Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

MATRIX REPRESENTATION OF BI-PERIODIC PELL SEQUENCE

S. Uygun ✉ and E. Akinci

Article type: Research Article
(Received: 14 July 2022, Received in revised form 29 October 2022)
(Accepted: 03 March 2023, Published Online: 10 March 2023)

Abstract. In this study, a generalization of the Pell sequence called bi-periodic Pell sequence is carried out to matrix theory. Therefore, we call this matrix sequence the bi-periodic Pell matrix sequence whose entries are bi-periodic Pell numbers. Then the generating function, Binet formula and some basic properties and sum formulas are examined.

Keywords: Pell sequence, Generating function, Binet formula, Generalized sequences
2020 MSC: Primary 11B39, 11B83, 15A24, 15B36.

1. Introduction

In the literature, especially in mathematics and physics, there are a lot of integer sequences, which are used in almost every kind modern science. Fundamental importance of special integer sequences is encountered in the fields of combinatorics and number theory. The use of such special sequences has increased significantly in quantum mechanics, quantum physics, etc. One of the popular integer sequences is the Pell sequence. The authors investigated the Catalan transform of the $k-$Pell, $k-$Pell-Lucas and modified $k-$Pell sequence in [5]. The authors found new recurrences on Pell numbers, Pell-Lucas numbers, Jacobsthal numbers, and Jacobsthal-Lucas numbers in [8]. The Pell sequence denoted by $\{p_n\}_{n=0}^{\infty}$ is defined recursively by $p_n = 2p_{n-1} + p_{n-2}$ with initial conditions $p_0 = 0,\ p_1 = 1$ for $n \geq 2$ in [2, 14, 15]. The first some elements of the sequence are $1, 2, 5, 12, 29, 70, 169, 408, 985, 2378$. The Pell numbers were named after the English mathematician John Pell.

Matrix sequences obtained by the elements of special integer sequences are also come attraction to the researchers. As an example of the usage of the matrix approach, we can exemplify to obtain the Simpson formula for the special integer sequences, namely, which may, of course, be established by means of the Binet form. There are many generalizations of the special sequences. For example, in [6], the authors studied some properties of $k-$generalized Fibonacci numbers. Bi-periodic number and matrix sequences can be given examples for generalized sequences.

✉ suygun@gantep.edu.tr, ORCID: 0000-0002-7878-2175
DOI: 10.22103/jmmr.2023.19887.1299
Publisher: Shahid Bahonar University of Kerman
How to cite: S. Uygun, E. Akinci, *Matrix Representation of Bi-Periodic Pell Sequence*, J. Mahani Math. Res. 2023; 12(2): 565-583.

Edson and Yayenie defined firstly a new generalization of the Fibonacci sequence called the bi-periodic Fibonacci sequence in [4,7]. Then, Bilgici defined bi-periodic Lucas sequence in [3]. Similarly, Uygun and Karatas found some properties of the bi-periodic Pell and Pell Lucas sequences [10,11]. Uygun and Owusu defined bi-periodic Jacobsthal and Jacobsthal Lucas sequences in [9,13].

In [1], Coskun and Taskara carried out bi-periodic sequences to matrix theory and defined the bi-periodic Fibonacci matrix sequence for any two non-zero real numbers a and b

$$F_n(a,b) = \begin{cases} aF_{n-1}(a,b) + F_{n-2}(a,b), & \text{if } n \text{ is even} \\ bF_{n-1}(a,b) + F_{n-2}(a,b), & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2,$$

with the initial conditions given as

$$F_0(a,b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ F_1(a,b) = \begin{pmatrix} b & \frac{b}{a} \\ 1 & 0 \end{pmatrix}.$$

Uygun and Owusu, in [12] defined the matrix representation of the bi-periodic Jacobsthal sequence as $J_n(a,b)$ recursively by for any two non-zero real numbers a and b

$$J_n(a,b) = \begin{cases} aJ_{n-1}(a,b) + 2J_{n-2}(a,b), & \text{if } n \text{ is even} \\ bJ_{n-1}(a,b) + 2J_{n-2}(a,b), & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2,$$

with the initial conditions given as

$$J_0(a,b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ J_1(a,b) = \begin{pmatrix} b & 2\frac{b}{a} \\ 1 & 0 \end{pmatrix}.$$

Definition 1.1. For any two non-zero real numbers a and b, the bi-periodic Pell sequence in [11] denoted by $\{P_n\}_{n=0}^{\infty}$ is defined recursively by

$$(1) \quad P_0 = 0, \ P_1 = 1, \ P_n = \begin{cases} 2aP_{n-1} + P_{n-2}, & \text{if } n \text{ is even} \\ 2bP_{n-1} + P_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2.$$

The recurrence equation of the bi-periodic Pell sequence is given as

$$x^2 - 2abx - ab = 0$$

and the roots of this equation are

$$(2) \qquad \alpha = ab + \sqrt{a^2b^2 + ab}, \quad \beta = ab - \sqrt{a^2b^2 + ab}.$$

α and β are defined by (2) satisfied the following properties

$$\begin{aligned} (2\alpha+1)(2\beta+1) &= 1, \\ \alpha+\beta &= 2ab, \qquad \alpha\beta = -ab \\ (2\alpha+1) &= \frac{\alpha^2}{ab}, \qquad (2\beta+1) = \frac{\beta^2}{ab}, \\ -(2\alpha+1)\beta &= \alpha, \qquad -(2\beta+1)\alpha = \beta. \end{aligned}$$

In this paper, we introduce the matrix representation of a new two-parameters generalization of Pell numbers. We shall call the bi-periodic Pell matrix sequence. We then proceed to obtain the nth general term of this new matrix sequence. By studying the distinct properties of this new matrix sequence, the well-known Cassini formula is obtained. The generating function together with the Binet formula are given. Some new properties, as well as some summation formulas for this new generalized matrix sequence are also investigated.

2. Bi-periodic Pell Matrix Sequence

Definition 2.1. For any two non-zero real numbers a and b, bi-periodic Pell matrix sequence denoted by $\left\{\tilde{P}_n(a,b)\right\}_{n=0}^{\infty}$ is defined recursively by

$$\tilde{P}_n(a,b) = \begin{cases} 2a\tilde{P}_{n-1}(a,b) + \tilde{P}_{n-2} & \text{if } n \text{ is even} \\ 2b\tilde{P}_{n-1}(a,b) + \tilde{P}_{n-2} & \text{if } n \text{ is odd} \end{cases} \tag{3}$$

with starting values

$$\tilde{P}_0(a,b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \tilde{P}_1(a,b) = \begin{pmatrix} 2b & \frac{b}{a} \\ 1 & 0 \end{pmatrix}.$$

The bi-periodic Pell matrix sequence $\left\{\tilde{P}_n\right\}_{n=0}^{\infty}$ satisfies the following properties

$$\begin{aligned} \tilde{P}_{2n} &= (4ab+2)\tilde{P}_{2n-2} - \tilde{P}_{2n-4}, \\ \tilde{P}_{2n+1} &= (4ab+2)\tilde{P}_{2n-1} - \tilde{P}_{2n-3}. \end{aligned} \tag{4}$$

Theorem 2.2. *The entries of the nth element of the bi-periodic Pell matrix sequence are the elements of the bi-periodic Pell number sequence as*

$$\tilde{P}_n(a,b) = \begin{pmatrix} \left(\frac{b}{a}\right)^{\xi(n)} P_{n+1} & \frac{b}{a} P_n \\ P_n & \left(\frac{b}{a}\right)^{\xi(n)} P_{n-1} \end{pmatrix}, \tag{5}$$

where $\lfloor n \rfloor$ *is the floor function of* n, $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ *is the parity function.*

Proof. We obtain the proof by means of mathematical induction. For $n = 0, 1$ the assertion is satisfied. Let the equation is true for $n = k$, where $k \in Z^+$

$$\tilde{P}_k(a,b) = \begin{pmatrix} \left(\frac{b}{a}\right)^{\xi(k)} P_{k+1} & \frac{b}{a} P_k \\ P_k & \left(\frac{b}{a}\right)^{\xi(k)} P_{k-1} \end{pmatrix}.$$

For $n = k + 1$, we have

$$\begin{aligned}
\tilde{P}_{k+1} &= \begin{cases} 2b\tilde{P}_k(a,b) + \tilde{P}_{k-1} & \text{if } k \text{ is even} \\ 2a\tilde{P}_k(a,b) + \tilde{P}_{k-1} & \text{if } k \text{ is odd} \end{cases} \\
&= (2a)^{\xi(k)}(2b)^{1-\xi(k)}\tilde{P}_k + \tilde{P}_{k-1} \\
&= 2a^{\xi(k)}b^{1-\xi(k)} \begin{pmatrix} \left(\frac{b}{a}\right)^{\xi(k)} P_{k+1} & \frac{b}{a}P_k \\ P_k & \left(\frac{b}{a}\right)^{\xi(k)} P_{k-1} \end{pmatrix} \\
&\quad + \begin{pmatrix} \left(\frac{b}{a}\right)^{\xi(k-1)} P_k & \frac{b}{a}P_{k-1} \\ P_{k-1} & \left(\frac{b}{a}\right)^{\xi(k-1)} P_{k-2} \end{pmatrix} \\
&= \begin{cases} \begin{pmatrix} \frac{b}{a}P_{k+2} & \frac{b}{a}P_{k+1} \\ P_{k+1} & \frac{b}{a}P_{k-1} \end{pmatrix}, & k \text{ is even} \\ \begin{pmatrix} P_{k+2} & \frac{b}{a}P_{k+1} \\ P_{k+1} & P_{k-1} \end{pmatrix}, & k \text{ is odd} \end{cases}
\end{aligned}$$

Thus, we get the result. □

Theorem 2.3. *The determinant of the elements of the bi-periodic Pell matrix sequence is*

$$\det \tilde{P}_n = (-1)^n \left(\frac{b}{a}\right)^{\xi(n)}.$$

(Cassini Property) *By the determinant of the bi-periodic Pell matrix sequence, we get the Cassini property for the bi-periodic Pell number sequence as*

$$\left(\frac{b}{a}\right)^{2\xi(n)} P_{n+1}P_{n-1} - \frac{b}{a}P_n^2 = \left(-\frac{b}{a}\right)^{\xi(n)}$$

or

$$a^{1-\xi(n)}b^{\xi(n)}P_{n+1}P_{n-1} - a^{\xi(n)}b^{1-\xi(n)}P_n^2 = a(-1)^n.$$

Theorem 2.4. *The following properties are satisfied by the bi-periodic Pell matrix sequence*

$$\begin{aligned}
\left(\frac{b}{a}\right)^{\xi(n)} \tilde{P}_{n+1} &= \tilde{P}_n\tilde{P}_1, && (6) \\
\tilde{P}_1^n &= \left(\frac{b}{a}\right)^{\lfloor \frac{n}{2} \rfloor} \tilde{P}_n.
\end{aligned}$$

Proof. First, we choose n as an even integer:

$$\tilde{P}_{n+1} = \begin{pmatrix} \frac{b}{a}P_{n+2} & \frac{b}{a}P_{n+1} \\ P_{n+1} & \frac{b}{a}P_n \end{pmatrix} = \begin{pmatrix} P_{n+1} & \frac{b}{a}P_n \\ P_n & P_{n-1} \end{pmatrix} \begin{pmatrix} 2b & \frac{b}{a} \\ 1 & 0 \end{pmatrix} = \tilde{P}_n\tilde{P}_1.$$

Then, we choose n as an odd integer:

$$\frac{b}{a}\tilde{P}_{n+1} = \frac{b}{a} \begin{pmatrix} P_{n+2} & \frac{b}{a}P_{n+1} \\ P_{n+1} & P_n \end{pmatrix} = \begin{pmatrix} \frac{b}{a}P_{n+1} & \frac{b}{a}P_n \\ P_n & \frac{b}{a}P_{n-1} \end{pmatrix} \begin{pmatrix} 2b & \frac{b}{a} \\ 1 & 0 \end{pmatrix} = \tilde{P}_n\tilde{P}_1.$$

For the second part of the proof, if n is an even integer, we get

$$\left(\frac{b}{a}\right)^{\frac{n}{2}} \tilde{P}_n = \left(\frac{b}{a}\right)^{\frac{n}{2}-1} \tilde{P}_{n-1}\tilde{P}_1 = \left(\frac{b}{a}\right)^{\frac{n}{2}-1} \tilde{P}_{n-2}\tilde{P}_1^2 = \ldots = \tilde{P}_1^n.$$

if n is an odd integer, we have

$$\left(\frac{b}{a}\right)^{\frac{n}{2}} \tilde{P}_n = \left(\frac{b}{a}\right)^{\frac{n}{2}} \tilde{P}_{n-1}\tilde{P}_1 = \left(\frac{b}{a}\right)^{\frac{n}{2}-1} \tilde{P}_{n-2}\tilde{P}_1^2 = \ldots = \tilde{P}_1^n.$$

The results display us the accuracy of the theorem. □

Theorem 2.5. *The following properties are also satisfied by the bi-periodic Pell matrix sequence*

i. $\tilde{P}_m\tilde{P}_n = \left(\frac{b}{a}\right)^{\xi(mn)} \tilde{P}_{m+n}$;
ii. $\tilde{P}_n^m = \left(\frac{b}{a}\right)^{\lfloor \frac{m}{2} \rfloor \xi(n)} \tilde{P}_{mn}$;
iii. $\tilde{P}_{n+1}^m = \left(\frac{a}{b}\right)^{\lfloor \frac{m+1}{2} \rfloor \xi(n)} \tilde{P}_1^m\tilde{P}_{mn}$;
iv. $\tilde{P}_{n-r}\tilde{P}_{n+r} = \left(\frac{b}{a}\right)^{\xi(n-r)} \tilde{P}_2^n = \left(\frac{b}{a}\right)^{(-1)^n\xi(r)} \tilde{P}_n^2$.

Proof. For the proof i., by Theorem 2.4, we get

$$\tilde{P}_m\tilde{P}_n = \left(\frac{a}{b}\right)^{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor} \tilde{P}_1^{m+n} = \left(\frac{a}{b}\right)^{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor - \lfloor \frac{m+n}{2} \rfloor} \tilde{P}_{m+n}.$$

By the property $\lfloor \frac{n}{2} \rfloor = \frac{n-\xi(n)}{2}$, we have

$$\begin{aligned} \left(\frac{a}{b}\right)^{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor - \lfloor \frac{m+n}{2} \rfloor} \tilde{P}_{m+n} &= \left(\frac{a}{b}\right)^{\frac{\xi(m+n)-\xi(m)-\xi(n)}{2}} \tilde{P}_{m+n} \\ &= \left(\frac{a}{b}\right)^{-\xi(mn)} \tilde{P}_{m+n} = \left(\frac{b}{a}\right)^{\xi(mn)} \tilde{P}_{m+n}. \end{aligned}$$

For the proof ii., by Theorem 2.4, we obtain

$$\tilde{P}_n^m = \left(\frac{a}{b}\right)^{m\lfloor \frac{n}{2} \rfloor} \tilde{P}_1^{mn} = \left(\frac{a}{b}\right)^{m\lfloor \frac{n}{2} \rfloor} \left(\frac{b}{a}\right)^{\lfloor \frac{mn}{2} \rfloor} \tilde{P}_{mn} = \left(\frac{b}{a}\right)^{\lfloor \frac{m}{2} \rfloor \xi(n)} \tilde{P}_{mn}.$$

For the proof iii., we get

$$\begin{aligned} \tilde{P}_{n+1}^m &= \left(\frac{a}{b}\right)^{m\lfloor \frac{n+1}{2} \rfloor} \tilde{P}_1^{m(n+1)} = \left(\frac{a}{b}\right)^{m\lfloor \frac{n+1}{2} \rfloor} \tilde{P}_1^{mn}\tilde{P}_1^m \\ &= \left(\frac{a}{b}\right)^{m\lfloor \frac{n+1}{2} \rfloor} \left(\frac{b}{a}\right)^{\lfloor \frac{mn}{2} \rfloor} \tilde{P}_{mn}\tilde{P}_1^m = \left(\frac{a}{b}\right)^{\lfloor \frac{m+1}{2} \rfloor \xi(n)} \tilde{P}_1^m\tilde{P}_{mn}. \end{aligned}$$

For the proof iv., we have

$$\begin{aligned}
\tilde{P}_{n-r}\tilde{P}_{n+r} &= \left(\frac{b}{a}\right)^{\xi((n-r)(n+r))}\tilde{P}_{2n} = \left(\frac{b}{a}\right)^{\xi((n+r))}\tilde{P}_{2n} \\
&= \left(\frac{b}{a}\right)^{\xi(n+r)}\left(\frac{a}{b}\right)^{\xi(n)}\tilde{P}_{n}^{2} = \left(\frac{b}{a}\right)^{(-1)^n\xi(r)}\tilde{P}_{n}^{2} \\
&= \left(\frac{b}{a}\right)^{\xi(n+r)}\tilde{P}_{2n} = \left(\frac{b}{a}\right)^{\xi((n+r))}\left(\frac{a}{b}\right)^{\lfloor\frac{n}{2}\rfloor\xi(2)}\tilde{P}_{2}^{n}.
\end{aligned}$$

□

Theorem 2.6. ***(Generating function)*** *Let us suppose that $\tilde{P}_i$ are coefficients of a power series with center at the origin and $\tilde{P}(x)$ is the sum of this series. Such an analytic function $\tilde{P}(x)$ is generating function for this sequence. The generating function $\tilde{P}(x)$ for the bi-periodic Pell matrix sequence is in the following*

$$\tilde{P}(x) = \sum_{i=0}^{\infty}\tilde{P}_i x^i = \frac{\tilde{P}_0 + \tilde{P}_1 x + x^2\left(2a\tilde{P}_1 - (4ab+1)\tilde{P}_0\right) + x^3\left(2b\tilde{P}_0 - \tilde{P}_1\right)}{1-(4ab+2)x^2+x^4}.$$

Proof. The generating function $\tilde{P}(x)$ for the bi-periodic Pell matrix sequence is displayed as

$$\begin{aligned}
\tilde{P}(x) &= \sum_{i=0}^{\infty}\tilde{P}_i x^i = \sum_{i=0}^{\infty}\tilde{P}_{2i}x^{2i} + \sum_{i=0}^{\infty}\tilde{P}_{2i+1}x^{2i+1} \\
&= \tilde{P}_0(x) + \tilde{P}_1(x).
\end{aligned}$$

For the even terms of the series

$$\tilde{P}_0(x) = \tilde{P}_0 + \tilde{P}_2 x^2 + \sum_{i=2}^{\infty}\tilde{P}_{2i}x^{2i}.$$

We multiply $\tilde{P}_0(x)$ by $(4ab+2)x^2$ and x^4, respectively,

$$(4ab+2)x^2\tilde{P}_0(x) = -(4ab+2)\tilde{P}_0 x^2 - (4ab+2)\sum_{i=2}^{\infty}\tilde{P}_{2i-2}x^{2i}.$$

$$x^4\tilde{P}_0(x) = \sum_{i=0}^{\infty}\tilde{P}_{2i}x^{2i+4} = \sum_{i=2}^{\infty}\tilde{P}_{2i-4}x^{2i}.$$

By these equalities, we obtain the following function of the matrix for even powers of the series

$$\begin{aligned}
[1-(4ab+2)x^2+x^4]\tilde{P}_0(x) &= \tilde{P}_0 + \tilde{P}_2 x^2 - (4ab+2)x^2\tilde{P}_0 \\
&\quad + \sum_{i=2}^{\infty}(\tilde{P}_{2i} - (4ab+2)\tilde{P}_{2i-2} + \tilde{P}_{2i-4})x^{2i}
\end{aligned}$$

$$\tilde{P}_0(x)=\frac{\tilde{P}_0+x^2\left(2a\tilde{P}_1-(4ab+1)\tilde{P}_0\right)}{1-(4ab+2)x^2+x^4}.$$

Similarly, for odd powers of the series, we get

$$\begin{aligned}\tilde{P}_1(x) &= \tilde{P}_1x+\tilde{P}_3x^3+\sum_{i=2}^{\infty}\tilde{P}_{2i+1}x^{2i+1},\\ -(4ab+2)x^2\tilde{P}_1(x) &= -(4ab+2)x^3\tilde{P}_1-(4ab+2)\sum_{i=2}^{\infty}\tilde{P}_{2i-1}x^{2i+1},\\ x^4\tilde{P}_1(x) &= \sum_{i=0}^{\infty}\tilde{P}_{2i+1}x^{2i+5}=\sum_{i=2}^{\infty}\tilde{P}_{2i-3}x^{2i+1}.\end{aligned}$$

By these equalities, we obtain

$$\begin{aligned}[1-(4ab+2)x^2+x^4]\tilde{P}_1(x) &= \tilde{P}_1x+\tilde{P}_3x^3-(4ab+2)x^3\tilde{P}_1\\ &\quad+\sum_{i=2}^{\infty}(\tilde{P}_{2i+1}-(4ab+2)\tilde{P}_{2i-1}+\tilde{P}_{2i-3})x^{2i+1}\end{aligned}$$

$$\tilde{P}_1(x)=\frac{\tilde{P}_1x+x^3\left(2b\tilde{P}_0-\tilde{P}_1\right)}{1-(4ab+2)x^2+x^4}.$$

By combining the results, we complete the proof:

$$\begin{aligned}\tilde{P}(x) &= \tilde{P}_0(x)+\tilde{P}_1(x)\\ &= \frac{\tilde{P}_0+\tilde{P}_2x^2-(4ab+2)\tilde{P}_0x^2+\tilde{P}_1x+\tilde{P}_3x^3-(4ab+2)\tilde{P}_1x^3}{1-(4ab+2)x^2+x^4}.\end{aligned}$$

Simplifying this equation using (3) is demonstrated by

$$\tilde{P}(x)=\frac{\tilde{P}_0+\tilde{P}_1x+\left[2a\tilde{P}_1-(4ab+1)\tilde{P}_0\right]x^2+\left[2b\tilde{P}_0-\tilde{P}_1\right]x^3}{1-(4ab+2)x^2+x^4}.$$

□

Theorem 2.7. ***(Binet Formula)*** *Binet formula gives us opportunity to obtain any element of the sequence easily. The Binet formula for the bi-periodic Pell matrix sequence is*

$$\begin{aligned}\tilde{P}_n &= \left(\tilde{P}_0\right)^{1-\xi(n)}\left(\tilde{P}_1\right)^{\xi(n)}\frac{\alpha^{2\lfloor\frac{n}{2}\rfloor+1}-\beta^{2\lfloor\frac{n}{2}\rfloor+1}}{(ab)^{\lfloor\frac{n}{2}\rfloor}(\alpha-\beta)}\\ &\quad+(b\tilde{P}_0)^{\xi(n)}(a\tilde{P}_1-2ab\tilde{P}_0)^{1-\xi(n)}\frac{\alpha^{2\lfloor\frac{n}{2}\rfloor}-\beta^{2\lfloor\frac{n}{2}\rfloor}}{(ab)^{\lfloor\frac{n}{2}\rfloor}(\alpha-\beta)}.\end{aligned}$$

Proof. From the generating function for the bi-periodic Pell matrix sequence, we have

$$\begin{aligned}\tilde{P}(x) &= \frac{\tilde{P}_0+\tilde{P}_1x+x^2\left(2a\tilde{P}_1-(4ab+1)\tilde{P}_0\right)+x^3\left(2b\tilde{P}_0-\tilde{P}_1\right)}{\left(x^2-(2\alpha+1)\right)\left(x^2-(2\beta+1)\right)} \\ &= \frac{Ax+B}{x^2-(2\alpha+1)}+\frac{Cx+D}{x^2-(2\beta+1)}.\end{aligned}$$

By the above equality, we find the coefficients as

$$\begin{aligned} A+C &= 2b\tilde{P}_0-\tilde{P}_1 \\ -A(2\beta+1)-C(2\alpha+1) &= \tilde{P}_1 \end{aligned}$$

$$\begin{aligned} A &= \frac{(2\alpha+1)\,b\tilde{P}_0-\alpha\tilde{P}_1}{\alpha-\beta} \\ C &= \frac{\beta\tilde{P}_1-(2\beta+1)\,b\tilde{P}_0}{\alpha-\beta} \end{aligned}$$

$$\begin{aligned} B+D &= 2a\tilde{P}_1-(4ab+1)\tilde{P}_0 \\ -B(2\beta+1)-D(2\alpha+1) &= \tilde{P}_0 \end{aligned}$$

$$\begin{aligned} B &= \frac{a(2\alpha+1)(\tilde{P}_1-2b\tilde{P}_0)-\alpha\tilde{P}_0}{\alpha-\beta} \\ D &= \frac{a(2\beta+1)(-\tilde{P}_1+2b\tilde{P}_0)+\beta\tilde{P}_0}{\alpha-\beta}. \end{aligned}$$

We know that by Maclauren series expansion

$$\frac{Ax+B}{x^2-C}=-\sum_{i=1}^{\infty}AC^{-n-1}x^{2n+1}-\sum_{i=1}^{\infty}BC^{-n-1}x^{2n}.$$

If we apply this expansion to the fraction $\frac{Ax+B}{x^2-(2\alpha+1)}$, we obtain

$$\frac{Ax+B}{x^2-(2\alpha+1)}= \frac{-1}{\alpha-\beta}\left(\sum_{i=1}^{\infty}\frac{(2\alpha+1)\,b\tilde{P}_0-\alpha\tilde{P}_1}{(2\alpha+1)^{n+1}}x^{2n+1}+\sum_{i=1}^{\infty}\frac{a(2\alpha+1)(\tilde{P}_1-2b\tilde{P}_0)-\alpha\tilde{P}_0}{(2\alpha+1)^{n+1}}x^{2n}\right),$$

and similarly if we apply this expansion to the fraction $\frac{Cx+D}{x^2-(2\beta+1)}$, we obtain

$$\frac{Cx+D}{x^2-(2\beta+1)}= \frac{-1}{\alpha-\beta}\left(\sum_{i=1}^{\infty}\frac{\beta\tilde{P}_1-(2\beta+1)\,b\tilde{P}_0}{(2\beta+1)^{n+1}}x^{2n+1}+\sum_{i=1}^{\infty}\frac{a(2\beta+1)(-\tilde{P}_1+2b\tilde{P}_0)+\beta\tilde{P}_0}{(2\beta+1)^{n+1}}x^{2n}\right).$$

Firstly, we examine the series with even powers

$$\begin{aligned}
E(x) &= -\frac{1}{\alpha-\beta}\sum_{i=1}^{\infty}\left[\begin{array}{c}(2\beta+1)^{n+1}\left\{a\,(2\alpha+1)\,(\tilde{P}_1-2b\tilde{P}_0)-\alpha\tilde{P}_0\right\}\\ +(2\alpha+1)^{n+1}\left(a\,(2\beta+1)\,(-\tilde{P}_1+2b\tilde{P}_0)+\beta\tilde{P}_0\right)\end{array}\right]x^{2n}\\
&= -\frac{1}{\alpha-\beta}\sum_{i=1}^{\infty}\left[\begin{array}{c}(2\beta+1)^{n}\,a(\tilde{P}_1-2b\tilde{P}_0)+(2\beta+1)^{n}\,\beta\tilde{P}_0\\ +(2\alpha+1)^{n}\,a(-\tilde{P}_1+2b\tilde{P}_0)-(2\alpha+1)^{n}\,\alpha\tilde{P}_0\end{array}\right]x^{2n}\\
&= -\frac{1}{\alpha-\beta}\sum_{i=1}^{\infty}\left[\begin{array}{c}\frac{\beta^{2n}}{(ab)^n}a(\tilde{P}_1-2b\tilde{P}_0)+\frac{\beta^{2n+1}}{(ab)^n}\tilde{P}_0\\ +\frac{\alpha^{2n}}{(ab)^n}a(-\tilde{P}_1+2b\tilde{P}_0)-\frac{\alpha^{2n+1}}{(ab)^n}\tilde{P}_0\end{array}\right]x^{2n}\\
&= \frac{1}{\alpha-\beta}\sum_{i=1}^{\infty}\left[\frac{\alpha^{2n}-\beta^{2n}}{(ab)^n}\left(a\tilde{P}_1-2ab\tilde{P}_0\right)+\frac{\alpha^{2n+1}-\beta^{2n+1}}{(ab)^n}\tilde{P}_0\right]x^{2n}.
\end{aligned}$$

Then, we examine the series with odd powers

$$\begin{aligned}
O(x) &= -\frac{1}{\alpha-\beta}\sum_{i=1}^{\infty}\left[\begin{array}{c}-\alpha\,(2\beta+1)^{n+1}\,\tilde{P}_1+b\tilde{P}_0\,(2\beta+1)^{n}\\ +\beta\,(2\alpha+1)^{n+1}\,\tilde{P}_1-b\tilde{P}_0\,(2\alpha+1)^{n}\end{array}\right]x^{2n+1}\\
&= \frac{1}{\alpha-\beta}\sum_{i=1}^{\infty}-\frac{\beta^{2n+1}}{(ab)^n}\tilde{P}_1-b\tilde{P}_0\frac{\beta^{2n}}{(ab)^n}x^{2n+1}\\
&\quad+\frac{1}{\alpha-\beta}\sum_{i=1}^{\infty}\frac{\alpha^{2n+1}}{(ab)^n}\tilde{P}_1+b\tilde{P}_0\frac{\alpha^{2n}}{(ab)^n}x^{2n+1}\\
&= \frac{1}{\alpha-\beta}\sum_{i=1}^{\infty}\left(\tilde{P}_1\frac{\alpha^{2n+1}-\beta^{2n+1}}{(ab)^n}+b\tilde{P}_0\frac{\alpha^{2n}-\beta^{2n}}{(ab)^n}\right)x^{2n+1}.
\end{aligned}$$

By combining the above two sums and Theorem 2.6, we get the desired result.

$$\begin{aligned}
\tilde{P}\,(x) &= \sum_{i=0}^{\infty}\tilde{P}_i x^i=\frac{1}{\alpha-\beta}\sum_{n=1}^{\infty}\left(\tilde{P}_1\frac{\alpha^{2n+1}-\beta^{2n+1}}{(ab)^n}+b\tilde{P}_0\frac{\alpha^{2n}-\beta^{2n}}{(ab)^n}\right)x^{2n+1}\\
&\quad+\frac{1}{\alpha-\beta}\sum_{n=1}^{\infty}\left(\frac{\alpha^{2n}-\beta^{2n}}{(ab)^n}\left(a\tilde{P}_1-2ab\tilde{P}_0\right)+\frac{\alpha^{2n+1}-\beta^{2n+1}}{(ab)^n}\tilde{P}_0\right)x^{2n}.
\end{aligned}$$

□

Theorem 2.8. ***(Summation Formula)*** *For any positive integer n, and $ab\neq 0$, the sum of the first n terms of the bi-periodic Pell matrix sequence is computed as*

$$\sum_{k=0}^{n-1}\tilde{P}_k=\frac{\left(a\tilde{P}_n+b\tilde{P}_{n-1}\right)^{\xi(n)}\left(a\tilde{P}_{n-1}+b\tilde{P}_n\right)^{1-\xi(n)}-\left(2a\tilde{P}_1+4ab\tilde{P}_0-b\tilde{P}_0\right)^{1-\xi(n)}\left(\tilde{P}_1-b\tilde{P}_0\right)^{\xi(n)}}{2ab}.$$

Proof. If n is even, it is obtained that by Binet formula

$$\begin{aligned}\sum_{k=0}^{n-1}\tilde{P}_k &= \sum_{k=0}^{\frac{n-2}{2}}\tilde{P}_{2k}+\sum_{k=0}^{\frac{n-2}{2}}\tilde{P}_{2k+1}\\ &= \sum_{k=0}^{\frac{n-2}{2}}\frac{\tilde{P}_0}{(ab)^k}\frac{\alpha^{2k+1}-\beta^{2k+1}}{\alpha-\beta}+\sum_{k=0}^{\frac{n-2}{2}}\frac{a\tilde{P}_1-2ab\tilde{P}_0}{(ab)^k}\frac{\alpha^{2k}-\beta^{2k}}{\alpha-\beta}\\ &\quad+\sum_{k=0}^{\frac{n-2}{2}}\frac{\tilde{P}_1}{(ab)^k}\frac{\alpha^{2k+1}-\beta^{2k+1}}{\alpha-\beta}+\sum_{k=0}^{\frac{n-2}{2}}\frac{b\tilde{P}_0}{(ab)^k}\frac{\alpha^{2k}-\beta^{2k}}{\alpha-\beta}.\end{aligned}$$

By the sum of geometric series, it is computed that

$$\begin{aligned}\sum_{k=0}^{n-1}\tilde{P}_k &= \frac{\tilde{P}_0}{(ab)^{\frac{n}{2}-1}(\alpha-\beta)}\left[\frac{\alpha^{n+1}-\alpha(ab)^{\frac{n}{2}}}{(\alpha^2-ab)}-\frac{\beta^{n+1}-\beta(ab)^{\frac{n}{2}}}{(\beta^2-ab)}\right]\\ &\quad+\frac{a\tilde{P}_1-2ab\tilde{P}_0}{(ab)^{\frac{n}{2}-1}(\alpha-\beta)}\left[\frac{\alpha^{n}-(ab)^{\frac{n}{2}}}{(\alpha^2-ab)}-\frac{\beta^{n}-(ab)^{\frac{n}{2}}}{(\beta^2-ab)}\right]\\ &\quad+\frac{\tilde{P}_1}{(ab)^{\frac{n}{2}-1}(\alpha-\beta)}\left[\frac{\alpha^{n+1}-\alpha(ab)^{\frac{n}{2}}}{(\alpha^2-ab)}-\frac{\beta^{n+1}-\beta(ab)^{\frac{n}{2}}}{(\beta^2-ab)}\right]\\ &\quad+\frac{b\tilde{P}_0}{(ab)^{\frac{n}{2}-1}(\alpha-\beta)}\left[\frac{\alpha^{n}-(ab)^{\frac{n}{2}}}{(\alpha^2-ab)}-\frac{\beta^{n}-(ab)^{\frac{n}{2}}}{(\beta^2-ab)}\right].\end{aligned}$$

After some algebraic operations, the following result is evaluated

$$\begin{aligned}&= \frac{\tilde{P}_0}{4(ab)^{\frac{n}{2}+2}(\alpha-\beta)}\left[-a^2b^2(\alpha^{n-1}-\beta^{n-1})+ab(\alpha^{n+1}-\beta^{n+1})\right]\\ &\quad+\frac{a\tilde{P}_1-2ab\tilde{P}_0}{4(ab)^{\frac{n}{2}+2}(\alpha-\beta)}\left[\begin{array}{c}-a^2b^2(\alpha^{n-2}-\beta^{n-2})+ab(\alpha^{n}-\beta^{n})\\ -(ab)^{\frac{n}{2}}(\alpha^2-\beta^2)\end{array}\right]\\ &\quad+\frac{\tilde{P}_1}{4(ab)^{\frac{n}{2}+2}(\alpha-\beta)}\left[-a^2b^2(\alpha^{n-1}-\beta^{n-1})+ab(\alpha^{n+1}-\beta^{n+1})\right]\\ &\quad+\frac{b\tilde{P}_0}{4(ab)^{\frac{n}{2}+2}(\alpha-\beta)}\left[\begin{array}{c}-a^2b^2(\alpha^{n-2}-\beta^{n-2})+ab(\alpha^{n}-\beta^{n})\\ -(ab)^{\frac{n}{2}}(\alpha^2-\beta^2)\end{array}\right].\end{aligned}$$

By the definition of the Pell matrix sequence, it is obtained that

$$\sum_{k=0}^{n-1}\tilde{P}_k=\frac{2a\tilde{P}_{n-1}+2b\tilde{P}_n-4a\tilde{P}_1+8ab\tilde{P}_0-2b\tilde{P}_0}{4ab}.$$

Similarly if n is odd, we obtain

$$\begin{aligned}\sum_{k=0}^{n-1}\tilde{P}_k &= \sum_{k=0}^{\frac{n-1}{2}}\tilde{P}_{2k}+\sum_{k=0}^{\frac{n-3}{2}}\tilde{P}_{2k+1}\\ &= \sum_{k=0}^{\frac{n-1}{2}}\frac{\tilde{P}_0}{(ab)^k}\frac{\alpha^{2k+1}-\beta^{2k+1}}{\alpha-\beta}+\sum_{k=0}^{\frac{n-1}{2}}\frac{a\tilde{P}_1-2ab\tilde{P}_0}{(ab)^k}\frac{\alpha^{2k}-\beta^{2k}}{\alpha-\beta}\\ &\quad+\sum_{k=0}^{\frac{n-3}{2}}\frac{\tilde{P}_1}{(ab)^k}\frac{\alpha^{2k+1}-\beta^{2k+1}}{\alpha-\beta}+\sum_{k=0}^{\frac{n-3}{2}}\frac{b\tilde{P}_0}{(ab)^k}\frac{\alpha^{2k}-\beta^{2k}}{\alpha-\beta}.\end{aligned}$$

By using the sum of geometric series, it is computed that

$$\begin{aligned}\sum_{k=0}^{n-1}\tilde{P}_k &= \frac{\tilde{P}_0}{(ab)^{\frac{n-1}{2}}(\alpha-\beta)}\left[\frac{\alpha^{n+2}-\alpha(ab)^{\frac{n+1}{2}}}{(\alpha^2-ab)}-\frac{\beta^{n+2}-\beta(ab)^{\frac{n+1}{2}}}{(\beta^2-ab)}\right]\\ &\quad+\frac{a\tilde{P}_1-2ab\tilde{P}_0}{(ab)^{\frac{n-1}{2}}(\alpha-\beta)}\left[\frac{\alpha^{n+1}-(ab)^{\frac{n+1}{2}}}{(\alpha^2-ab)}-\frac{\beta^{n+1}-(ab)^{\frac{n+1}{2}}}{(\beta^2-ab)}\right]\\ &\quad+\frac{\tilde{P}_1}{(ab)^{\frac{n-3}{2}}(\alpha-\beta)}\left[\frac{\alpha^{n}-\alpha(ab)^{\frac{n-1}{2}}}{(\alpha^2-ab)}-\frac{\beta^{n}-\beta(ab)^{\frac{n-1}{2}}}{(\beta^2-ab)}\right]\\ &\quad+\frac{b\tilde{P}_0}{(ab)^{\frac{n-3}{2}}(\alpha-\beta)}\left[\frac{\alpha^{n-1}-(ab)^{\frac{n-1}{2}}}{(\alpha^2-ab)}-\frac{\beta^{n-1}-(ab)^{\frac{n-1}{2}}}{(\beta^2-ab)}\right].\end{aligned}$$

After some algebraic operations, the following result is evaluated

$$\begin{aligned}&= \frac{\tilde{P}_0}{4(ab)^{\frac{n+5}{2}}(\alpha-\beta)}\left[-a^2b^2(\alpha^n-\beta^n)+ab(\alpha^{n+2}-\beta^{n+2})\right]\\ &\quad+\frac{a\tilde{P}_1-2ab\tilde{P}_0}{4(ab)^{\frac{n+5}{2}}(\alpha-\beta)}\left[\begin{array}{c}-a^2b^2(\alpha^{n-1}-\beta^{n-1})+ab(\alpha^{n+1}-\beta^{n+1})\\ -(ab)^{\frac{n+1}{2}}(\alpha^2-\beta^2)\end{array}\right]\\ &\quad+\frac{\tilde{P}_1}{4(ab)^{\frac{n+3}{2}}(\alpha-\beta)}\left[-a^2b^2(\alpha^{n-2}-\beta^{n-2})+ab(\alpha^{n}-\beta^{n})\right]\\ &\quad+\frac{b\tilde{P}_0}{4(ab)^{\frac{n+3}{2}}(\alpha-\beta)}\left[\begin{array}{c}-a^2b^2(\alpha^{n-3}-\beta^{n-3})+ab(\alpha^{n-1}-\beta^{n-1})\\ -(ab)^{\frac{n+1}{2}}(\alpha^2-\beta^2)\end{array}\right].\end{aligned}$$

By the definition of the Pell matrix sequence, it is obtained that

$$\begin{aligned}\sum_{k=0}^{n-1}\tilde{P}_k &= \frac{\tilde{P}_{n+1}-\tilde{P}_{n-1}+\tilde{P}_n-\tilde{P}_{n-2}-2\tilde{P}_1-2b\tilde{P}_0}{4ab}\\ &= \frac{2a\tilde{P}_n+2b\tilde{P}_{n-1}-2\tilde{P}_1-2b\tilde{P}_0}{4ab}.\end{aligned}$$

By the above two results, we get the desired result. □

Example 2.9. *For $n = 100$, the sum of the first 100 terms of the bi-periodic Pell matrix sequence is*

$$\sum_{k=0}^{99} \tilde{P}_k = \frac{a\tilde{P}_{99} + b\tilde{P}_{100} - (2a\tilde{P}_1 + 4ab\tilde{P}_0 - b\tilde{P}_0)}{2ab},$$

For $n = 101$, the sum of the first 101 terms of the bi-periodic Pell matrix sequence is

$$\sum_{k=0}^{100} \tilde{P}_k = \frac{a\tilde{P}_{101} + b\tilde{P}_{100} - (\tilde{P}_1 - b\tilde{P}_0)}{2ab}.$$

Theorem 2.10. *For any positive integer n, and $ab \neq 0$, the sum of the square of the first n terms of the bi-periodic Pell matrix sequence is computed as*

$$\begin{aligned}\sum_{k=0}^{n-1} \tilde{P}_k^2 &= \frac{\left(\frac{b}{a}\right)^{1-\xi(n)} \tilde{P}_{2n+2} + \left(\frac{b}{a}\right)^{\xi(n)} \tilde{P}_{2n} - \left(\frac{b}{a}\right)^{\xi(n)} \tilde{P}_{2n-4} - \left(\frac{b}{a}\right)^{1-\xi(n)} \tilde{P}_{2n-2}}{16ab(ab+1)} \\ &\quad - \frac{(1+2ab)(a\tilde{P}_1 - 2ab\tilde{P}_0)}{8(ab+1)}.\end{aligned}$$

Proof. If n is even, it is obtained that by using Binet formula

$$\begin{aligned}\sum_{k=0}^{n-1} \tilde{P}_k^2 &= \sum_{k=0}^{\frac{n-2}{2}} \tilde{P}_{2k}^2 + \sum_{k=0}^{\frac{n-2}{2}} \tilde{P}_{2k+1}^2 = \sum_{k=0}^{\frac{n-2}{2}} \tilde{P}_{4k} + \frac{b}{a} \sum_{k=0}^{\frac{n-2}{2}} \tilde{P}_{4k+2} \\ &= \sum_{k=0}^{\frac{n-2}{2}} \frac{\tilde{P}_0}{(ab)^{2k}} \frac{\alpha^{4k+1} - \beta^{4k+1}}{\alpha - \beta} + \sum_{k=0}^{\frac{n-2}{2}} \frac{a\tilde{P}_1 - 2ab\tilde{P}_0}{(ab)^{2k}} \frac{\alpha^{4k} - \beta^{4k}}{\alpha - \beta} \\ &\quad + \frac{b}{a} \sum_{k=0}^{\frac{n-2}{2}} \frac{\tilde{P}_0}{(ab)^{2k+1}} \frac{\alpha^{4k+3} - \beta^{4k+3}}{\alpha - \beta} + \frac{b}{a} \sum_{k=0}^{\frac{n-2}{2}} \frac{a\tilde{P}_1 - 2ab\tilde{P}_0}{(ab)^{2k+1}} \frac{\alpha^{4k+2} - \beta^{4k+2}}{\alpha - \beta}.\end{aligned}$$

By using the sum of geometric series, it is computed that

$$\begin{aligned}\sum_{k=0}^{n-1} \tilde{P}_k^2 &= \frac{\tilde{P}_0}{(ab)^{n-2}(\alpha - \beta)} \left[\frac{\alpha^{2n+1} - \alpha (ab)^n}{\alpha^4 - a^2b^2} - \frac{\beta^{2n+1} - \beta (ab)^n}{\beta^4 - a^2b^2} \right] \\ &\quad + \frac{a\tilde{P}_1 - 2ab\tilde{P}_0}{(ab)^{n-2}(\alpha - \beta)} \left[\frac{\alpha^{2n} - (ab)^n}{\alpha^4 - a^2b^2} - \frac{\beta^{2n} - (ab)^n}{\beta^4 - a^2b^2} \right] \\ &\quad + \frac{b}{a} \frac{\tilde{P}_0}{(ab)^{n-1}(\alpha - \beta)} \left[\frac{\alpha^{2n+3} - \alpha^3 (ab)^n}{\alpha^4 - a^2b^2} - \frac{\beta^{2n+3} - \beta^3 (ab)^n}{\beta^4 - a^2b^2} \right] \\ &\quad + \frac{b}{a} \frac{b\tilde{P}_0}{(ab)^{n-1}(\alpha - \beta)} \left[\frac{\alpha^{2n+2} - \alpha^2 (ab)^n}{\alpha^4 - a^2b^2} - \frac{\beta^{2n+2} - \beta^2 (ab)^n}{\beta^4 - a^2b^2} \right].\end{aligned}$$

After some algebraic operations, the following result is evaluated

$$\sum_{k=0}^{n-1}\tilde{P}_k^2 = \frac{\tilde{P}_0}{16\,(ab)^{n+3}\,(\alpha-\beta)}\left[\frac{\begin{array}{c}-a^4b^4(\alpha^{2n-3}-\beta^{2n-3})+a^2b^2(\alpha^{2n+1}-\beta^{2n+1})\\ -\,(ab)^{n+2}\,(\alpha-\beta)+(ab)^{n+1}\,(\alpha^3-\beta^3)\end{array}}{ab+1}\right]$$

$$+\frac{a\tilde{P}_1-2ab\tilde{P}_0}{16\,(ab)^{n+3}\,(\alpha-\beta)}\left[\frac{\begin{array}{c}-a^4b^4(\alpha^{2n-4}-\beta^{2n-4})+a^2b^2(\alpha^{2n}-\beta^{2n})\\ -\,(ab)^{n}\,(\alpha^4-\beta^4)\end{array}}{ab+1}\right]$$

$$+\frac{b}{a}\frac{\tilde{P}_0}{16\,(ab)^{n+4}\,(\alpha-\beta)}\left[\frac{\begin{array}{c}-a^4b^4(\alpha^{2n-1}-\beta^{2n-1})+a^2b^2(\alpha^{2n+3}-\beta^{2n+3})\\ +\,(ab)^{n+3}\,(\alpha-\beta)-(ab)^{n+2}\,(\alpha^3-\beta^3)\end{array}}{ab+1}\right]$$

$$+\frac{b}{a}\frac{a\tilde{P}_1-2ab\tilde{P}_0}{16\,(ab)^{n+4}\,(\alpha-\beta)}\left[\frac{\begin{array}{c}-a^4b^4(\alpha^{2n-2}-\beta^{2n-2})+a^2b^2(\alpha^{2n+2}-\beta^{2n+2})\\ +2\,(ab)^{n+2}\,(\alpha^2-\beta^2)\end{array}}{ab+1}\right]$$

By the definition of the Pell matrix sequence, it is obtained that

$$\begin{aligned}\sum_{k=0}^{n-1}\tilde{P}_k^2 &= \frac{\frac{b}{a}\tilde{P}_{2n+2}+\tilde{P}_{2n}-\tilde{P}_{2n-4}-\frac{b}{a}\tilde{P}_{2n-2}}{16ab(ab+1)}\\ &\quad-\frac{\tilde{P}_0(1-\frac{b}{a})+\frac{a\tilde{P}_1-2ab\tilde{P}_0}{ab}((-2ab-1-\frac{b}{a})}{4(ab+1)}\end{aligned}$$

Similarly if n is odd, we obtain

$$\begin{aligned}\sum_{k=0}^{n-1}\tilde{P}_k^2 &= \sum_{k=0}^{\frac{n-1}{2}}\tilde{P}_{2k}^2+\sum_{k=0}^{\frac{n-3}{2}}\tilde{P}_{2k+1}^2=\sum_{k=0}^{\frac{n-1}{2}}\tilde{P}_{4k}+\frac{b}{a}\sum_{k=0}^{\frac{n-3}{2}}\tilde{P}_{4k+2}\\ &= \sum_{k=0}^{\frac{n-1}{2}}\frac{\tilde{P}_0}{(ab)^{2k}}\frac{\alpha^{4k+1}-\beta^{4k+1}}{\alpha-\beta}+\sum_{k=0}^{\frac{n-1}{2}}\frac{a\tilde{P}_1-2ab\tilde{P}_0}{(ab)^{2k}}\frac{\alpha^{4k}-\beta^{4k}}{\alpha-\beta}\\ &\quad+\frac{b}{a}\sum_{k=0}^{\frac{n-3}{2}}\frac{\tilde{P}_0}{(ab)^{2k+1}}\frac{\alpha^{4k+3}-\beta^{4k+3}}{\alpha-\beta}+\frac{b}{a}\sum_{k=0}^{\frac{n-3}{2}}\frac{a\tilde{P}_1-2ab\tilde{P}_0}{(ab)^{2k+1}}\frac{\alpha^{4k+2}-\beta^{4k+2}}{\alpha-\beta}.\end{aligned}$$

By using the sum of geometric series, it's computed that

$$\begin{aligned}\sum_{k=0}^{n-1}\tilde{P}_k^2 &= \frac{\tilde{P}_0}{(ab)^{n-1}(\alpha-\beta)}\left[\frac{\alpha^{2n+3}-\alpha(ab)^{n+1}}{\alpha^4-a^2b^2}-\frac{\beta^{2n+3}-\beta(ab)^{n+1}}{\beta^4-a^2b^2}\right]\\ &+\frac{a\tilde{P}_1-2ab\tilde{P}_0}{(ab)^{n-1}(\alpha-\beta)}\left[\frac{\alpha^{2n+2}-(ab)^{n+1}}{\alpha^4-a^2b^2}-\frac{\beta^{2n+2}-(ab)^{n+1}}{\beta^4-a^2b^2}\right]\\ &+\frac{b}{a}\frac{\tilde{P}_0}{(ab)^{n-2}(\alpha-\beta)}\left[\frac{\alpha^{2n+1}-\alpha^3(ab)^{n-1}}{\alpha^4-a^2b^2}-\frac{\beta^{2n+1}-\beta^3(ab)^{n-1}}{\beta^4-a^2b^2}\right]\\ &+\frac{b}{a}\frac{a\tilde{P}_1-2ab\tilde{P}_0}{(ab)^{n-2}(\alpha-\beta)}\left[\frac{\alpha^{2n}-\alpha^2(ab)^{n-1}}{\alpha^4-a^2b^2}-\frac{\beta^{2n}-\beta^2(ab)^{n-1}}{\beta^4-a^2b^2}\right].\end{aligned}$$

After some algebraic operations, the following result is evaluated

$$\begin{aligned}\sum_{k=0}^{n-1}\tilde{P}_k^2 &= \frac{\tilde{P}_0}{16(ab)^{n+4}(\alpha-\beta)}\left[\frac{\begin{array}{c}-a^4b^4(\alpha^{2n-1}-\beta^{2n-1})+a^2b^2(\alpha^{2n+3}-\beta^{2n+3})\\ -(ab)^{n+3}(\alpha-\beta)+(ab)^{n+2}(\alpha^3-\beta^3)\end{array}}{ab+1}\right]\\ &+\frac{a\tilde{P}_1-2ab\tilde{P}_0}{16(ab)^{n+4}(\alpha-\beta)}\left[\frac{\begin{array}{c}-a^4b^4(\alpha^{2n-2}-\beta^{2n-2})+a^2b^2(\alpha^{2n+2}-\beta^{2n+2})\\ -(ab)^{n+1}(\alpha^4-\beta^4)\end{array}}{ab+1}\right]\\ &+\frac{b}{a}\frac{\tilde{P}_0}{16(ab)^{n+3}(\alpha-\beta)}\left[\frac{\begin{array}{c}-a^4b^4(\alpha^{2n-3}-\beta^{2n-3})+a^2b^2(\alpha^{2n+1}-\beta^{2n+1})\\ +(ab)^{n+2}(\alpha-\beta)-(ab)^{n+1}(\alpha^3-\beta^3)\end{array}}{ab+1}\right]\\ &+\frac{b}{a}\frac{a\tilde{P}_1-2ab\tilde{P}_0}{16(ab)^{n+3}(\alpha-\beta)}\left[\frac{\begin{array}{c}-a^4b^4(\alpha^{2n-4}-\beta^{2n-4})+a^2b^2(\alpha^{2n}-\beta^{2n})\\ -2(ab)^{n+1}(\alpha^2-\beta^2)\end{array}}{ab+1}\right].\end{aligned}$$

By similar procedure for the case n is even, we get the result. If we compare two results, we complete the proof. □

Example 2.11. *For $n = 101$, the sum of the square of the first 101 terms of the bi-periodic Pell matrix sequence is*

$$\sum_{k=0}^{100}\tilde{P}_k^2 = \frac{\frac{b}{a}\tilde{P}_{204}+\frac{b}{a}\tilde{P}_{202}-\frac{b}{a}\tilde{P}_{198}-\tilde{P}_{200}}{16ab(ab+1)}-\frac{(1+2ab)(a\tilde{P}_1-2ab\tilde{P}_0)}{8(ab+1)}.$$

For $n = 100$, the sum of the square of the first 100 terms of the bi-periodic Pell matrix sequence is

$$\sum_{k=0}^{99} \tilde{P}_k^2 = \frac{\frac{b}{a}\tilde{P}_{202} + \tilde{P}_{200} - \tilde{P}_{196} - \frac{b}{a}\tilde{P}_{198}}{16ab(ab+1)} - \frac{(1+2ab)(a\tilde{P}_1 - 2ab\tilde{P}_0)}{8(ab+1)}.$$

Theorem 2.12. *For any positive integer n, we have*

$$\sum_{k=0}^{n-1} \frac{\tilde{P}_k}{x^k} = \frac{1}{x^4 - x^2(4ab+2)+1} \left[\begin{array}{c} \frac{\tilde{P}_{n-1}}{x^{n-1}} - \frac{\tilde{P}_{n+1}}{x^{n-3}} + \frac{\tilde{P}_{n-2}}{x^{n-2}} - \frac{\tilde{P}_n}{x^{n-4}} + x^4\tilde{P}_0 + x^3\tilde{P}_1 \\ + x^2(2a\tilde{P}_1 - 4ab\tilde{P}_0 - \tilde{P}_0) + x(2b\tilde{P}_0 - \tilde{P}_1) \end{array} \right].$$

Proof. If n is even, we get

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{\tilde{P}_k}{x^k} &= \sum_{k=0}^{\frac{n-2}{2}} \frac{\tilde{P}_{2k}}{x^{2k}} + \sum_{k=0}^{\frac{n-2}{2}} \frac{\tilde{P}_{2k+1}}{x^{2k+1}} \\ &= \sum_{k=0}^{\frac{n-2}{2}} \frac{\tilde{P}_0}{(abx^2)^k} \frac{\alpha^{2k+1} - \beta^{2k+1}}{\alpha - \beta} + \sum_{k=0}^{\frac{n-2}{2}} \frac{a\tilde{P}_1 - 2ab\tilde{P}_0}{(abx^2)^k} \frac{\alpha^{2k} - \beta^{2k}}{\alpha - \beta} \\ &\quad + \sum_{k=0}^{\frac{n-2}{2}} \frac{\tilde{P}_1}{x\,(abx^2)^k} \frac{\alpha^{2k+1} - \beta^{2k+1}}{\alpha - \beta} + \sum_{k=0}^{\frac{n-2}{2}} \frac{b\tilde{P}_0}{x\,(abx^2)^k} \frac{\alpha^{2k} - \beta^{2k}}{\alpha - \beta}. \end{aligned}$$

By using the sum of geometric series, it is computed that

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{\tilde{P}_k}{x^k} &= \frac{\tilde{P}_0}{(\alpha - \beta)} \left[\frac{\alpha^{n+1} - \alpha\left(abx^2\right)^{\frac{n}{2}}}{\alpha^2 - abx^2} - \frac{\beta^{n+1} - \beta\left(abx^2\right)^{\frac{n}{2}}}{\beta^2 - abx^2} \right] \\ &\quad + \frac{a\tilde{P}_1 - 2ab\tilde{P}_0}{x^{n-2}\,(ab)^{\frac{n}{2}+1}(\alpha - \beta)} \left[\frac{\alpha^n - \left(abx^2\right)^{\frac{n}{2}}}{\alpha^2 - abx^2} - \frac{\beta^n - \left(abx^2\right)^{\frac{n}{2}}}{\beta^2 - abx^2} \right] \\ &\quad + \frac{\tilde{P}_1}{x^{n-1}\,(ab)^{\frac{n}{2}+1}(\alpha - \beta)} \left[\frac{\alpha^{n+1} - \alpha\left(abx^2\right)^{\frac{n}{2}}}{\alpha^2 - abx^2} - \frac{\beta^{n+1} - \beta\left(abx^2\right)^{\frac{n}{2}}}{\beta^2 - abx^2} \right] \\ &\quad + \frac{b\tilde{P}_0}{x^{n-1}\,(ab)^{\frac{n}{2}+1}(\alpha - \beta)} \left[\frac{\alpha^n - \left(abx^2\right)^{\frac{n}{2}}}{\alpha^2 - abx^2} - \frac{\beta^n - \left(abx^2\right)^{\frac{n}{2}}}{\beta^2 - abx^2} \right]. \end{aligned}$$

By some algebraic operations, we have

$$= \frac{\tilde{P}_0}{x^{n-2}(ab)^{\frac{n}{2}+1}(\alpha-\beta)}\left[\frac{\begin{array}{c}a^2b^2(\alpha^{n-1}-\beta^{n-1})-abx^2(\alpha^{n+1}-\beta^{n+1})\\+\left(abx^2\right)^{\frac{n}{2}+1}(\alpha-\beta)-\left(abx^2\right)^{\frac{n}{2}}(\alpha-\beta)\end{array}}{x^4-x^2(4ab+2)+1}\right]$$

$$+\frac{a\tilde{P}_1-2ab\tilde{P}_0}{x^{n-2}(ab)^{\frac{n}{2}+1}(\alpha-\beta)}\left[\frac{\begin{array}{c}a^2b^2(\alpha^{n-2}-\beta^{n-2})-abx^2(\alpha^{n}-\beta^{n})\\+\left(abx^2\right)^{\frac{n}{2}}(\alpha^2-\beta^2)\end{array}}{x^4-x^2(4ab+2)+1}\right]$$

$$+\frac{\tilde{P}_1}{x^{n-1}(ab)^{\frac{n}{2}+1}(\alpha-\beta)}\left[\frac{\begin{array}{c}a^2b^2(\alpha^{n-1}-\beta^{n-1})-abx^2(\alpha^{n+1}-\beta^{n+1})\\+\left(abx^2\right)^{\frac{n}{2}+1}(\alpha-\beta)-\left(abx^2\right)^{\frac{n}{2}}(\alpha-\beta)\end{array}}{x^4-x^2(4ab+2)+1}\right]$$

$$+\frac{b\tilde{P}_0}{x^{n-1}(ab)^{\frac{n}{2}+1}(\alpha-\beta)}\left[\frac{\begin{array}{c}+a^2b^2(\alpha^{n-2}-\beta^{n-2})-abx^2(\alpha^{n}-\beta^{n})\\+\left(abx^2\right)^{\frac{n}{2}}(\alpha^2-\beta^2)\end{array}}{x^4-x^2(4ab+2)+1}\right].$$

By the definition of the Pell matrix sequence, it is obtained that

$$\sum_{k=0}^{n-1}\frac{\tilde{P}_k}{x^k}=\frac{1}{x^4-x^2(4ab+2)+1}\left[\begin{array}{c}-\frac{\tilde{P}_{n+1}}{x^{n-3}}+\frac{\tilde{P}_{n-1}}{x^{n-1}}-\frac{\tilde{P}_{n}}{x^{n-4}}+\frac{\tilde{P}_{n-2}}{x^{n-2}}+x^4\tilde{P}_0+x^3\tilde{P}_1\\ \\+x^2(2a\tilde{P}_1-4ab\tilde{P}_0-\tilde{P}_0)+x(2b\tilde{P}_0-\tilde{P}_1)\end{array}\right]$$

Similarly, if n is odd, we obtain

$$\begin{aligned}\sum_{k=0}^{n-1}\frac{\tilde{P}_k}{x^k} &= \sum_{k=0}^{\frac{n-1}{2}}\frac{\tilde{P}_{2k}}{x^{2k}}+\sum_{k=0}^{\frac{n-3}{2}}\frac{\tilde{P}_{2k+1}}{x^{2k+1}}\\ &= \sum_{k=0}^{\frac{n-1}{2}}\frac{\tilde{P}_0}{\left(abx^2\right)^k}\frac{\alpha^{2k+1}-\beta^{2k+1}}{\alpha-\beta}+\sum_{k=0}^{\frac{n-1}{2}}\frac{a\tilde{P}_1-2ab\tilde{P}_0}{\left(abx^2\right)^k}\frac{\alpha^{2k}-\beta^{2k}}{\alpha-\beta}\\ &\quad+\sum_{k=0}^{\frac{n-3}{2}}\frac{\tilde{P}_1}{x\left(abx^2\right)^k}\frac{\alpha^{2k+1}-\beta^{2k+1}}{\alpha-\beta}+\sum_{k=0}^{\frac{n-3}{2}}\frac{b\tilde{P}_0}{x\left(abx^2\right)^k}\frac{\alpha^{2k}-\beta^{2k}}{\alpha-\beta}.\end{aligned}$$

By using the sum of geometric series, it is computed that

$$\begin{aligned}\sum_{k=0}^{n-1}\frac{\tilde{P}_k}{x^k} &= \frac{\tilde{P}_0}{\left(abx^2\right)^{\frac{n-1}{2}}(\alpha-\beta)}\left[\frac{\alpha^{n+2}-\alpha\left(abx^2\right)^{\frac{n+1}{2}}}{\alpha^2-abx^2}-\frac{\beta^{n+2}-\beta\left(abx^2\right)^{\frac{n+1}{2}}}{\beta^2-abx^2}\right]\\
&+\frac{a\tilde{P}_1-2ab\tilde{P}_0}{\left(abx^2\right)^{\frac{n-1}{2}}(\alpha-\beta)}\left[\frac{\alpha^{n+1}-\left(abx^2\right)^{\frac{n+1}{2}}}{\alpha^2-abx^2}-\frac{\beta^{n+1}-\left(abx^2\right)^{\frac{n+1}{2}}}{\beta^2-abx^2}\right]\\
&+\frac{\tilde{P}_1}{x\left(abx^2\right)^{\frac{n-3}{2}}(\alpha-\beta)}\left[\frac{\alpha^{n}-\alpha\left(abx^2\right)^{\frac{n-1}{2}}}{\alpha^2-abx^2}-\frac{\beta^{n}-\beta\left(abx^2\right)^{\frac{n-1}{2}}}{\beta^2-abx^2}\right]\\
&+\frac{b\tilde{P}_0}{x\left(abx^2\right)^{\frac{n-3}{2}}(\alpha-\beta)}\left[\frac{\alpha^{n-1}-\left(abx^2\right)^{\frac{n-1}{2}}}{\alpha^2-abx^2}-\frac{\beta^{n-1}-\left(abx^2\right)^{\frac{n-1}{2}}}{\beta^2-abx^2}\right].\end{aligned}$$

After some algebraic operations, the following result is evaluated

$$\begin{aligned}&= \frac{\tilde{P}_0}{x^{n-1}(ab)^{\frac{n+3}{2}}(\alpha-\beta)}\left[\frac{\begin{array}{c}a^2b^2(\alpha^n-\beta^n)-abx^2(\alpha^{n+2}-\beta^{n+2})\\ +\left(abx^2\right)^{\frac{n+3}{2}}(\alpha-\beta)-\left(abx^2\right)^{\frac{n+1}{2}}ab(\alpha-\beta)\end{array}}{x^4-x^2(4ab+2)+1}\right]\\
&+\frac{a\tilde{P}_1-2ab\tilde{P}_0}{x^{n-1}(ab)^{\frac{n+3}{2}}((\alpha-\beta)}\left[\frac{\begin{array}{c}a^2b^2(\alpha^{n-1}-\beta^{n-1})-abx^2(\alpha^{n+1}-\beta^{n+1})\\ +\left(abx^2\right)^{\frac{n+1}{2}}(\alpha^2-\beta^2)\end{array}}{x^4-x^2(4ab+2)+1}\right]\\
&+\frac{\tilde{P}_1}{x^{n-2}(ab)^{\frac{n+1}{2}}((\alpha-\beta)}\left[\frac{\begin{array}{c}a^2b^2(\alpha^{n-2}-\beta^{n-2})-abx^2(\alpha^{n}-\beta^{n})\\ +\left(abx^2\right)^{\frac{n+1}{2}}(\alpha-\beta)-\left(abx^2\right)^{\frac{n-1}{2}}ab(\alpha-\beta)\end{array}}{x^4-x^2(4ab+2)+1}\right]\\
&+\frac{b\tilde{P}_0}{x^{n-2}(ab)^{\frac{n+3}{2}}((\alpha-\beta)}\left[\frac{\begin{array}{c}a^2b^2(\alpha^{n-3}-\beta^{n-3})-abx^2(\alpha^{n-1}-\beta^{n-1})\\ +\left(abx^2\right)^{\frac{n-1}{2}}(\alpha^2-\beta^2)\end{array}}{x^4-x^2(4ab+2)+1}\right].\end{aligned}$$

By the definition of the Pell matrix sequence, it is obtained that

$$\sum_{k=0}^{n-1}\frac{\tilde{P}_k}{x^k}=\frac{1}{x^4-x^2(4ab+2)+1}\left[\begin{array}{c}\frac{\tilde{P}_{n-1}}{x^{n-1}}-\frac{\tilde{P}_{n+1}}{x^{n-3}}+\frac{\tilde{P}_{n-2}}{x^{n-2}}-\frac{\tilde{P}_{n}}{x^{n-4}}+x^4\tilde{P}_0+x^3\tilde{P}_1\\ +x^2(2a\tilde{P}_1-4ab\tilde{P}_0-\tilde{P}_0)+x(2b\tilde{P}_0-\tilde{P}_1)\end{array}\right].$$

We find the same results either n is an even or odd number. □

Example 2.13. *For* $n = 101$, *the result of Theorem 2.12 is obtained by*

$$\sum_{k=0}^{100} \frac{\tilde{P}_k}{x^k} = \frac{1}{x^4 - x^2(4ab+2)+1} \left[\begin{array}{c} \frac{\tilde{P}_{100}}{x^{n-1}} - \frac{\tilde{P}_{101}}{x^{n-3}} + \frac{\tilde{P}_{99}}{x^{n-2}} - \frac{\tilde{P}_{101}}{x^{n-4}} + x^4\tilde{P}_0 + x^3\tilde{P}_1 \\ +x^2(2a\tilde{P}_1 - 4ab\tilde{P}_0 - \tilde{P}_0) + x(2b\tilde{P}_0 - \tilde{P}_1) \end{array} \right].$$

3. Conclusion

There are many generalizations for the special integer sequences in the literature. Generalized sequences give us the opportunity for finding the properties of many sequences in the same time. In this study, we carried out generalized Pell sequences to matrix theory. We call this sequence bi periodic Pell matrix sequence. We investigate the properties of this sequence.

4. Author Contributions

Formal analysis, S. Uygun and E. Akinci; resources, S. Uygun and E. Akinci; writing—original draft preparation, S. Uygun and E. Akinci; writing—review and editing, S. Uygun; supervision, S. Uygun. All authors have read and agreed to the published version of the manuscript.

5. Data Availability Statement

Not applicable.

6. Conflict of interest

The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

References

[1] A. Coskun, N. Taskara, *A note on the bi-periodic Fibonacci and Lucas matrix sequences*, Applied Mathematics and Computation vol.320 (2018) 400–406.

[2] A. F. Horadam, B. J. M. Mahon, *Pell and Pell Lucas Polynomials*, The Fibonacci Quarterly vol.23, no.1 (1985) 7–20.

[3] G. Bilgici, *Two generalizations of Lucas sequence*, Applied Mathematics and Computation vol.245 (2014) 526–538.

[4] M. Edson, O. Yayenie, *A new generalization of Fibonacci sequences and the extended Binet's formula*, INTEGERS Electron. J. Comb. Number Theory vol.9 (2009) 639–654.

[5] M. Tastan, E. Ozkan, *Catalan transform of the* $k-$*Pell,* $k-$*Pell–Lucas and modified* $k-$*Pell sequence*, Notes on Number Theory and Discrete Mathematics vol.27, no.1 (2021) 198–207.

[6] N. Yilmaz, A. Aydogdu, E. Ozkan, *Some Properties of* $k-$*Generalized Fibonacci Numbers*, Mathematica Montisnigri vol.L (2021) 73–79.

[7] O. Yayenie, *A note on generalized Fibonacci sequence*, Appl. Math. Comput. vol.217 (2011) 5603–5611.

[8] S. Celik, I. Durukan, E. Ozkan, *New recurrences on Pell numbers, Pell-Lucas numbers, Jacobsthal numbers, and Jacobsthal Lucas numbers*, Chaos, Solitons & Fractals vol.150, no.111173 (2021) 1–8.

[9] S. Uygun, E. Owusu, *A New Generalization of Jacobsthal Numbers (Bi-Periodic Jacobsthal Sequences)*, Journal of Mathematical Analysis vol.7, no.5 (2016) 28–39.
[10] S. Uygun, H. Karatas, *A New Generalization of Pell-Lucas Numbers (Bi-Periodic Pell-Lucas Sequence)*, Communications in Mathematics and Applications vol.10, no.3 (2019) 1–12.
[11] S. Uygun, H. Karatas, *Bi-Periodic Pell Sequence*, Academic Journal of Applied Mathematical Sciences vol.6, no.7 (2020) 136–144.
[12] S. Uygun, E. Owusu, *Matrix Representation of bi-Periodic Jacobsthal Sequence*, Journal of Advances in Mathematics and Computer Science vol.34, no.6 (2020) 1–12.
[13] S. Uygun, E. Owusu, *A New Generalization of Jacobsthal Lucas Numbers (Bi-Periodic Jacobsthal Lucas Sequence)*, Journal of Advances in Mathematics and Computer Science vol.34, no.5 (2020) 1–13.
[14] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons Inc., NY, 2001.
[15] T. Koshy, *Pell and Pell-Lucas Numbers with Applications*, Springer, Berlin, 2014.

Sukran Uygun
Orcid number: 0000-0002-7878-2175
Department of Mathematics
Faculty of Science and Arts
Gaziantep University
Gaziantep, Turkiye
Email address: suygun@gantep.edu.tr

Ersen Akinci
Orcid number: 0000-0001-8517-3987
Department of Mathematics
Faculty of Science and Arts
Gaziantep University
Gaziantep, Turkiye
Email address: ersenakinci33@gmail.com