

Proof. (1) ⇒ (2). If K is strongly left stabilizing, then by [1, Proposition 10], S/K is principally weakly po-flat and so, by the assumption, it satisfies condition $(PWP_E)_w$. Thus, by Theorem 3.4, K is E-SLA.
(2) ⇒ (1). Let S/K be principally weakly po-flat. Then, by [1, Proposition 10], K is strongly left stabilizing and so, by the assumption, it is E-SLA. Thus, by Theorem 3.4, S/K satisfies condition $(PWP_E)_w$.
The case condition (PWP_E) could be proved similarly. □

Theorem 3.19. *For any ideal K of S, the following statements are equivalent.*

(1) *All principally weakly flat right Rees factor S-posets satisfy condition $(PWP_E)_w$ $\big((PWP_E)\big)$.*

(2) *If for every $k \in K$ there exist $k', k'' \in K$ such that $k'k \leq k \leq k''k$, then K is strongly left stabilizing and E-SLA (SE-SLA).*

Proof. (1) $\Rightarrow$ (2). If for every $k \in K$ there exist $k', k'' \in K$ such that $k'k \leq k \leq k''k$, then by [1, Proposition 9], S/K is principally weakly flat and so, by the assumption, it satisfies condition $(PWP_E)_w$. Thus, by Theorem 3.4, K is strongly left stabilizing and E-SLA.
(2) $\Rightarrow$ (1). Let S/K be principally weakly flat. Then, by [1, Proposition 9], for every $k \in K$ there exist $k', k'' \in K$ such that $k'k \leq k \leq k''k$. So, by the assumption, K is strongly left stabilizing and E-SLA. Thus, by Theorem 3.4, S/K satisfies condition $(PWP_E)_w$.
The case condition (PWP_E) could be proved similarly. □

Theorem 3.20. *For any ideal K of S, the following statements are equivalent.*

(1) *All weakly flat right Rees factor S-posets satisfy condition $(PWP_E)_w$ $\big((PWP_E)\big)$.*
(2) *If S is weakly right reversible and S/K is principally weakly flat, then K is strongly left stabilizing and E-SLA (SE-SLA).*

Proof. (1) $\Rightarrow$ (2). Let S be weakly right reversible, and suppose that S/K is principally weakly flat. Then, by [1, Proposition 14], S/K is weakly flat and so, by the assumption it satisfies condition $(PWP_E)_w$. Thus, by Theorem 3.4, K is strongly left stabilizing and E-SLA.
(2) $\Rightarrow$ (1). Let S/K be weakly flat. Then, by [1, Proposition 14], S is weakly right reversible and S/K is principally weakly flat. So, by the assumption, K is strongly left stabilizing and E-SLA. Thus, by Theorem 3.4, S/K satisfies condition $(PWP_E)_w$.
The case condition (PWP_E) could be proved similarly. □

By an argument similar to the proof of Theorem 3.20, we obtain the following theorems.

Theorem 3.21. *For any ideal K of S, the following statements are equivalent.*

(1) *All weakly po-flat right Rees factor S-posets satisfy condition $(PWP_E)_w$ $\big((PWP_E)\big)$.*
(2) *If S is weakly right reversible and S/K is principally weakly po-flat, then K is strongly left stabilizing and E-SLA (SE-SLA).*

Theorem 3.22. *For any ideal K of S, the following statements are equivalent.*

(1) *All po-torsion free right Rees factor S-posets satisfy condition $(PWP_E)_w$ $\big((PWP_E)\big)$.*
(2) *If for every $s \in S$ and any right po-cancellable element $c \in S$, $sc \in (K] \Rightarrow s \in (K]$ and $sc \in [K) \Rightarrow s \in [K)$, then K is strongly left stabilizing and E-SLA (SE-SLA).*

Theorem 3.23. *For any ideal K of S, the following statements are equivalent.*

(1) *All weakly torsion free right Rees factor S-posets satisfy condition $(PWP_E)_w$ $\big((PWP_E)\big)$.*
(2) *If for every $s \in S$ and any right po-cancellable element $c \in S$, $sc \in K \Rightarrow s \in K$, then K is strongly left stabilizing and E-SLA (SE-SLA).*

4. Conclusion

Although we feel that the results of this paper are significant progress to the complete understanding of classifications of pomonoids over which all Rees factor S-posets satisfying conditions (PWP_E) or $(PWP_E)_w$ have a certain property, and vice versa, but there obviously remain a number of unsolved problems. We believe it is a worthy goal to obtain result similar to Theorem 3.12, for condition $(WP)_w$, similar to Theorem 3.13, for condition (WP) and similar to Theorem 3.21, for property (po-) flat.

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