Journal of Mahani Mathematical Research
Shahid Bahonar University of Kerman
Print ISSN: 2251-7952 Online ISSN: 2645-4505

BIFURCATION OF BIG PERIODIC ORBITS THROUGH SYMMETRIC HOMOCLINICS, APPLICATION TO DUFFING EQUATION

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Article type: Research Article
(Received: 09 October 2023, Received in revised form 13 March 2023)
(Accepted: 25 March 2023, Published Online: 26 March 2023)

Abstract. We consider a planar symmetric vector field that undergoes a homoclinic bifurcation. In order to study the existence of exterior periodic solutions of the vector field around broken symmetric homoclinic orbits, we investigate the existence of fixed points of the exterior Poincaré map around these orbits. This Poincaré map is the result of the combination of flows inside and outside the homoclinic orbits. It shows how a big periodic orbit converts to two small periodic orbits by passing through a double homoclinic structure. Finally, we use the results to investigate the existence of periodic solutions of the perturbed Duffing equation.

Keywords: Poincaré map, homoclinic bifurcation, fixed point, periodic solution.
2020 MSC: 37G15, 34C37.

1. Introduction

The problem of global bifurcations of homoclinic orbits leading to the appearance of periodic solutions is an interesting subject in the theory of differential equations and dynamical systems. Today, there exist several methods for investigating the bifurcations of homoclinic orbits (see for example [1,2,4,10,18] and references within). One of the main tools in the study of these bifurcations is the Poincaré map, such that investigating the behavior of orbits of vector field near homoclinic is equivalent to investigating the behavior of orbits of the Poincaré map. For example, the existence of the closed orbits for vector fields is equivalent to the existence of the fixed points for the Poincaré map and vice versa.

In 1983, Glendinning and Sparrow considered the autonomous system $\dot{x} = f(x) + g(x,\mu)$ where $x \in \mathbb{R}^3$, $f(x)$ is the linear part of the system, g is an analytic function w.r.t. x and $\mu \in \mathbb{R}$ is a parameter. The assumption was the existence of a homoclinic solution based on the saddle equilibrium point $x = 0$ for the system. They investigated the existence, number, and behavior of periodic solutions of the system near the homoclinic orbit by constructing

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DOI: 10.22103/jmmr.2023.20343.1349
Publisher: Shahid Bahonar University of Kerman
How to cite: L. Soleimani, O. RabieiMotlagh, *Bifurcation of big periodic orbits through symmetric homoclinics, application to Duffing equation*, J. Mahani Math. Res. 2024; 13(1): 1-11.

the Poincaré map [8]. Deng and Zhu studied the bifurcations of homoclinic loops with double orbit flips of the system $\dot{x} = f(x) + g(x, \mu)$ where $x \in \mathbb{R}^4$, $\mu \in \mathbb{R}^{l\geq 3}$ and $f, g \in C^{r\geq 3}$. The assumption was the existence of a homoclinic solution based on the saddle equilibrium point $x = 0$ for the system. They computed the Poincaré map to discuss numbers, existence, coexistence, and non-coexistence of Various homoclinic and periodic orbits [7]. The bifurcation of generalized homoclinic loops of the piecewise smooth systems was verified by Liang, Han, and Zhang in [13]. They calculated the Poincaré map and imposed some conditions on the system for getting one or two limit cycles near the loops. Li and Huang concentrated on homoclinic bifurcations of the planar perturbed discontinuous Filippov systems. They gave some conditions for an unperturbed system to show the existence and stability of homoclinic orbits. Furthermore, they proved the existence of stable and unstable limit cycles near the homoclinic orbit by analyzing the Poincaré map near the homoclinic [12]. Xiong and Han considered a non-smooth system. They made the Poincaré map to verify the bifurcation of the limit cycle near a homoclinic (heteroclinic) loop. Furthermore, they discussed the number of limit cycles that can appear from homoclinic (heteroclinic) bifurcation [19].

In this paper, we will investigate the bifurcation of periodic solutions of a planar vector field near two symmetric homoclinic orbits w.r.t. origin. For this purpose, we consider the planar system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_1(x, y, \mu) \\ f_2(x, y, \mu) \end{pmatrix} \tag{1}$$

where $\lambda_1 < 0$, $\lambda_2 > 0$, $f_{1,2} \in C^2$ are the nonlinear parts of the system, and $\mu \in \mathbb{R}$ is a parameter. We assume that for $\mu = 0$, the system is symmetric w.r.t. the origin and has two homoclinic orbits based on it (see Fig. 1.a). Here, the main assumption is that for $\mu \neq 0$, the system has passed a homoclinic bifurcation (see Fig. 1.b and 1.c). Previously, the interior Poincaré map $P_i = p_1' \circ p_0'$ was constructed (see Fig. 2 a) and its results were summarized in [18] as bellow:

Theorem 1.1. *([18, pp.144]) Consider (1) and let* $0 < |\mu| \ll 1$.

- **A1**: *If* $(-\lambda_1/\lambda_2) < 1$ *and* $\mu < 0$, *then (1) has a unique stable periodic orbit inside the broken homoclinic orbit.*
- **A2**: *If* $(-\lambda_1/\lambda_2) > 1$ *and* $\mu > 0$, *then (1) has a unique unstable periodic orbit inside the broken homoclinic orbit.*

There exist many restrictions, especially when $\mu \neq 0$, which turn the investigation of the exterior Poincaré map into an interesting subject in dynamical systems. This map is the result of a combination of four local Poincaré maps p_1, p_2, p_3, and p_4 (see Fig. 2.b). For constructing the exterior Poincaré map, we define four cross sections as below:

$$\pi_r^- = \{(x, y) \in \mathbb{R}^2 : x = \epsilon,\ -\epsilon \leqslant y < 0\},\ \pi_d^+ = \{(x, y) \in \mathbb{R}^2 : 0 < x \leqslant \epsilon,\ y = -\epsilon\},$$
$$\pi_l^+ = \{(x, y) \in \mathbb{R}^2 : x = -\epsilon,\ 0 < y \leqslant \epsilon\},\ \pi_t^- = \{(x, y) \in \mathbb{R} : -\epsilon \leqslant x < 0,\ y = \epsilon\}.$$

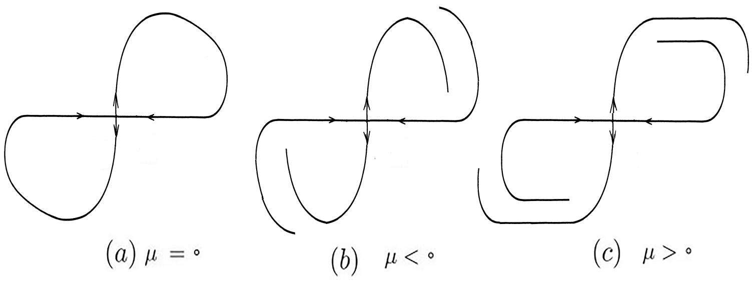

FIGURE 1. The behavior of homoclinic orbits for $\mu = 0$, $\mu > 0$ and $\mu < 0$

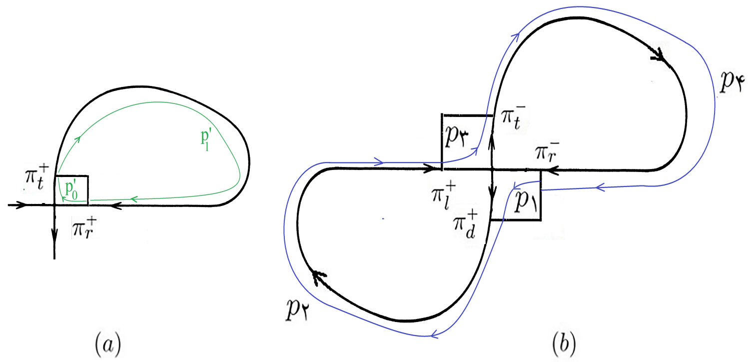

FIGURE 2. The generated flow (a): inside the homoclinic orbit. (b): outside the homoclinic orbits

In the next sections, we will construct the exterior Poincaré map. Then in Section 3, we will show that there exists a big periodic solution[1] that breaks into two small periodic solutions[2] by passing through symmetric homoclinic orbits (see Fig. 5 and its description in Section 5). In Section 4, as an application of the results, we will discuss the existence of periodic solutions of the perturbed Duffing equation

$$\begin{cases} \dot{x} = y - \nu u(x) y^2 \\ \dot{y} = x - x^3 + \nu(x + by) + \mu \sin x. \end{cases}$$

Studying such equations in science and engineering has attracted the attention of many researchers. For example, in 2002, Chung et al. focused on the bifurcation and limit cycles of nonlinear oscillators $\ddot{x}_i + g_i(x_i) = \mu f_i(x_1, ..., x_n, \dot{x}_1, ..., \dot{x}_n)$,

[1]It means a periodic solution that has a homoclinic structure in its inner region.
[2]It means the periodic solutions enclosed by the homoclinic orbits

$i = 1, ..., n$. Especially they investigated coupled generalized Van der Pol and Rayleigh oscillators [5]. In 2009, Kuznetsov and Roman studied the regimes of possible oscillation and structure of parameter space in the dissipative coupled Van der Pol and Van der Pol-Duffing oscillators with different controlling parameters [11]. In 2017 Chen et al. discussed homoclinic solutions of a power-law nonlinear oscillator with the equation $\ddot{x}+c_1x+c_px^p = \epsilon f(x,\dot{x},\mu)$ [3]. Recently, such equations have been taken into consideration, such as [9, 14, 15, 17].

2. Construction of the exterior Poincaré map

To begin we consider the Poincaré map $p_1 : \pi_r^- \to \pi_d^+$ which obtained by following the flow produced by the linear part of the system near the origin. A direct calculation shows that

$$p_1 : (\epsilon, y) \mapsto M_1(\epsilon, y)^t = (x, -\epsilon), \quad M_1 = \begin{pmatrix} (\frac{-\epsilon}{y})^{\frac{\lambda_1}{\lambda_2}} & 0 \\ 0 & \frac{-\epsilon}{y} \end{pmatrix}, \quad y < 0.$$

For defining p_2, we assume

$$p_2(0, -\epsilon, 0) = \big(p_{20}(0, -\epsilon, 0), p_{21}(0, -\epsilon, 0)\big) = (-\epsilon, 0).$$

By calculating the Taylor expansion of p_{21} around $(0, -\epsilon, 0)$, we get

$$\begin{aligned} p_{21}(x, y, \mu) &= p_{21}(0, -\epsilon, 0) + \tfrac{\partial p_{21}}{\partial x}(0, -\epsilon, 0)(x - 0) + \tfrac{\partial p_{21}}{\partial y}(0, -\epsilon, 0)(-\epsilon + \epsilon) \\ &\quad + \tfrac{\partial p_{21}}{\partial \mu}(0, -\epsilon, 0)(\mu - 0) + O(2) = ax + b\mu + O(2) \end{aligned}$$

where, from Fig. 1, $b < 0$ and $a > 0$. So, we can rescale μ such that $b = -1$. Thus, for $x > 0$ the function $p_2 : \pi_d^+ \to \pi_l^+$ can be defined as

$$p_2 : (x, -\epsilon) \mapsto (-\epsilon, ax - \mu) + O(2) = (-\epsilon, 0) + M_2(x, -\epsilon)^t + \mu(0, -1) + O(2) = (-\epsilon, \tilde{y}),$$

$$M_2 = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}.$$

Because (1) is symmetric w.r.t. the origin, $p_3 : \pi_l^+ \to \pi_t^-$ and $p_4 : \pi_t^- \to \pi_r^-$ are respectively defined as p_1 and p_2; that is:

$$p_3 : (-\epsilon, \tilde{y}) \mapsto (-\epsilon(\frac{\epsilon}{\tilde{y}})^{\frac{\lambda_1}{\lambda_2}}, \epsilon) + O(2) = M_3(-\epsilon, \tilde{y})^t + O(2), \quad \tilde{y} > 0,$$

$$M_3 = \begin{pmatrix} (\frac{\epsilon}{\tilde{y}})^{\frac{\lambda_1}{\lambda_2}} & 0 \\ 0 & \frac{\epsilon}{\tilde{y}} \end{pmatrix},$$

$$p_4 : (\tilde{x}, \epsilon) \mapsto (\epsilon, a\tilde{x} + \mu) + O(2) = (\epsilon, 0) + M_4(\tilde{x}, \epsilon)^t + \mu(0, 1) + O(2), \quad \tilde{x} < 0,$$

$$M_4 = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}.$$

Thus, the exterior Poincaré map, which is the result of the combination of the four functions p_1, p_2, p_3, and p_4, is given by:

$$(2) \qquad P_o : (\epsilon, y) \mapsto p_4 \circ p_3 \circ p_2 \circ p_1(\epsilon, y) = (\epsilon, \mu) + M(-\epsilon, -\mu)^t,$$

$$M = \begin{pmatrix} 0 & 0 \\ a\big(a\epsilon^{\frac{\lambda_1}{\lambda_2}}(-y)^{\frac{-\lambda_1}{\lambda_2}} - \frac{\mu}{\epsilon}\big)^{\frac{-\lambda_1}{\lambda_2}} & 0 \end{pmatrix}.$$

Note that in (2), for combining $p_2 \circ p_1$ with p_3, the image of $p_2 \circ p_1$ must be limited in the domain of p_3; it means $0 < p_2 \circ p_1(\epsilon, y) < \epsilon$ or equivalently

$$-\epsilon < \mu - a\epsilon\big(a\epsilon^{\frac{\lambda_1}{\lambda_2}}(-y)^{-\frac{\lambda_1}{\lambda_2}} - \frac{\mu}{\epsilon}\big)^{-\frac{\lambda_1}{\lambda_2}} < 0.$$

In other words,

$$\max\{-\epsilon, -\epsilon a^{\frac{\lambda_2}{\lambda_1}}\Big((\frac{\mu+\epsilon}{a\epsilon})^{-\frac{\lambda_2}{\lambda_1}} + \frac{\mu}{\epsilon}\Big)^{-\frac{\lambda_2}{\lambda_1}}\} < y \ < \min\{0, -\epsilon a^{\frac{\lambda_2}{\lambda_1}}\Big((\frac{\mu}{a\epsilon})^{-\frac{\lambda_2}{\lambda_1}} + \frac{\mu}{\epsilon}\Big)^{-\frac{\lambda_2}{\lambda_1}}\}.$$

Now by considering
(3)

$$y_{\max}(\mu) := -\epsilon a^{\frac{\lambda_2}{\lambda_1}}\Big((\frac{\mu}{a\epsilon})^{-\frac{\lambda_2}{\lambda_1}} + \frac{\mu}{\epsilon}\Big)^{-\frac{\lambda_2}{\lambda_1}}, \quad y_{\min}(\mu) := -\epsilon a^{\frac{\lambda_2}{\lambda_1}}\Big((\frac{\mu+\epsilon}{a\epsilon})^{-\frac{\lambda_2}{\lambda_1}} + \frac{\mu}{\epsilon}\Big)^{-\frac{\lambda_2}{\lambda_1}},$$

the domain of $p_2 \circ p_1$ is obtained as

$$\pi^-_{r,\mu} = \{(\epsilon, y) : \ \max\{-\epsilon, y_{\min}(\mu)\} < y \ < \min\{0, y_{\max}(\mu)\}$$

Thus, the Poincaré map $\mathcal{P} : \pi^-_{r,\mu} \cup \pi^+_r \to \pi^-_r \cup \pi^+_r$ is defined as

$$\mathcal{P}(\epsilon, y) = \begin{cases} P_i(\epsilon, y), & if \quad y > 0 \\ P_o(\epsilon, y), & if \quad \max\big(-\epsilon, y_{\min}\big) < y \ < \min(y_{\max}, 0). \end{cases}$$

Note that the Poincaré map $P_i : \pi_0^{r+} \to \pi_0^{r+}$ defined in [18] can also be calculated by combining $p_1^{'}$ with p_4:

$$P_i : (\epsilon, y) \mapsto (\epsilon, 0) + M^{'}(\epsilon, y) + \mu(0, 1), \quad M^{'} = \begin{pmatrix} 0 & 0 \\ a(\frac{\epsilon}{y})^{\frac{\lambda_1}{\lambda_2}} & 0 \end{pmatrix}.$$

Since the two homoclinic orbits are symmetric w.r.t. the origin, we can generalize the results of the Poincaré map inside one of the homoclinic orbits to the other. In the next section, we will investigate the existence of fixed points of the Poincaré map $\mathcal{P}$.

3. Existence of fixed points

It is well-known that the fixed points of $\mathcal{P}$ can be found by solving the equation $\mathcal{P}(\epsilon, y) = (\epsilon, y)$. For $y > 0$ the fixed points of $\mathcal{P}$ are equivalent to the fixed points of P_i, the existence of these fixed points has already been studied in [18] and the results have been described in Theorem 1.1.

Here, we investigate the fixed points of $\mathcal{P}$ when $\max\{-\epsilon, y_{\min}\} < y < \min\{y_{\max}, 0\}$. If $\mu < 0$ then P_o is defined for $y < 0$ and if $\mu > 0$ then P_o

can be defined for $\max\{-\epsilon, y_{\min}\} < y \ < \min\{y_{\max}, 0\}$. On the other hand,

$$\frac{dy_{\max}(\mu)}{d\mu} = \frac{\lambda_2 \epsilon a^{\frac{\lambda_2}{\lambda_1}} \left((\frac{\mu}{a\epsilon})^{-\frac{\lambda_2}{\lambda_1}} + \frac{\mu}{\epsilon} \right)^{-\frac{\lambda_2}{\lambda_1}-1} \left(\frac{1}{\epsilon} - \frac{\lambda_2 (\frac{\mu}{a\epsilon})^{-\frac{\lambda_2}{\lambda_1}-1}}{a\epsilon\lambda_1} \right)}{\lambda_1} < 0,$$

$$(4) \quad \frac{dy_{\min}(\mu)}{d\mu} = \frac{\lambda_2 \epsilon a^{\frac{\lambda_2}{\lambda_1}} \left((\frac{\mu+\epsilon}{a\epsilon})^{-\frac{\lambda_2}{\lambda_1}} + \frac{\mu}{\epsilon} \right)^{-\frac{\lambda_2}{\lambda_1}-1} \left(\frac{1}{\epsilon} - \frac{\lambda_2 (\frac{\mu+\epsilon}{a\epsilon})^{-\frac{\lambda_2}{\lambda_1}-1}}{a\lambda_1\epsilon} \right)}{\lambda_1} < 0.$$

Therefore, $y_{\max}(\mu)$ and $y_{\min}(\mu)$ are strictly decreasing functions; In addition, in the case $(-\lambda_1/\lambda_2) > 1$, by a direct calculation, we can see $dy_{\min}(0)/d\mu < 0$, $dy_{\max}(0)/d\mu = -\infty$ and $d^2 y_{\min}(0)/d\mu^2 > 0$. Also, in the case $(-\lambda_1/\lambda_2) < 1$, we obtain $dy_{\max}(0)/d\mu = 0$, $dy_{\min}(0)/d\mu < 0$ and $d^2 y_{\min}(0)/d\mu^2 < 0$. Thus, the domain of P_o in the plan (y, μ) is as Fig. 3. For finding the fixed points of

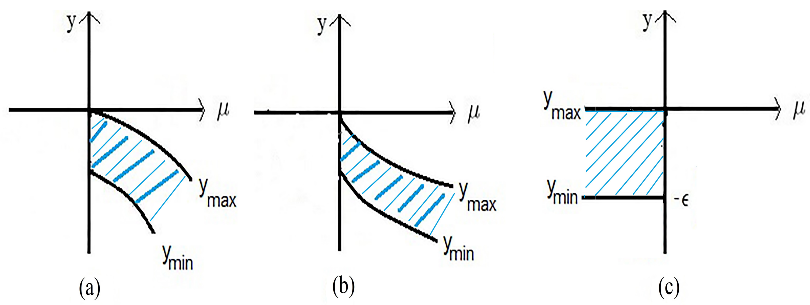

FIGURE 3. Domain of Poincaré map P_o, (a) When $\mu > 0$ and $(-\lambda_1/\lambda_2) < 1$. (b) when $\mu > 0$ and $(-\lambda_1/\lambda_2) > 1$. (c) when $\mu < 0$.

P_o, we need a generalization of the implicit function theorem.

Definition 3.1. A region is called uniformly connected if there exists a constant G such that any two points of this region can be connected by a broken line that is entirely inside the region, and the length of this broken line is not greater than G times of the length of a straight line connecting two points.

Theorem 3.2. *(The implicit function theorem on a boundary [6])* *Consider the nonlinear system* $F_i(x_1, ..., x_n, u_1, ..., u_m) = 0$, $1 \leq i \leq n$ *on the uniformly connected region* S*. Also, assume the point* $(a, c) = (a_1, ..., a_n, c_1, ..., c_m)$ *on the boundary of* S *be a solution of the system such that* $\det\Big((\partial F_i/\partial x_j)(a, c) \Big) \neq 0$. *So, in the neighborhood of* (a, c)*, the system* $F_i = 0$ *has a solution* $x_i = f_i(u_1, u_2, ..., u_m)$ *such that* $a_i = f_i(c_1, c_2, ..., c_m)$.

As seen in Fig. 3, the domain of P_o is a uniformly connected region. On the other hand, according to the relation (2), the existence of a fixed point for

P_o is equivalent to the roots of equation $h(y,\mu) := y - \mu - g(y,\mu) = 0$ where $g(y,\mu) = -a\epsilon\big(a\epsilon^{\lambda_1/\lambda_2}(-y)^{-\lambda_1/\lambda_2} - \mu/\epsilon\big)^{-\lambda_1/\lambda_2}$ and this is equivalent to the intersection points of line $x = y - \mu$ and $x = g(y,\mu)$.

Theorem 3.3. *Consider the system (1) and assume* $0 < |\mu| \ll 1$.

B1: *If* $(-\lambda_1/\lambda_2) < 1$, *then for* $\mu > 0$, *the system (1) has a unique stable big periodic solution outside the broken homoclinic orbit.*

B2: *If* $(-\lambda_1/\lambda_2) > 1$, *then for* $\mu < 0$, *the system (1) has a unique unstable big periodic solution outside the broken homoclinic orbits.*

Proof. **B1:** Let $-\lambda_1/\lambda_2 < 1$. In this case, the intersection points of $x = y - \mu$ with $x = g(y,\mu)$ are equivalent to the intersection points of the corresponding inverse functions, i.e. $x = y + \mu$ with $x = g^{-1}(y,\mu)$. We put

$$k(y,\mu) := y + \mu - g^{-1}(y,\mu), \tag{5}$$

where

$$g^{-1}(y,\mu) = -\Big[a^{-1}\epsilon^{\frac{-\lambda_1}{\lambda_2}}\Big(\big(\frac{y}{-a\epsilon}\big)^{\frac{-\lambda_2}{\lambda_1}} + \frac{\mu}{\epsilon}\Big)\Big]^{\frac{-\lambda_2}{\lambda_1}}.$$

Therefore,

$$k(y_{max},0) = 0, \qquad \big(\partial k/\partial\mu\big)(y_{\max},0) = 1, \qquad \big(\partial k/\partial y\big)(y_{\max},0) = 1.$$

Thus, from Theorem (3.2), for sufficiently small μ, the equation $k(y,\mu) = 0$ has a solution; i.e. $y + \mu = g^{-1}(y,\mu)$. On the other hand, for $y < y_{max}(\mu)$,

$$\frac{\partial g(y,0)}{\partial\mu} = a(\frac{-\lambda_1}{\lambda_2})(a\epsilon^{\frac{\lambda_1}{\lambda_2}}(-y)^{\frac{-\lambda_1}{\lambda_2}})^{\frac{-(\lambda_1+\lambda_2)}{\lambda_2}} > 0.$$

Hence, $g(y,\mu)$ is not constant w.r.t. μ. So, for sufficiently small μ, the line $x = y - \mu$ and $x = g(y,\mu)$ have an intersection point. Thus, for $\mu > 0$, the Poincaré map P_o has a big periodic solution outside the broken homoclinic orbits. Since $\big(\partial k/\partial y\big)(y_{\max},0) = 1$, thus the big periodic orbit is unstable for the inverse system, which implies that it is stable for the system itself.

B2: Let $(-\lambda_1/\lambda_2) > 1$. A direct calculation shows that

$$h(0,0) = 0, \qquad \frac{\partial h}{\partial\mu}(0,0) = -1, \qquad \frac{\partial h}{\partial y}(0,0) = 1.$$

Thus, Theorem (3.2) implies that for $\mu < 0$, the Poincaré map P_o has a big unique periodic solution outside the broken homoclinic orbits. Finally, since $\big(\partial h/\partial y\big)(0,0) = 1$, thus the big periodic orbit is unstable.

□

4. Application to perturbed Duffing equation

In this section, we implement the result of Theorem 3.3 to investigate the existence of periodic solutions around broken homoclinic orbits of the perturbed

Duffing equation

$$\begin{cases} \dot{x} = y - \nu u(x) y^2 \\ \dot{y} = x - x^3 + \nu(x + by) + \mu \sin x \end{cases} \tag{6}$$

where ν, μ are positive parameters, $b \in \mathbb{R}$ and $u(x)$ is an odd function.

The unperturbed part of (6), i.e. $\nu, \mu = 0$, is a Hamiltonian system with the Hamiltonian function

$$H(x, y) = \frac{1}{2}\big(y^2 - x^2 + \frac{x^4}{2}\big).$$

Also, it has three equilibrium points $p_0 = (0, 0)$ and $p_{1,2} = (\pm 1, 0)$ that p_0 is a saddle point and $p_{1,2}$ are center points. Furthermore, there exist two symmetric homoclinic orbits (see Fig. $4(a)$) $\gamma^{\pm}$ indicated by $y^2 = x^2 - x^4/2$ and respectively for

$$\begin{aligned} \gamma^+ \quad &: \quad y = \pm x\sqrt{1 - \frac{x^2}{2}}, \qquad 0 < x \leq \sqrt{2}, \\ \gamma^- \quad &: \quad y = \pm x\sqrt{1 - \frac{x^2}{2}}, \qquad -\sqrt{2} \leq x < 0. \end{aligned} \tag{7}$$

It is easy to check that $\gamma^{\pm}$ intersects the x-axis at $x_0^{\pm} = \pm\sqrt{2}$. Since $\gamma^{\pm}$ are symmetric w.r.t. the origin, so we investigate the behavior of the system around γ^+. Consider (6) for $\mu = 0$ i.e.

$$\begin{cases} \dot{x} = y + \nu u(x) y^2 \\ \dot{y} = x - x^3 + \nu(x + by). \end{cases} \tag{8}$$

By implementing the Melnikov method (see [18, Sec. 28.4]), we can obtain the Melnikov function:

$$M(b) = \int_{\mathbb{R}} \big(u(x) y^2(-x + x^3) + y(x + by)\big) dt.$$

Considering this fact that $y(t) = x^{'}(t)$ and using (7), we can rewrite the above integral in the form:

$$\begin{aligned} M(b) &= \int_{\gamma^+} \big(u(x) y(-x + x^3) + x + by\big) dx \\ &= \int_0^{\sqrt{2}} \big(u(x) y_+(-x + x^3) + x + by_+\big) dx - \int_{\sqrt{2}}^0 \big(u(x) y_-(-x + x^3) + x + by_-\big) dx. \end{aligned}$$

Here $y_+ = x\sqrt{1 - x^2/2}$ and $y_- = -x\sqrt{1 - x^2/2}$ are respectively the functions of graph of γ^+ on the intervals $(0, \sqrt{2}]$ and $[\sqrt{2}, 0)$. Since $y_+ = -y_-$, so

$$M(b) = 2\int_0^{\sqrt{2}} \big(u(x) y_+(-x + x^3) + by_+\big) dx.$$

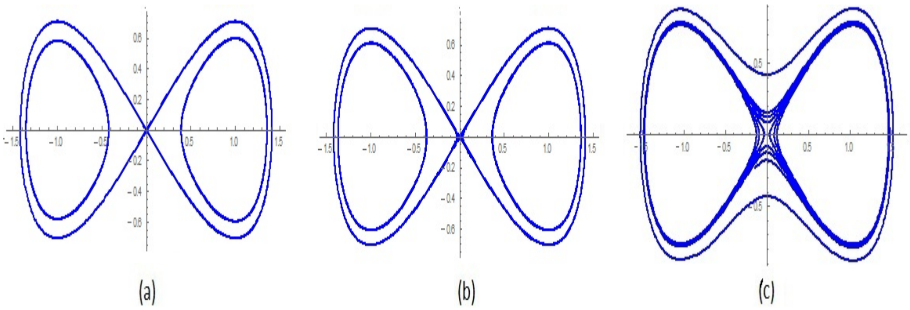

FIGURE 4. (a): The double homoclinic solutions for (6) when $\mu = \nu = 0$. (b): The double homoclinic solutions for (6) when $\mu = 0$. (c): Big periodic orbit for (6) when $\mu = 0.01$, $\nu = 0.001$, $u(x) = x$ and $b = \frac{4(29\sqrt{2}-1)}{35(2\sqrt{2}-1)}$.

Hence, by a direct calculation, for

$$b_0 = \frac{\int_0^{\sqrt{2}} u(x) y_+ (x - x^3) dx}{\int_0^{\sqrt{2}} y_+ dx},$$

we obtain

$$M(b_0) = 0, \qquad \frac{\partial M}{\partial b}(b_0) \neq 0.$$

Therefore, for $0 < |\nu| \ll 1$, there exists $b \in \mathbb{R}$ such that (8) has two symmetric (w.r.t. origin) homoclinic orbits (see Fig. 4(b)) with corresponding eigenvalues

$$\lambda_1 = \frac{1}{2}\left(b\nu - \sqrt{b^2\nu^2 + 4\nu + 4}\right), \qquad \lambda_2 = \frac{1}{2}\left(b\nu + \sqrt{b^2\nu^2 + 4\nu + 4}\right).$$

Note that (8) is an autonomous system, so the simple zeroes of the Melnikov function indicate homoclinic orbits of the system. By Theorem 3.3, since $(-\lambda_1/\lambda_2) < 1$, so for $0 < \nu, \mu$ sufficiently small, the system has a unique stable big periodic solution outside the broken homoclinic orbits (see Fig 4(c)).

5. Conclusion

This paper investigates the existence of periodic solutions bifurcated from symmetric homoclinic orbits for a planar system by applying the Poincaré maps. It is well-known that the system's periodic orbits correspond to the Poincaré map's fixed points.

The existence of these periodic solutions was discussed in Theorem 3.3. In fact, in the case of $-\lambda_1 > \lambda_2$ for sufficiently small $\mu < 0$, the system (1) has a unique unstable big periodic orbit outside the broken homoclinic orbits that merge to homoclinics in $\mu = 0$ and finally, for $\mu > 0$, they convert to two unstable periodic orbits inside the broken homoclinic orbits (see Fig. 5). Similarly, in the case of $-\lambda_1 < \lambda_2$, for $\mu < 0$ there exist two stable periodic orbits inside the broken homoclinics that merge to the homoclinic orbits in $\mu = 0$, and finally, for $\mu > 0$, they convert to a stable big periodic orbit outside the broken homoclinics. This is precisely what we have seen for perturbed Duffing equation (6), where the broken homoclinics $\gamma^{\pm}$ are surrounded by a stable big periodic orbit (see Fig. 4(c) again).

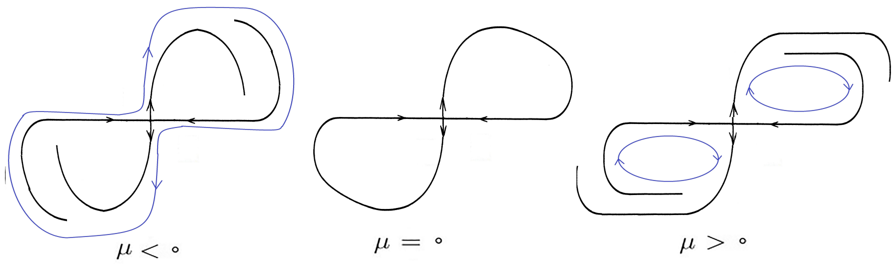

Figure 5. The global bifurcation of periodic solutions in the case $-\lambda_1 > \lambda_2$

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