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A DB ESTIMATION METHOD FOR THE E-MN PROBABILITY DISTRIBUTION PARAMETERS WITH APPLICATIONS IN HUMANITIES

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Abstract. One of the humanities' most basic topics is the response time to creative problem-solving and decision-making in this field. In recent years, response time modeling by fitting an exponentially-modified normal (E-MN) probability distribution and the results obtained from this process have been widely used. The E-MN probability distribution results from the convolution of a normal probability distribution and an exponential probability distribution and contains three parameters. In this paper, a developed Bayesian (DB) estimation method is introduced to estimate the parameters of an E-MN probability distribution. This new estimation method uses the adaptive rejection Metropolis-Hastings (ARM-H) sampling method. The reason for this is that in normal mode and based on the classical Bayesian estimation method, the chosen prior probability density functions (pdfs) lead to posterior pdfs with unknown form and, they are not always logarithmically concave. Also, respectively, simulation and real data sets study have been done to demonstrate the better performance of the DB estimation method than the two other well-known estimation methods used in this context, including the maximum likelihood (ML) estimation method and the quantile maximum likelihood (QML) estimation method. To show the better efficiency of the proposed estimation method compared with the two other estimation methods, the root mean squared error (RMSE) criterion is used.

Keywords: Response time, Exponentially-modified normal probability distribution, Developed Bayesian estimation method, Adaptive rejection Metropolis-Hastings sampling method, Root mean squared error, Quantile maximum likelihood estimation method.
2020 MSC: 62F15

1. Introduction and motivation

One of the most fundamental topics in the humanities is response time (RT) [23, 28], that is, the time it takes a person to respond to a stimulant [6]. The deep attention of humanities in this important index comes from the fact that it provides the possibility of discovering human cognition with the help of the experimental handwork of the properties of the used stimuli [32, 36].

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Broadly speaking, it is accepted that RT affects the length of time needed to perform the three perceptual, cognitive and motor stages of response preparation [35]. Suppose a linguist assumes that the words of a language are organized in memory according to their frequency of use. Experimental manipulation of this factor should affect cognitive processes. Specifically, a human should be more accurate and faster at processing the maximum-frequency words (for example, mom) than the minimum-frequency words (for example, solipsism). That is, as soon as a word is presented audibly, handwriting on a graphics tablet should be reflected with a shorter RT for the maximum-frequency words than for the minimum-frequency words. Whenever the durations required to perceive the stimulant and prepare motor responses are constant, comparing RTs between lexical frequency conditions allows researchers to test the hypothesis that words are organized in memory by their frequency in language [14, 20].

On the other hand, upon the fisher's suggestions, humanities base its results on the use of hypothesis testing [39]. In recent years, researchers have found that many RTs have very long lengths [10]. Also, the characteristic of positive skewness of probability distribution of RTs has been shown in different papers [4, 9, 11, 13, 27, 30, 33]. That the existence of this feature has a significance to improve the estimation process and, following it, on the strength of the results obtained from the hypothesis testing [12, 37]. This issue is mostly handled in humanities by using strategies to normalize the RTs probability distribution, such as transforming the data using link functions or as a method to right-censoring this probability distribution without ignoring all the problems that these strategies can cause [17, 31, 38]. It has also been tried to obtain the probability distribution of RTs from the sum of two independent random variables. One of these two random variables follows the exponential probability distribution, and the second random variable follows the normal probability distribution. This probability distribution, called exponentially-modified normal (E-MN) probability distribution [1, 2, 29], allows humanities researchers to work with a theoretical probability distribution function close to that of RTs [8, 24, 34].

The main goal of this paper is to introduce a new, efficient and robust method to estimate the parameters of an E-MN probability distribution. Because the empirical conditions governing humanities research are a limitation for common estimators of E-MN probability distribution parameters. For example, when a researcher examines a population of minors, it is difficult to derive a large number RTs for this age group. On the other hand, many proposed solutions to estimate the E-MN probability distribution parameters are based on various methods to maximize the log-likelihood function. But, the big problem is that the results of all these methods are valid for sample sizes greater than 100 [15, 18]. The aim of this paper is to introduce an efficient alternative method to estimating E-MN probability distribution parameters for sample sizes less than 100. In such a way that is propose a developed Bayesian (DB) estimation method in which each parameter is considered as a random variable. In this method, it chooses a prior probability density function (pdf) for each of the variables, and simultaneously the likelihood function of the related computed has been computed. Then, the conditional posterior pdf of each parameter is obtained using the selected prior pdfs and the calculated likelihood function. Also, since the calculated posterior pdfs do not have a known form and are not always logarithmically concave, the adaptive rejection Metropolis-Hastings (ARM-H) sampling method is used to sample from these conditional posterior pdfs. To compare the efficiency and robustness of the DB estimation method introduced in this paper relative to two other important and practical estimation methods used in estimating the E-MN probability distribution parameters, including the maximum likelihood (ML) and the quantile maximum likelihood (QML) estimation method, simultaneously, simulation and analysis of real data have been used. Then, the root mean square error (RMSE) of E-MN probability distribution parameters estimators obtained through three estimation methods has been compared. Results show that the RMSE of the parameters obtained using the DB estimation method is closer to zero than the RMSE of the parameters obtained by the QML estimation method and the ML estimation method. This shows that the proposed estimation method is much more efficient and better.

The order of the contents of this paper is as follows: in section 2, the E-MN probability distribution is defined. Section 3, in which a DB method is introduced to estimate the parameters of the E-MN probability distribution, is divided into two subsections. In the first subsection, since the parameters are considered as random variables in the framework of the Bayesian estimation method, a prior pdf is selected for each parameter and a likelihood function is calculated. Then the conditional posterior pdfs of these parameters are calculated using these prior pdfs and the likelihood function. In the second subsection, an algorithm for estimating E-MN probability distribution parameters based on the ARM-H sampling method is presented. In section 4, two ML and QML estimation methods are reminded to compare later to the proposed DB estimation method. The efficiency and robustness of the proposed estimation method are shown by the simulation study in section 5 and through the real data sets analysis in section 6. In these two sections, the results obtained from the DB estimation method are compared with the results obtained from the QML and ML estimation methods. Finally, in section 7, discussion and conclusion are presented.

2. Definition of the E-MN probability distribution

In this section, the E-MN probability distribution is defined. The E-MN probability distribution is a famous and very practical probability distribution in the humanities [21,40]. This probability distribution is the sum of the two independent random variables, one with normal probability distribution and the other with exponential probability distribution. In other words, a continuous random variable Y with an E-MN probability distribution can be written as a random variable $Y = Y_1 + Y_2$, where two random variables Y_1 and Y_2 are independent, Y_1 follows the normal probability distribution with two parameters α as the mean and β as the variance and, Y_2 follows the exponential probability distribution with parameter δ. That the joint pdf for $\boldsymbol{y}' = (y_1, y_2, y_3, \ldots, y_n)$ as a vector of observations with length n of the random variable Y is as follows:

$$(1)\quad g(\boldsymbol{y}; \alpha, \beta, \delta) = \delta \left\{ \exp\left[\delta\left(\alpha - \boldsymbol{y}\right) + \frac{1}{2}\left(\delta^2 \beta\right) \right] \right\} \left\{ \Psi\left[\frac{(\boldsymbol{y} - \alpha)}{\theta} - \delta\theta \right] \right\}, \begin{cases} \alpha \epsilon \mathbb{R} \\ \beta \epsilon \mathbb{R}^+ \\ \delta \epsilon \mathbb{R}^+ \\ \boldsymbol{y} \epsilon \mathbb{R}^n \\ n \in \mathbb{N} \end{cases}$$

that $\theta = |\sqrt{\beta}|$ and $\Psi\left[\frac{(\boldsymbol{y}-\alpha)}{\theta} - \delta\theta\right] = \frac{1}{\sqrt{2\pi}} \int_{\boldsymbol{w}=-\infty}^{\left[\frac{(\boldsymbol{y}-\alpha)}{\theta} - \delta\theta\right]} \exp\left(-\frac{1}{2}\boldsymbol{w}^2\right) d\boldsymbol{w}$ is the cumulative distribution function (cdf) of the standard normal probability distribution. Also, it is assumed that $\boldsymbol{\varphi}' = (\alpha, \beta, \delta)$ the vector includes unknown parameters.

3. DB estimation method

It is clear that in common statistical inference methods, parameters are considered to be fixed values and the observations are random quantities. In fact, in these types of methods, probability distributions are used to display observations. But, in Bayesian statistical inference, probability distributions are often used to represent more than observations. They also show the uncertainty of the prior pdf in the parameters. These are then updated with current observations using Bayes' theorem to generate probability distributions of the posterior pdf.

This section, which introduces a DB method to estimate the parameters of the E-MN probability distribution, is divided into two subsections. In the first subsection, since the parameters are considered as random variables in the framework of the Bayesian estimation method, a prior pdf is selected for each parameter and a likelihood function is calculated. Then the conditional posterior pdfs of these parameters are calculated using these prior pdfs and the likelihood function. In the second subsection, an algorithm for estimating E-MN probability distribution parameters based on the ARH-M sampling method is presented.

3.1. **Finding of the conditional posterior pdfs of E-MN probability distribution parameters.** The Bayesian inference method begins by choosing a prior probability distribution for the parameter vector $\boldsymbol{\varphi}' = (\alpha, \beta, \delta)$. This paper, use a conjugate prior pdf for the vector $\boldsymbol{\varphi}$ to simplify the calculations.

This prior pdf quantifies the available information about $\boldsymbol{\varphi}$:

(2)
$$\begin{cases} \alpha \sim Normal(a_1, a_2),\ \text{where: } g(\alpha) = \frac{1}{\sqrt{2\pi a_2}} \left\{ \exp\left[-\frac{(\alpha - a_1)^2}{2a_2} \right] \right\}, & \begin{cases} a_1 \in \mathbb{R} \\ a_2 \in \mathbb{R}^+ \\ \alpha \in \mathbb{R} \end{cases} \\ \beta \sim Inverse\ Gamma(b_1, b_2),\ \text{where: } g(\beta) = \frac{b_2^{b_1}}{\Gamma(b_1)} \frac{1}{\beta^{b_1+1}} \left[\exp\left(-\frac{b_2}{\beta} \right) \right], & \begin{cases} b_1 \in \mathbb{R}^+ \\ b_2 \in \mathbb{R}^+ \\ \beta \in \mathbb{R}^+ \end{cases} \\ \delta \sim Gamma(c_1, c_2),\ \text{where: } g(\delta) = \frac{c_2^{c_1}}{\Gamma(c_1)} \delta^{c_1 - 1} [\exp(-c_2 \delta)], & \begin{cases} c_1 \in \mathbb{R}^+ \\ c_2 \in \mathbb{R}^+ \\ \delta \in \mathbb{R}^+ \end{cases} \end{cases} .$$

The prior probability distribution is then updated by calculating the likelihood function to obtain the posterior probability distribution under Bayes' rule:

$$g(\alpha, \beta, \delta; \boldsymbol{y}) = \frac{g(\boldsymbol{y}; \alpha, \beta, \delta g(\alpha) g(\beta) g(\delta)}{g(\boldsymbol{y})}, \tag{3}$$

that $g(\boldsymbol{y}; \alpha, \beta, \delta)$ is the joint pdf defined in relation (1). Also, $g(\alpha)$, $g(\beta)$ and $g(\delta)$ given in relation (2), are the prior pdfs for the parameters α, β and δ, respectively. The marginal likelihood function $g(\boldsymbol{y})$ is the joint pdf of the observations $\boldsymbol{y}$, and it does not include any parameters. Therefore, the relation (3) can be written as:

$$g(\alpha, \beta, \delta; \boldsymbol{y}) \propto g(\boldsymbol{y}; \alpha, \beta, \delta) g(\alpha) g(\beta) g(\delta), \tag{4}$$

that $\propto$ is a symbol to denote proportionality.

The likelihood function is the joint pdf of the observations $\boldsymbol{y}' = (y_1, y_2, y_3, \dots, y_n)$ given the parameters $\boldsymbol{\varphi}' = (\alpha, \beta, \delta)$. In other words, it is a function of parameters:

$$\begin{aligned} L(\alpha, \beta, \delta) &= \prod_{k=1}^{n} g(y_k; \alpha, \beta, \delta) \\ &= \prod_{k=1}^{n} \left[\frac{1}{\delta} \left\{ \exp\left[\frac{1}{\delta}(v - y_k) \right] + \frac{1}{2} \frac{\beta}{\delta^2} \right\} \left\{ \Psi \left[\frac{(v - y_k)}{\theta} - \delta\theta \right] \right\} \right] \\ &= \frac{1}{\delta^n} \left\{ \exp\left[\frac{1}{\delta} \sum_{k=1}^{n} (v - y_k) \right] + \frac{1}{2} \frac{n\beta}{\delta^2} \right\} \prod_{k=1}^{n} \left\{ \Psi \left[\frac{(v - y_k)}{\theta} - \delta\theta \right] \right\}. \end{aligned} \tag{5}$$

At this point, for calculating the joint pdf of the observations $\boldsymbol{y}' = (y_1, y_2, y_3, \dots, y_n)$ and the parameters $\boldsymbol{\varphi}' = (\alpha, \beta, \delta)$, the pdf $g(\boldsymbol{y}; \alpha, \beta, \delta)$ is multiplied by prior pdfs. Which can be calculated from relations (3), (4) and

(5):

$$g(\boldsymbol{y},\alpha,\beta,\delta) \propto \frac{1}{\delta^n}\left\{\exp\left[\frac{1}{\delta}\sum_{k=1}^{n}(v-y_k)\right]+\frac{1}{2}\frac{n\beta}{\delta^2}\right\}\prod_{k=1}^{n}\left\{\Psi\left[\frac{(v-y_k)}{\theta}-\delta\theta\right]\right\}$$

(6)

$$\times\frac{1}{\sqrt{2\pi a_2}}\left\{\exp\left[-\frac{(\alpha-a_1)^2}{2a_2}\right]\right\}\frac{{b_2}^{b_1}}{\Gamma(b_1)}\frac{1}{\beta^{b_1+1}}\left\{\exp\left(-\frac{b_2}{\beta}\right)\right\}\frac{{c_2}^{c_1}}{\Gamma(c_1)}\delta^{c_1-1}[\exp(-c_2\delta)].$$

Now, according to relation (4), it can be shown that the conditional posterior pdf is proportional to the joint pdf in relation (6):

$$g(\alpha,\beta,\delta;\boldsymbol{y}) \propto g(\boldsymbol{y},\alpha,\beta,\delta). \tag{7}$$

Consequently, the conditional pdf of α is:

$$g(\alpha;\boldsymbol{y},\beta,\delta) = \frac{g(\alpha,\beta,\delta;\boldsymbol{y})}{g(\beta,\delta;\boldsymbol{y})} \propto g(\boldsymbol{y},\alpha,\beta,\delta). \tag{8}$$

The above relation is calculated using relation (7) and considering that the pdf $g(\beta,\delta;\boldsymbol{y})$ does not include parameter α. Therefore, according to relation (8) and by ignoring all the expressions containing parameter α, it is obtained:

$$g(\alpha;\boldsymbol{y},\beta,\delta)\propto\left\{\exp\left[\tfrac{1}{\delta}\sum_{k=1}^{n}(v-y_k)\right]\right\}\left\{\exp\left[-\tfrac{(\alpha-a_1)^2}{2a_2}\right]\right\}\prod_{k=1}^{n}\left\{\Psi\left[\tfrac{(v-y_k)}{\theta}-\delta\theta\right]\right\}. \tag{9}$$

In the same way as above, the conditional posterior pdf of parameter β is calculated as the following:

$$g(\beta;\boldsymbol{y},\alpha,\delta)\propto\left[\exp\left(\frac{1}{2}\frac{n\beta}{\delta^2}\right)\right]\left\{\tfrac{1}{\beta^{b_1+1}}\left[exp\left(-\tfrac{b_2}{\beta}\right)\right]\right\}\prod_{k=1}^{n}\left\{\Psi\left[\tfrac{(v-y_k)}{\theta}-\delta\theta\right]\right\}. \tag{10}$$

Similarly, the conditional posterior pdf of parameter δ is obtained as follows:

$$\begin{aligned} g(\delta;\boldsymbol{y},\alpha,\beta) \propto & \left[\tfrac{1}{\delta^n}\left\{\exp\left[\tfrac{1}{\delta}\sum_{k=1}^{n}(v-y_k)\right]\right\}\right]\left[\exp\left(\tfrac{1}{2}\frac{n\beta}{\delta^2}\right)\right] \\ & \left\{\delta^{c_1-1}\left[\exp\left(-c_2\delta\right)\right]\right\}\prod_{k=1}^{n}\left\{\Psi\left[\tfrac{(v-y_k)}{\theta}-\delta\theta\right]\right\}. \end{aligned} \tag{11}$$

Sampling from the probability distributions of random variables presented in the relations (9), (10), and (11) is somewhat complicated since they do not have a known pdf form. Because the chosen prior pdfs do not always lead to logarithmically concave conditional pdfs [3].

3.2. **The ARM-H sampling method algorithm for finding estimators of parameters.** In this section, to solve the problem discussed in the previous section, the ARM-H sampling method is used for sampling from the probability distributions of calculated conditional pdfs in the relations (9), (10) and (11). This sampling method have an efficient and suitable algorithm for sampling from probability distributions that do not has a closed form [5, 25]. It should be noted that this sampling method is calculated in the "HI" package of "R" software with the "arms" command [19]. The ARM-H sampling method algorithm used in this paper is as follows:

REQUIRE: I and J.
1: Random initialization of $\boldsymbol{\varphi}'_{(0)} = \left(\alpha_{(0)}, \beta_{(0)}, \delta_{(0)}\right)$.
2: **for** $k = 1, 2, 3, \ldots, J$ **do**
3: $\alpha_{(k)} \sim g\left(\alpha; \boldsymbol{y}, \beta_{(k-1)}, \delta_{(k-1)}\right)$,
4: $\beta_{(k)} \sim g\left(\beta; \boldsymbol{y}, \alpha_{(k-1)}, \delta_{(k-1)}\right)$,
5: $\delta_{(k)} \sim g\left(\delta; \boldsymbol{y}, \alpha_{(k-1)}, \beta_{(k-1)}\right)$,
6: **end for**
7: $\widetilde{\alpha} = \frac{\sum_{k=I}^{J} \alpha_{(k)}}{I-J}$, $\widetilde{\beta} = \frac{\sum_{k=I}^{J} \beta_{(k)}}{I-J}$, $\widetilde{\delta} = \frac{\sum_{k=I}^{J} \delta_{(k)}}{I-J}$,
Ensure: $\widetilde{\boldsymbol{\varphi}}' = \left(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\delta}\right)$.

That I is the number of iterations and J is the burn-in period steps. Steps 3,4 and 5 are carried out using "arms" command in the "HI" package of the software "R". The conditional posterior pdfs $g\left(\alpha; \boldsymbol{y}, \beta_{(k-1)}, \delta_{(k-1)}\right)$, $g(\beta; \boldsymbol{y}, \alpha_{(k-1)}, \delta_{(k-1)})$ and $g(\delta; \boldsymbol{y}, \alpha_{(k-1)}, \beta_{(k-1)})$ are given in relations (9), (10) and (11) respectively. Also, in step 3, averages of all estimators after the burn-in period have been obtained.

4. Reminder of the estimation methods of ML and QML

In this section, two ML and QML estimation methods are reminded to compare later to the proposed DB estimation method [7,16,22]. The ML estimation method includes calculation of the log-likelihood function of the observation vector $\boldsymbol{y}' = (y_1, y_2, y_3, \ldots, y_n)$ sampled from the E-MN probability distribution and, then, maximization of this function for estimating $\boldsymbol{\varphi}' = (\alpha, \beta, \delta)$ the vector of the probability distribution parameters. On the other hand, the QML estimation method is a variant of the ML estimation method that combines the robustness of quantiles and the efficiency and consistency of ML estimators. In this estimation method, the first step is to transform the observation vector $\boldsymbol{y}' = (y_1, y_2, y_3, \ldots, y_n)$ sampled from the E-MN probability distribution into a vector of quantile estimators and, a vector of observation counts that occur in each inter-quantile range. In the next step, the quantile log-likelihood function will be calculated and, then, this function maximizes to estimate the parameters of the E-MN probability distribution.

5. Simulation study

In this section, a simulation study is presented to compare the efficiency of the proposed DB estimation method, the ML estimation method and the QML estimation method. For this purpose, $N = 100$ observation vectors $\boldsymbol{y}' = (y_1, y_2, y_3, \ldots, y_n)$ of size $n = 40, 80, 100, 110, 200, 300, 500$ and 1000 is sampled from an E-MN probability distribution. The parameters used are $\alpha = 13$,

$\theta = 7$ and $\delta = 9$. Also, it is used a burn-in of 2500 iterations out of 5000 Gibbs samples and, then, posterior pdfs are estimated at a further 2500 iterations. For the DB estimation method, the initialization of the parameters is chosen as:

$$\begin{cases} \alpha_{(0)} = \text{mean}(\boldsymbol{y}) - [0.8sd(\boldsymbol{y})] \\ \beta_{(0)} = \text{var}(\boldsymbol{y}) - [0.8sd(\boldsymbol{y})]^2 \\ \delta_{(0)} = \frac{1}{[0.8sd(\boldsymbol{y})]} \end{cases} . \tag{12}$$

Finally, the RMSEs of the parameters of the E-MN probability distribution are defined as:

$$\begin{cases} \text{RMSE}(\alpha) = \sqrt{\frac{1}{N}\sum_{r=1}^{N}\left[\widetilde{\alpha}_{(r)} - \alpha\right]^2} \\ \text{RMSE}(\theta) = \sqrt{\frac{1}{N}\sum_{r=1}^{N}\left[\widetilde{\theta}_{(r)} - \theta\right]^2} \\ \text{RMSE}(\delta) = \sqrt{\frac{1}{N}\sum_{r=1}^{N}\left[\widetilde{\delta}_{(r)} - \delta\right]^2} \end{cases} , \tag{13}$$

that subscript r labels the estimators obtained in the simulation of the r stage.

Tables 1, 2, 3 and 4 present the RMSEs of the E-MN probability distribution parameters for $n = 40$ and 80, $n = 100$ and 110, $n = 200$ and 300 and, $n = 500$ and 1000, respectively.

TABLE 1. RMSEs of E-MN probability distribution parameters for simulated observations of size $n = 40$ and 80 using the ML, QML and DB estimation methods.

$n \rightarrow$	$n = 40$			$n = 80$		
Estimation Method $\downarrow$	RMSE(α)	RMSE(θ)	RMSE(δ)	RMSE(α)	RMSE(θ)	RMSE(δ)
ML	3.3	2.1	3.3	2.3	1.4	2.3
QML	2.6	1.6	2.5	2.2	1.2	2.1
DB	2.8	1.7	3.5	2.0	1.3	2.8

TABLE 2. RMSEs of E-MN probability distribution parameters for simulated observations of size $n = 100$ and 110 using the ML, QML and DB estimation methods.

$n \rightarrow$	$n = 100$			$n = 110$		
Estimation Method $\downarrow$	RMSE(α)	RMSE(θ)	RMSE(δ)	RMSE(α)	RMSE(θ)	RMSE(δ)
ML	2.3	1.4	2.2	2.0	1.3	1.9
QML	2.1	1.3	1.9	2.3	1.5	2.1
DB	1.7	1.1	2.4	1.5	1.1	2.2

Table 3. RMSEs of E-MN probability distribution parameters for simulated observations of size $n = 200$ and 300 using the ML, QML and DB estimation methods.

$n \rightarrow$	$n = 200$			$n = 300$		
Estimation Method $\downarrow$	RMSE(α)	RMSE(θ)	RMSE(δ)	RMSE(α)	RMSE(θ)	RMSE(δ)
ML	1.2	1.0	1.1	1.1	0.9	1.1
QML	1.6	1.1	1.6	1.2	1.2	1.3
DB	1.1	0.9	1.2	0.9	0.7	1.0

Table 4. RMSEs of E-MN probability distribution parameters for simulated observations of size $n = 500$ and 1000 using the ML, QML and DB estimation methods.

$n \rightarrow$	$n = 500$			$n = 1000$		
Estimation Method $\downarrow$	RMSE(α)	RMSE(θ)	RMSE(δ)	RMSE(α)	RMSE(θ)	RMSE(δ)
ML	0.7	0.8	0.9	0.6	0.7	0.7
QML	0.9	1.1	1.0	0.7	0.7	0.8
DB	0.6	0.5	0.5	0.4	0.5	0.3

The values of all 4 Tables above show that when n is becomes greater, the RMSEs of the E-MN probability distribution estimated parameters obtained using the DB estimation method than the RMSEs of parameter estimates obtained using ML and QML estimation methods are closer to zero. This shows that the proposed estimation method is more efficient and robust than the other two estimation methods. Also, these Tables show that for n less than or equal to 100, the QML estimation method is more efficient than the ML estimation.

Figure 1 shows RMSEs of the estimated parameters of the E-MN probability distribution using the DB estimation method for different sizes of n. It is clear from the figure that RMSEs approach zero as n increases. This indicates the proper performance of the proposed estimation method.

Figure 2 shows the convergence of the estimated parameters of the E-MN probability distribution using the DB estimation method for different sizes of n. Clearly, the estimated parameters $\widetilde{\alpha}$, $\widetilde{\theta}$ and $\widetilde{\delta}$ converge to the true values of the parameters $\alpha = 13$, $\theta = 7$, and $\delta = 9$, respectively. This also indicates that the proposed estimation method is more efficient.

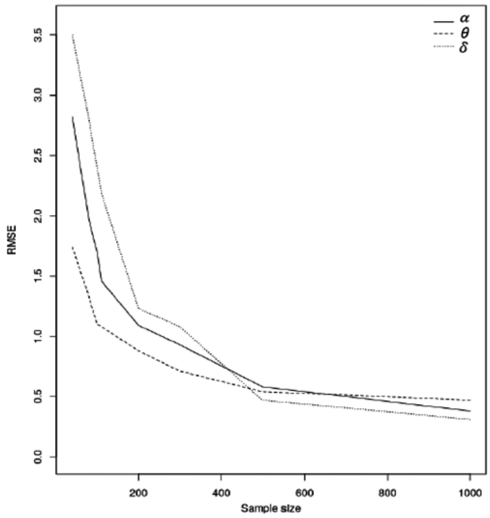

FIGURE 1. RMSEs of the estimated parameters of the E-MN probability distribution using the DB estimation method for different sizes of n.

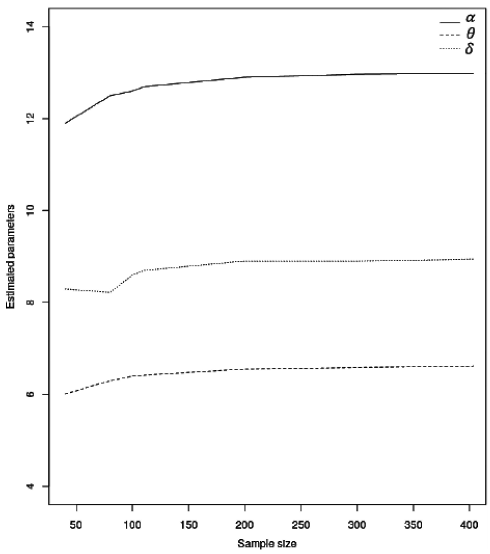

FIGURE 2. Convergence of the estimated parameters of the E-MN probability distribution using the DB estimation method for different sizes of n.

6. Real data sets analysis

In this section, two examples of real data sets are presented to compare the proposed DB estimation method with the QML and the MLE estimation methods, one for n less than 100 and the other for n greater than 100.

The first data set includes observations of 50 students. Observations from a study in which each student had to copy 28 letters of the English alphabet twice in capital letters on a Wacom, and then their RTs were recorded. The order of presentation of the letters was random. A white page on the Wacom allowed manuscripts to be collected to check answers. The student could not see or monitor the output because a separate screen hid their hands and the Wacom. That by the using the DB, ML and QML estimation methods for the data sets of this example, the results of Table 5 are obtained.

While the second data set includes observations of 230 students. Observations are from a study in which each student had to handwrite the label of a specific image on a Wacom. Namely, it was a picture naming task. All the RTs were then recorded. Also, a white page on the Wacom allowed manuscripts to be collected to check answers. By using the DB, ML and QML estimation methods for data sets of this example, the results of Table 6 are obtained.

TABLE 5. Estimated parameters of the E-MN probability distribution using the DB, ML and QML estimation methods for the first example.

$n \rightarrow$	$n = 50$		
Estimation Method $\downarrow$	$\widetilde{\alpha}$	$\widetilde{\theta}$	$\widetilde{\delta}$
ML	448.7	53.3	100
QML	440.7	60.8	106.3
DB	366.3	14.2	179.9

TABLE 6. Estimated parameters of the E-MN probability distribution using the DB, ML and QML estimation methods for the second example.

$n \rightarrow$	$n = 230$		
Estimation Method $\downarrow$	$\widetilde{\alpha}$	$\widetilde{\theta}$	$\widetilde{\delta}$
ML	927.9	198.6	100
QML	828.9	162.6	216.7
DB	466.4	180.1	540.5

The interesting thing is that although for both sample sizes, the parameter values estimated by ML and QML estimation methods are relatively close to each other, the parameter values estimated by the proposed DB estimation method are significantly different from their values.

On the other hand, Table 7 contains convergence diagnostic values $\hat{R}$ [26] of the DB estimated parameters $\widetilde{\alpha}$, $\widetilde{\theta}$, and $\widetilde{\delta}$. This is a criterion to compare the between-chain variability to the within-chain variability. To prove that the chains are properly converged, all the values of $\hat{R}$ must be less than 1.1. It should be noted that it is available in the "rstan" package of "R" software with the "Rhat" command.

TABLE 7. Convergence diagnostic values $\hat{R}$ of the DB estimated parameters $\widetilde{\alpha}$, $\widetilde{\theta}$, and $\widetilde{\delta}$ for sample sizes $n = 50$ and 230.

$n \to$	$n = 50$			$n = 230$		
Parameters	$\widetilde{\alpha}$	$\widetilde{\theta}$	$\widetilde{\delta}$	$\widetilde{\alpha}$	$\widetilde{\theta}$	$\widetilde{\delta}$
$\hat{R}$	1.00	0.99	0.99	1	1	0.99

All values in Table 7 are less than 1.1, and this also shows the effectiveness of the proposed estimation method.

Therefore, it seems reasonable to conclude that the proposed DB estimation method provides the closest estimates to true values of the parameters.

7. Discussion and conclusion

The use of RTs as one of the important behavioral criterions in humanities causes problems in practice due to the presence of positive skewness in its probability distribution. But various studies in recent years have shown that the probability distribution of RTs corresponds well with the E-MN probability distribution. As explained in the Introduction section of the paper, in humanities and in experimental conditions, the sample size is a limitation for choosing the appropriate method for estimating the E-MN probability distribution parameters. Since the use of the ML method to estimate the parameters of the E-MN probability distribution requires a greater sample size than those found in experimental conditions, the resulting estimators are not highly accurate. The aim of this paper is to propose an efficient and robust method for estimating the parameters of the E-MN probability distribution, named as the DB estimation method, for sample sized less than 100. For this, with the simultaneous use of simulation study and real data sets study analysis, it is compared the efficiency and robustness of the proposed DB estimation method than to two other important and practical estimation methods included ML and QML. Simulation results shows that although for sample sizes less than 100 the QML estimation method should be preferred for estimate of the parameters, for sample sizes greater than 100 the propose DB estimation method is more efficient than the two other estimation methods. In general, it is concluded that the RMSEs of the estimated parameters of the E-MN probability distribution using the DB estimation method approaches zero as n increases. This indicates the proper performance and excellent efficiency of the proposed estimation method. Finally, three estimation methods were studied on two real data sets, the first one with a sample size of $n = 50$ and one with a sample size of $n = 230$. The interesting thing is that although for both sample sizes, the parameter values estimated by ML and QML estimation methods are relatively close to each other, the parameter values estimated by the proposed DB estimation method are significantly different from their values. Therefore, it seems reasonable to conclude that, at least for great sample sizes, the proposed DB estimation method provides the closest estimates to true values. Thus, in general, it can be said that the proposed DB estimation method is the most trustworthy method for estimating the E-MN probability distribution parameters.

Conflict of interest

The author has no conflicts to disclose.

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