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NUMERICAL SOLUTIONS FOR A CLASS OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. The aim of this manuscript is to introduce and analyze a stochastic finite difference scheme for Itô stochastic partial differential equations. We also discuss the consistency, stability, and convergence for the stochastic finite difference scheme. The numerical simulations obtained from the proposed stochastic finite difference scheme show the efficiency of the suggested stochastic finite difference scheme.

Keywords: Stochastic partial differential equations, Stochastic finite difference scheme, Stability, Consistency, Convergence.
2020 MSC: 60H15, 65M12.

1. **Introduction**

Over the past few decades, stochastic partial differential equations (SPDEs) have been considered for the modeling of realistic phenomena due to their possession of real behaviors. The footsteps of SPDEs are found in various branches of sciences, such as, chemical physics, fluid mechanics, biology, and economics. Thus, the modelings related to such phenomena are worthy to be studied more deeply and therefore many researchers have devoted growing attention to the analysis of SPDEs. It is also should be noted that finding the exact solutions of SPDEs in many cases is formidable, and therefore it is necessary to explore their numerical solutions by using some effective numerical methods.

The most popular numerical methods for the approximation of the solution of SPDEs are stochastic finite difference (SFD) schemes, finite element methods (FEMs) [1, 13] and spectral methods [5]. In [2, 9, 10], Namjoo et al. investigated the convergence and stability of three stochastic finite difference schemes for a class of SPDEs. Yoo presented the semi-discretization of SPDEs by SFD scheme [15]. Roth used an explicit SFD scheme to approximate the solution of some stochastic hyperbolic equations [11]. In [3], a compact finite difference technique is employed for detecting the numerical solutions of

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SPDEs. In [14], a numerical scheme for solving a stochastic Fitzhugh-Nagumo equation was proposed. Youssri and Muttardi suggested a mingle Tau-finite difference method for stochastic first-order partial differential equations [16]. In [6], wavelet optimized finite difference B-spline polynomial chaos method has been applied for solving SPDEs. In [8], a numerical scheme based on finite difference scheme and radial basis functions was suggested for solving fractional stochastic advection-diffusion equations. In [4], the homotopy perturbation method was used to solve fractional partial differential equations. This manuscript provides an SFD scheme for solving the SPDEs with one dimensional white noise process. The main structure of this paper is summarized as follows: In Section 2, we introduce an explicit SFD scheme to approximate the solution of SPDEs. In Section 3, we explore the consistency, stability, and convergence for the proposed SFD scheme. Finally, we terminate the paper in the last section by some numerical results.

2. Numerical scheme for SPDEs

Consider the following SPDE [12]

$$U_t(x,t) + \alpha U_x(x,t) = \lambda U_{xx}(x,t) + \eta U(x,t)\dot{B}(t), \ x \in [0,1], \ t \in [0,1], \tag{1}$$

with the initial and boundary conditions

$$U(x,0) = U_0(x),$$

and

$$U(0,t) = U_1(t), \quad U(1,t) = U_2(t),$$

where the functions $U_0(x)$, $U_1(t)$ and $U_2(t)$ are known. Here the parameters α, λ are positive constants and η is a real number. Also, $B(t)$ is called the Brownian motion [7]. A discretization scheme transforms finding a solution to an SPDE into a finite dimensional problem. One class of numerical methods for the approximation of the solution of an SPDE is standard finite difference schemes. These schemes use fixed equidistant grid points to convert differential operators into discrete ones by using neighboring points. Let us consider a uniform space and time gird in the time-space lattice, with the stepsizes Δx and Δt, respectively. Let u_m^n denotes the value of numerical approximation at the node point $(x_m, t_n) = (m\Delta x, n\Delta t)$. In order to get an SFD scheme, the time and space partial derivatives in the SPDE (1) are approximated as follows:

$$\begin{aligned}
U_t(x_m,t_n) &\approx \frac{u_m^{n+1} - u_m^n}{\Delta t},\\
U_x(x_m,t_n) &\approx \frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x},\\
U_{xx}(x_m,t_n) &\approx \frac{1}{\Delta x^2}\left(-\frac{1}{560}u_{m-4}^n + \frac{8}{315}u_{m-3}^n - \frac{1}{5}u_{m-2}^n + \frac{8}{5}u_{m-1}^n\right.\\
&\qquad \left. - \frac{205}{72}u_m^n + \frac{8}{5}u_{m+1}^n - \frac{1}{5}u_{m+2}^n + \frac{8}{315}u_{m+3}^n - \frac{1}{560}u_{m+4}^n\right).
\end{aligned} \tag{2}$$

In fact, the estimation of (2) is an eighth order approximation of $U_{xx}(m\Delta x, n\Delta t)$, showing the local truncation error is $O((\Delta x)^8)$. Thus, the stochastic finite difference scheme approximates the SPDE (1) to

$$\begin{aligned} u_m^{n+1} &= \left(\frac{8}{5}\sigma+\gamma\right)u_{m-1}^n + \left(1-\frac{205}{72}\sigma\right)u_m^n + \left(\frac{8}{5}\sigma-\gamma\right)u_{m+1}^n \\ &+\sigma\left(-\frac{1}{560}u_{m-4}^n + \frac{8}{315}u_{m-3}^n - \frac{1}{5}u_{m-2}^n - \frac{1}{5}u_{m+2}^n\right. \\ &\left.+\frac{8}{315}u_{m+3}^n - \frac{1}{560}u_{m+4}^n\right) + \eta\, u_m^n \Delta B_n, \\ & u(x_m,0)=u_0(x_m), \quad u(0,t_n)=u_1(t_n), \quad u(1,t_n)=u_2(t_n), \end{aligned} \tag{3}$$

where $\gamma = \alpha\frac{\Delta t}{2\Delta x}$, $\sigma = \lambda\frac{\Delta t}{(\Delta x)^2}$ and $\Delta B_n = B((n+1)\Delta t) - B(n\Delta t)$ is called the Brownian motion increments with the property $\Delta B_n \sim \mathcal{N}(0, \Delta t)$. Here $\Delta x = x_{m+1} - x_m$, $\Delta t = t_{n+1} - t_n$, where $0 \le m \le M-1$ and $0 \le n \le N-1$.

3. **Consistency, stability and convergence for the proposed stochastic finite difference scheme**

In this section, we declare and prove the main results. For this purpose, by integrating both sides of the SPDE (1) with respect to t from 0 to t, we get

$$\begin{aligned} U(x,t) - U(x,0) + \alpha\int_0^t U_x(x,s)\,\mathrm{d}s &= \lambda\int_0^t U_{xx}(x,s)\;\mathrm{d}s \\ &+ \eta\int_0^t U(x,s)\;\mathrm{d}B(s). \end{aligned} \tag{4}$$

Now, by substituting the value $t = t_{n+1}$ and $t = t_n$, into (4) we obtain

$$\begin{aligned} U(x,t_{n+1}) - U(x,0) + \alpha\int_0^{t_{n+1}} U_x(x,s)\;\mathrm{d}s &= \lambda\int_0^{t_{n+1}} U_{xx}(x,s)\;\mathrm{d}s \\ &+ \eta\int_0^{t_{n+1}} U(x,s)\;\mathrm{d}B(s), \end{aligned} \tag{5}$$

and

$$\begin{aligned} U(x,t_n) - U(x,0) + \alpha\int_0^{t_n} U_x(x,s)\;\mathrm{d}s &= \lambda\int_0^{t_n} U_{xx}(x,s)\,\mathrm{d}s \\ &+ \eta\int_0^{t_n} U(x,s)\,\mathrm{d}B(s). \end{aligned} \tag{6}$$

Subtracting (6) from (5) and letting $x = x_m$ we arrive at

$$\begin{aligned} & U(x_m,t_{n+1}) - U(x_m,t_n) + \alpha\int_{t_n}^{t_{n+1}} U_x(x_m,s)\,\mathrm{d}s \\ & - \lambda\int_{t_n}^{t_{n+1}} U_{xx}(x_m,s)\,\mathrm{d}s - \eta\int_{t_n}^{t_{n+1}} U(x_m,s)\,\mathrm{d}B(s) = 0. \end{aligned} \tag{7}$$

The equation (7) can be written as

$$\mathcal{L}U(x_m, t_n) = \mathcal{F},$$

where

$$\begin{aligned}
\mathcal{L}U(x_m, t_n) &= U(x_m, t_{n+1}) - U(x_m, t_n) \\
&+ \alpha \int_{t_n}^{t_{n+1}} U_x(x_m, s)\,\mathrm{d}s - \lambda \int_{t_n}^{t_{n+1}} U_{xx}(x_m, s)\,\mathrm{d}s \\
&- \eta \int_{t_n}^{t_{n+1}} U(x_m, s)\,\mathrm{d}B(s),
\end{aligned} \tag{8}$$

and

$$\mathcal{F} = 0.$$

In order to get a numerical approximation for the equation (7), we consider the following approximations

$$\begin{aligned}
\int_{t_n}^{t_{n+1}} u_x(x_m, s)\,\mathrm{d}s &\approx \int_{t_n}^{t_{n+1}} u_x(x_m, t_n)\,\mathrm{d}s, \\
\int_{t_n}^{t_{n+1}} u_{xx}(x_m, s)\,\mathrm{d}s &\approx \int_{t_n}^{t_{n+1}} u_{xx}(x_m, t_n)\,\mathrm{d}s, \\
\int_{t_n}^{t_{n+1}} u(x_m, s)\,\mathrm{d}B(s) &\approx \int_{t_n}^{t_{n+1}} u(x_m, t_n)\,\mathrm{d}B(s).
\end{aligned} \tag{9}$$

Substituting the approximations (2) into (9) and using the equation (7), we have

$$\begin{aligned}
&u(x_m, t_{n+1}) - u(x_m, t_n) + \frac{\alpha \Delta t}{2\Delta x}(u(x_{m+1}, t_n) - u(x_{m-1}, t_n)) \\
&- \frac{\lambda \Delta t}{(\Delta x)^2}\Bigg\{ \frac{-1}{560} u(x_{m-4}, t_n) + \frac{8}{315} u(x_{m-3}, t_n) \\
&- \frac{1}{5} u(x_{m-2}, t_n) + \frac{8}{5} u(x_{m-1}, t_n) - \frac{205}{72} u(x_m, t_n) \\
&+ \frac{8}{5} u(x_{m+1}, t_n) - \frac{1}{5} u(x_{m+2}, t_n) + \frac{8}{315} u(x_{m+3}, t_n) \\
&- \frac{1}{560} u(x_{m+4}, t_n) \Bigg\} - \eta u(x_m, t_n)\Delta B_n = 0.
\end{aligned} \tag{10}$$

The difference equation (10) can be written as

$$\mathcal{L}_m^n u_m^n = \mathcal{F}_m^n,$$

where

$$\begin{aligned}
\mathcal{L}_m^n u_m^n &= u(x_m, t_{n+1}) - u(x_m, t_n) + \frac{\alpha \Delta t}{2\Delta x}(u(x_{m+1}, t_n) - u(x_{m-1}, t_n)) \\
&- \frac{\lambda \Delta t}{(\Delta x)^2}\Bigg\{ \frac{-1}{560} u(x_{m-4}, t_n) + \frac{8}{315} u(x_{m-3}, t_n)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{5}u(x_{m-2},t_n)+\frac{8}{5}u(x_{m-1},t_n)-\frac{205}{72}u(x_m,t_n)\\
&+\frac{8}{5}u(x_{m+1},t_n)-\frac{1}{5}u(x_{m+2},t_n)+\frac{8}{315}u(x_{m+3},t_n)\\
&-\frac{1}{560}u(x_{m+4},t_n)\bigg\}-\eta u(x_m,t_n)\Delta B_n,
\end{aligned}
$$

and

$$\mathcal{F}_m^n=0.$$

Let us consider an SPDE of the form $\mathcal{L}U=\mathcal{F}$. Suppose that $\mathcal{L}_m^n u_m^n=\mathcal{F}_m^n$ denotes the proposed SFD scheme. In order to prove consistency, stability, and convergence of the stochastic finite difference scheme (3), it is necessary to consider a norm. Let $\{u_m^n\}$ be a sequence of numerical solutions that are obtained by a stochastic finite difference scheme (3). For this sequence, we define $\|u^n\|=(\sup_{0\le m\le M}|u_m^n|^2)^{\frac{1}{2}}$, where $u^n=(u_0^n,u_1^n,\dots,u_M^n)$. For additional details concerning the concepts of consistency, stability and convergence, see [11].

Definition 3.1. A stochastic finite difference scheme $\mathcal{L}_m^n u_m^n=\mathcal{F}_m^n$ is said to be consistent in mean square with SPDE $\mathcal{L}U=\mathcal{F}$ at point (x,t), if for any continuously differentiable $\Psi(x,t)$, we have

$$\mathbb{E}\|\left(\mathcal{L}\Psi(m\Delta x,n\Delta t)-\mathcal{F}(m\Delta x,n\Delta t)\right)-\left(\mathcal{L}_m^n\Psi(m\Delta x,n\Delta t)-\mathcal{F}_m^n\right)\|^2\to 0,$$

while $\Delta x\to 0$, $\Delta t\to 0$ and $(m\Delta x,(n+1)\Delta t)\to(x,t)$.

Theorem 3.2. *The stochastic finite difference scheme (3) is consistent in the sense of mean square.*

Proof. Let $\Psi(x,t)$ be a continuously differentiable function, then we have:

$$
\begin{aligned}
\mathcal{L}\Psi(m\Delta x,n\Delta t)&=\Psi(m\Delta x,(n+1)\Delta t)-\Psi(m\Delta x,n\Delta t)\\
&\quad+\alpha\int_{t_n}^{t_{n+1}}\Psi_x(m\Delta x,s)\,\mathrm{d}s-\lambda\int_{t_n}^{t_{n+1}}\Psi_{xx}(m\Delta x,s)\,\mathrm{d}s\\
&\quad-\eta\int_{t_n}^{t_{n+1}}\Psi(m\Delta x,s)\,\mathrm{d}B(s),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}_m^n\Psi&=\Psi(m\Delta x,(n+1)\Delta t)-\Psi(m\Delta x,n\Delta t)\\
&\quad+\frac{\alpha\Delta t}{2\Delta x}(\Psi((m+1)\Delta x,n\Delta t)-\Psi((m-1)\Delta x,n\Delta t))\\
&\quad-\frac{\lambda\Delta t}{(\Delta x)^2}\bigg(-\frac{1}{560}\Psi((m-4)\Delta x,n\Delta t)\\
&\quad+\frac{8}{315}\Psi((m-3)\Delta x,n\Delta t)-\frac{1}{5}\Psi((m-2)\Delta x,n\Delta t)
\end{aligned}
$$

$$+\frac{8}{5}\Psi((m-1)\Delta x,n\Delta t)-\frac{205}{72}\Psi(m\Delta x,n\Delta t)$$
$$+\frac{8}{5}\Psi((m+1)\Delta x,n\Delta t)-\frac{1}{5}\Psi((m+2)\Delta x,n\Delta t)$$
$$+\frac{8}{315}\Psi((m+3)\Delta x,n\Delta t)-\frac{1}{560}\Psi((m+4)\Delta x,n\Delta t)\Big)$$
$$-\eta\,\Psi(m\Delta x,n\Delta t)\Delta B_n.$$

Using the square property of the Itô integral, we deduce that

$$\mathbb{E}|\mathcal{L}\Psi(m\Delta x,n\Delta t)-\mathcal{L}_m^n\Psi|^2$$
$$=\mathbb{E}\bigg|\alpha\int_{t_n}^{t_{n+1}}\bigg(\Psi_x(m\Delta x,s)-\bigg(\frac{1}{2\Delta x}(\Psi((m+1)\Delta x,n\Delta t)$$
$$-\Psi((m-1)\Delta x,n\Delta t))\bigg)\bigg)\,\mathrm{d}s$$
$$-\lambda\int_{t_n}^{t_{n+1}}\bigg(\Psi_{xx}(m\Delta x,s)-\frac{1}{(\Delta x)^2}\bigg(-\frac{1}{560}\Psi((m-4)\Delta x,n\Delta t)$$
$$+\frac{8}{315}\Psi((m-3)\Delta x,n\Delta t)-\frac{1}{5}\Psi((m-2)\Delta x,n\Delta t)$$
$$+\frac{8}{5}\Psi((m-1)\Delta x,n\Delta t)-\frac{205}{72}\Psi(m\Delta x,n\Delta t)$$
$$+\frac{8}{5}\Psi((m+1)\Delta x,n\Delta t)-\frac{1}{5}\Psi((m+2)\Delta x,n\Delta t)$$
$$+\frac{8}{315}\Psi((m+3)\Delta x,n\Delta t)-\frac{1}{560}\Psi((m+4)\Delta x,n\Delta t)\bigg)\,\mathrm{d}s$$
$$-\eta\int_{t_n}^{t_{n+1}}(\Psi(m\Delta x,s)-\Psi(m\Delta x,n\Delta t))\,\mathrm{d}B(s)\bigg|^2$$
$$\leq 4\alpha^2\mathbb{E}\bigg|\int_{t_n}^{t_{n+1}}\bigg(\Psi_x(m\Delta x,s)-\bigg(\frac{1}{2\Delta x}(\Psi((m+1)\Delta x,n\Delta t)$$
$$-\Psi((m-1)\Delta x,n\Delta t))\bigg)\bigg)\,\mathrm{d}s\bigg|^2$$
$$+4\lambda^2\mathbb{E}\bigg|\int_{t_n}^{t_{n+1}}\bigg(\Psi_{xx}(m\Delta x,s)-\frac{1}{(\Delta x)^2}\bigg(-\frac{1}{560}\Psi((m-4)\Delta x,n\Delta t)$$
$$+\frac{8}{315}\Psi((m-3)\Delta x,n\Delta t)-\frac{1}{5}\Psi((m-2)\Delta x,n\Delta t)$$
$$+\frac{8}{5}\Psi((m-1)\Delta x,n\Delta t)-\frac{205}{72}\Psi(m\Delta x,n\Delta t)$$
$$+\frac{8}{5}\Psi((m+1)\Delta x,n\Delta t)-\frac{1}{5}\Psi((m+2)\Delta x,n\Delta t)$$

$$+\frac{8}{315}\Psi((m+3)\Delta x, n\Delta t) - \frac{1}{560}\Psi((m+4)\Delta x, n\Delta t)\Big) \, \mathrm{d}s\Bigg|^2$$

$$+ 4\eta^2 \int_{t_n}^{t_{n+1}} |\Psi(m\Delta x, s) - \Psi(m\Delta x, n\Delta t)|^2 \, \mathrm{d}s. \tag{11}$$

Therefore, we can see $\mathbb{E}|\mathcal{L}\Psi(m\Delta x, n\Delta t) - \mathcal{L}_m^n\Psi|^2$ tends to zero, when m, $n \to \infty$. □

Definition 3.3. A stochastic finite difference scheme $\mathcal{L}_m^n u_m^n = \mathcal{F}_m^n$ is called stable with respect to a norm in the sense of mean square if there exist positive constants $\overline{\Delta x}$, $\overline{\Delta t}$ and nonnegative constants C and θ such that

$$\mathbb{E}\|u^{n+1}\|^2 \le Ce^{\theta t}\mathbb{E}\|u^0\|^2,$$

for all $t = (n+1)\Delta t$, $0 \le \Delta x \le \overline{\Delta x}$, and $0 \le \Delta t \le \overline{\Delta t}$, where

$$u^{n+1} = \left(u_0^{n+1}, u_1^{n+1}, \dots, u_M^{n+1}\right)^T.$$

Our next objective is to investigate the stability analysis of the SFD scheme (3).

Theorem 3.4. *If* $\frac{5\gamma}{8} \le \sigma \le \frac{72}{205}$ *and* $t = (n+1)\Delta t$, *then the stochastic finite difference scheme (3) is stable with respect to sup-norm.*

Proof. By applying $\mathbb{E}|\cdot|^2$ from both sides of Eq. (3) and considering the independence of the Brownian motion increments, we have

$$\mathbb{E}\left|u_m^{n+1}\right|^2 = \mathbb{E}\Bigg|\left(\frac{8}{5}\sigma+\gamma\right)u_{m-1}^n + \left(1-\frac{205}{72}\sigma\right)u_m^n + \left(\frac{8}{5}\sigma-\gamma\right)u_{m+1}^n$$
$$+\sigma\left(-\frac{1}{560}u_{m-4}^n + \frac{8}{315}u_{m-3}^n - \frac{1}{5}u_{m-2}^n - \frac{1}{5}u_{m+2}^n + \frac{8}{315}u_{m+3}^n - \frac{1}{560}u_{m+4}^n\right)\Bigg|^2$$
$$+ \eta^2\Delta t\mathbb{E}|u_m^n|^2.$$

Now by the hypothesis of $\frac{5\gamma}{8} \le \sigma \le \frac{72}{205}$, we obtain that

$$\mathbb{E}\left|u_m^{n+1}\right|^2 \le \left(\frac{8}{5}\sigma+\gamma\right)^2\mathbb{E}|u_{m-1}^n|^2 + \left(1-\frac{205}{72}\sigma\right)^2\mathbb{E}|u_m^n|^2$$
$$+ \left(\frac{8}{5}\sigma-\gamma\right)^2\mathbb{E}|u_{m+1}^n|^2 + \sigma^2\mathbb{E}\Bigg|-\frac{1}{560}u_{m-4}^n + \frac{8}{315}u_{m-3}^n - \frac{1}{5}u_{m-2}^n$$
$$-\frac{1}{5}u_{m+2}^n + \frac{8}{315}u_{m+3}^n - \frac{1}{560}u_{m+4}^n\Bigg|^2$$
$$+2\left(\frac{8}{5}\sigma+\gamma\right)\left(1-\frac{205}{72}\sigma\right)\mathbb{E}|u_{m-1}^n u_m^n|$$
$$+ 2\left(\frac{8}{5}\sigma+\gamma\right)\left(\frac{8}{5}\sigma-\gamma\right)\mathbb{E}|u_{m-1}^n u_{m+1}^n|$$

$$
\begin{aligned}
&+2\left(\frac{8}{5}\sigma+\gamma\right)\sigma\mathbb{E}\left|u_{m-1}^n\left(-\frac{1}{560}u_{m-4}^n+\frac{8}{315}u_{m-3}^n-\frac{1}{5}u_{m-2}^n\right.\right.\\
&\left.\left.-\frac{1}{5}u_{m+2}^n+\frac{8}{315}u_{m+3}^n-\frac{1}{560}u_{m+4}^n\right)\right|\\
&+2\left(1-\frac{205}{72}\sigma\right)\left(\frac{8}{5}\sigma-\gamma\right)\mathbb{E}|u_m^n u_{m+1}^n|\\
&+2\left(1-\frac{205}{72}\sigma\right)\sigma\mathbb{E}\left|u_m^n\left(-\frac{1}{560}u_{m-4}^n+\frac{8}{315}u_{m-3}^n-\frac{1}{5}u_{m-2}^n\right.\right.\\
&\left.\left.-\frac{1}{5}u_{m+2}^n+\frac{8}{315}u_{m+3}^n-\frac{1}{560}u_{m+4}^n\right)\right|\\
&+2\left(\frac{8}{5}\sigma-\gamma\right)\sigma\mathbb{E}\left|u_{m+1}^n\left(-\frac{1}{560}u_{m-4}^n+\frac{8}{315}u_{m-3}^n-\frac{1}{5}u_{m-2}^n\right.\right.\\
&\left.\left.-\frac{1}{5}u_{m+2}^n+\frac{8}{315}u_{m+3}^n-\frac{1}{560}u_{m+4}^n\right)\right|+\eta^2\Delta t\mathbb{E}|u_m^n|^2\\
\leq&\left\{\left[\left(\frac{8}{5}\sigma+\gamma\right)+\left(1-\frac{205}{72}\sigma\right)+\left(\frac{8}{5}\sigma-\gamma\right)+\sigma\left(\frac{1}{560}+\frac{8}{315}+\frac{1}{5}\right.\right.\right.\\
&\left.\left.\left.+\frac{1}{5}+\frac{8}{315}+\frac{1}{560}\right)\right]^2+\eta^2\Delta t\right\}\sup_m\mathbb{E}|u_m^n|^2\\
=&\left(1+\frac{113}{70}\sigma+\frac{12769}{19600}\sigma^2+\eta^2\Delta t\right)\sup_m\mathbb{E}|u_m^n|^2.
\end{aligned}
$$

Therefore, we have

$$\sup_m\mathbb{E}|u_m^{n+1}|^2\leq\left(1+\frac{113}{70}\sigma+\frac{12769}{19600}\sigma^2+\eta^2\Delta t\right)\sup_m\mathbb{E}|u_m^n|^2.$$

It will thus be sufficient to choose δ such that

$$\frac{113}{70}\sigma+\frac{12769}{19600}\sigma^2+\eta^2\Delta t\leq\delta^2\Delta t,$$

and consequently

$$\sup_m\mathbb{E}|u_m^{n+1}|^2\leq(1+\delta^2\Delta t)\sup_m\mathbb{E}|u_m^n|^2\leq\cdots\leq(1+\delta^2\Delta t)^{n+1}\sup_m\mathbb{E}|u_m^0|^2. \tag{12}$$

Substituting $\Delta t=\frac{t}{(n+1)}$ in (12), we arrive at

$$\mathbb{E}\|u^{n+1}\|^2\leq\left(1+\frac{\delta^2 t}{n+1}\right)^{n+1}\mathbb{E}\|u^0\|^2\leq e^{\delta^2 t}\mathbb{E}\|u^0\|^2. \tag{13}$$

Taking $C=1$ and $\theta=\delta^2$, we conclude that the stochastic finite difference scheme (3) is stable. □

Throughout the remainder of the manuscript we assume u^{n+1} and U^{n+1} denote the numerical and exact solution at the time-level $n+1$, respectively.

Definition 3.5. The stochastic difference scheme $\mathcal{L}_m^n u_m^n = \mathcal{F}_m^n$, which approximates the SPDE $\mathcal{L}U = \mathcal{F}$, is convergent in mean square at time $t = (n+1)\Delta t$, if $\mathbb{E}\|u^{n+1} - U^{n+1}\|^2$ tends to zero when $\Delta x \to 0$, and $\Delta t \to 0$.

In order to get convergence results, we need the following definition.

Definition 3.6. Let Ω be a domain in $\mathbb{R}^n$ $(n \geq 2)$. The Sobolev space of order m, denoted by $H^m(\Omega)$, is defined to be the space consisting of those functions in $L^2(\Omega)$ that, together with all their weak partial derivatives up to and including those of order m, belong to $L^2(\Omega)$.

Theorem 3.7. *Suppose that* $U \in H^3((0,1) \times (0,1))$. *If* $\frac{5\gamma}{8} \leq \sigma \leq \frac{72}{205}$, *then the SFD scheme* (3) *with respect to* sup*–norm is convergent.*

Proof. From (3) it follows that

$$\begin{aligned} u_m^{n+1} = u_m^n - \alpha\Delta t \frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x} + \lambda \frac{\Delta t}{(\Delta x)^2} \Big(& -\frac{1}{560} u_{m-4}^n + \frac{8}{315} u_{m-3}^n \\ & -\frac{1}{5} u_{m-2}^n + \frac{8}{5} u_{m-1}^n - \frac{205}{72} u_m^n + \frac{8}{5} u_{m+1}^n \\ & -\frac{1}{5} u_{m+2}^n + \frac{8}{315} u_{m+3}^n - \frac{1}{560} u_{m+4}^n \Big) + \eta\, u_m^n \Delta B_n. \end{aligned}$$

According to the Taylor expansion $U_x(x,s)$ and $U_{xx}(x,s)$ with respect to the space variable,

$$\begin{aligned} U_m^{n+1} &= U_m^n - \alpha \int_{t_n}^{t_{n+1}} U_x(x_m, s)\,\mathrm{d}s + \lambda \int_{t_n}^{t_{n+1}} U_{xx}(x_m, s)\ \mathrm{d}s \\ &\quad + \eta \int_{t_n}^{t_{n+1}} U(x_m, s)\ \mathrm{d}B(s) \\ &= U_m^n - \alpha \int_{t_n}^{t_{n+1}} \Big(\frac{U_{m+1}^n - U_{m-1}^n}{2\Delta x} \\ &\quad - \frac{(\Delta x)^2}{12} \Big(U_{xxx}((m+\nu_1)\Delta x, s) + U_{xxx}((m+\nu_2)\Delta x, s) \Big) \\ &\quad + \lambda \int_{t_n}^{t_{n+1}} \Big(\frac{1}{\Delta x^2} \Big(-\frac{1}{560} U_{m+4}^n + \frac{8}{315} U_{m-3}^n - \frac{1}{5} U_{m-2}^n + \frac{8}{5} U_{m-1}^n \\ &\quad - \frac{205}{72} U_m^n + \frac{8}{5} U_{m+1}^n - \frac{1}{5} U_{m+2}^n + \frac{8}{315} U_{m+3}^n - \frac{1}{560} U_{m+4}^n \Big) \\ &\quad + \frac{256}{496125} (\Delta x)^8 \Big(U^{(10)}((m+\mu_1\Delta x), s) + U^{(10)}((m+\mu_2\Delta x), s) \Big) \\ &\quad - \frac{81}{196000} (\Delta x)^8 \Big(U^{(10)}((m+\mu_3\Delta x), s) + U^{(10)}((m+\mu_4\Delta x), s) \Big) \\ &\quad + \frac{4}{70875} (\Delta x)^8 \Big(U^{(10)}((m+\mu_5\Delta x), s) + U^{(10)}((m+\mu_6\Delta x), s) \Big) \end{aligned}$$

$$- \frac{1}{2268000}(\Delta x)^8\Big(U^{(10)}((m+\mu_7\Delta x),s)$$
$$+U^{(10)}((m+\mu_8\Delta x),s)\Big)\Bigg)\,\mathrm{d}s + \eta\int_{t_n}^{t_{n+1}} U(x_m,s)\ \mathrm{d}B(s).$$

where ν_1, ν_2, μ_1, μ_2, μ_3, μ_4, μ_5, μ_6, μ_7 and $\mu_8 \in [0,1]$. Let $e_m^n = U_m^n - u_m^n$ be the error at the node point (x_m, t_n). Hence, we get

$$e_m^{n+1} = e_m^n - \alpha\int_{t_n}^{t_{n+1}}\Bigg(\frac{e_{m+1}^n - e_{m-1}^n}{2\Delta x}$$
$$- \frac{(\Delta x)^2}{12}\Big(U_{xxx}((m+\nu_1)\Delta x,s) + U_{xxx}((m+\nu_2)\Delta x,s)\Big)$$
$$+ \lambda\int_{t_n}^{t_{n+1}}\Bigg(\frac{1}{\Delta x^2}\Bigg(-\frac{1}{560}e_{m-4}^n + \frac{8}{315}e_{m-3}^n - \frac{1}{5}e_{m-2}^n + \frac{8}{5}e_{m-1}^n$$
$$-\frac{205}{72}e_m^n + \frac{8}{5}e_{m+1}^n - \frac{1}{5}e_{m+2}^n + \frac{8}{315}e_{m+3}^n - \frac{1}{560}e_{m+4}^n\Bigg)$$
$$+ \frac{256}{496125}(\Delta x)^8\Big(U^{(10)}((m+\mu_1\Delta x),s) + U^{(10)}((m+\mu_2\Delta x),s)\Big)$$
$$- \frac{81}{196000}(\Delta x)^8\Big(U^{(10)}((m+\mu_3\Delta x),s) + U^{(10)}((m+\mu_4\Delta x),s)\Big)$$
$$+ \frac{4}{70875}(\Delta x)^8\Big(U^{(10)}((m+\mu_5\Delta x),s) + U^{(10)}((m+\mu_6\Delta x),s)\Big)$$
$$- \frac{1}{2268000}(\Delta x)^8\Big(U^{(10)}((m+\mu_7\Delta x),s)$$
$$+U^{(10)}((m+\mu_8\Delta x),s)\Big)\Bigg)\,\mathrm{d}s$$
$$+ \eta\int_{t_n}^{t_{n+1}}(U(x_m,s) - u_m^n)\ \mathrm{d}B(s).$$

Therefore, from the last equality we see that

$$e_m^{n+1} = \left(\frac{8}{5}\sigma + \gamma\right)e_{m-1}^n + \left(1 - \frac{205}{72}\sigma\right)e_m^n + \left(\frac{8}{5}\sigma - \gamma\right)e_{m+1}^n$$
$$+ \sigma\Bigg(-\frac{1}{560}e_{m-4}^n + \frac{8}{315}e_{m-3}^n - \frac{1}{5}e_{m-2}^n$$
$$-\frac{1}{5}e_{m+2}^n + \frac{8}{315}e_{m+3}^n - \frac{1}{560}e_{m+4}^n\Bigg)$$
$$- \alpha\int_{t_n}^{t_{n+1}}\Bigg[-\frac{(\Delta x)^2}{12}\Big(U_{xxx}((m+\nu_1)\Delta x,s)$$
$$+U_{xxx}((m+\nu_2)\Delta x,s)\Big)\Bigg]\,\mathrm{d}s$$

$$
\begin{aligned}
&+\lambda\int_{t_n}^{t_{n+1}}\Big[\frac{256}{496125}(\Delta x)^8\Big(U^{(10)}((m+\mu_1\Delta x),s)\\
&+U^{(10)}((m+\mu_2\Delta x),s)\Big)\\
&-\frac{81}{196000}(\Delta x)^8\Big(U^{(10)}((m+\mu_3\Delta x),s)+U^{(10)}((m+\mu_4\Delta x),s)\Big)\\
&+\frac{4}{70875}(\Delta x)^8\Big(U^{(10)}((m+\mu_5\Delta x),s)+U^{(10)}((m+\mu_6\Delta x),s)\Big)\\
&-\frac{1}{2268000}(\Delta x)^8\Big(U^{(10)}((m+\mu_7\Delta x),s)\\
&+U^{(10)}((m+\mu_8\Delta x),s)\Big)\Big]\,\mathrm{d}s\\
&+\eta\int_{t_n}^{t_{n+1}}(U(x_m,s)-u_m^n)\,\mathrm{d}B(s).
\end{aligned}
$$

Using the following inequality

$$
\mathbb{E}|W+X+Y+Z|^2\le 4\mathbb{E}|W|^2+8\mathbb{E}|X|^2+8\mathbb{E}|Y|^2+2\mathbb{E}|Z|^2,
$$

and taking $\mathbb{E}|\cdot|^2$ from both sides of the last equality we achieve

$$
\begin{aligned}
&\mathbb{E}\left|e_m^{n+1}\right|^2\le 4\mathbb{E}\Bigg|\left(\frac{8}{5}\sigma+\gamma\right)e_{m-1}^n+\left(1-\frac{205}{72}\sigma\right)e_m^n\\
&+\left(\frac{8}{5}\sigma-\gamma\right)e_{m+1}^n+\sigma\left(-\frac{1}{560}e_{m-4}^n+\frac{8}{315}e_{m-3}^n-\frac{1}{5}e_{m-2}^n\right.\\
&\quad\left.-\frac{1}{5}e_{m+2}^n+\frac{8}{315}e_{m+3}^n-\frac{1}{560}e_{m+4}^n\right)\Bigg|^2\\
&+8\mathbb{E}\Bigg|-\alpha\int_{t_n}^{t_{n+1}}\Big[-\frac{(\Delta x)^2}{12}\Big(U_{xxx}((m+\nu_1)\Delta x,s)\\
&\quad+U_{xxx}((m+\nu_2)\Delta x,s)\Big]\,\mathrm{d}s\Bigg|^2\\
&+8\mathbb{E}\Bigg|\lambda\int_{t_n}^{t_{n+1}}\Big[\frac{256}{496125}(\Delta x)^8\Big(U^{(10)}((m+\mu_1\Delta x),s)\\
&\quad+U^{(10)}((m+\mu_2\Delta x),s)\Big)\\
&-\frac{81}{196000}(\Delta x)^8\Big(U^{(10)}((m+\mu_3\Delta x),s)+U^{(10)}((m+\mu_4\Delta x),s)\Big)\\
&+\frac{4}{70875}(\Delta x)^8\Big(U^{(10)}((m+\mu_5\Delta x),s)+U^{(10)}((m+\mu_6\Delta x),s)\Big)\\
&-\frac{1}{2268000}(\Delta x)^8\Big(U^{(10)}((m+\mu_7\Delta x),s)\\
&\quad+U^{(10)}((m+\mu_8\Delta x),s)\Big)\Big]\,\mathrm{d}s\Bigg|^2
\end{aligned}
$$

$$+ 2\eta^2 \int_{t_n}^{t_{n+1}} \mathbb{E}|U(x_m, s) - U_m^n + U_m^n - u_m^n|^2 \, \mathrm{d}s.$$

After some calculations, we deduce that

$$\begin{aligned}
\mathbb{E}\left|e_m^{n+1}\right|^2 &\leq 4\left(1 + \frac{113}{70}\sigma + \frac{12769}{19600}\sigma^2 + \eta^2 \Delta t\right) \sup_m \mathbb{E}|e_m^n|^2 \\
&+ 8 \sup_m \mathbb{E}\bigg| - \alpha \int_{t_n}^{t_{n+1}} \bigg[- \frac{(\Delta x)^2}{12}\Big(U_{xxx}((m+\nu_1)\Delta x, s) \\
&\quad + U_{xxx}((m+\nu_2)\Delta x, s)\bigg] \, \mathrm{d}s\bigg|^2 \\
&+ 8 \sup_m \mathbb{E}\bigg|\lambda \int_{t_n}^{t_{n+1}} \bigg[\frac{256}{496125}(\Delta x)^8\Big(U^{(10)}((m+\mu_1\Delta x), s) \\
&\quad + U^{(10)}((m+\mu_2\Delta x), s)\Big) \\
&- \frac{81}{196000}(\Delta x)^8\Big(U^{(10)}((m+\mu_3\Delta x), s) + U^{(10)}((m+\mu_4\Delta x), s)\Big) \\
&+ \frac{4}{70875}(\Delta x)^8\Big(U^{(10)}((m+\mu_5\Delta x), s) + U^{(10)}((m+\mu_6\Delta x), s)\Big) \\
&- \frac{1}{2268000}(\Delta x)^8\Big(U^{(10)}((m+\mu_7\Delta x), s) \\
&\quad + U^{(10)}((m+\mu_8\Delta x), s)\Big)\bigg] \, \mathrm{d}s\bigg|^2 \\
&+ 4\eta^2 \sup_m \int_{t_n}^{t_{n+1}} \mathbb{E}|U(x_m, s) - U_m^n|^2 \, \mathrm{d}s + 4\eta^2 \Delta t \sup_m \mathbb{E}|e_m^n|^2.
\end{aligned}$$

Let us introduce the notations

$$\begin{aligned}
\alpha_{1m} &= U_{xxx}((m+\nu_1)\Delta x, s), \\
\alpha_{2m} &= U_{xxx}((m+\nu_2)\Delta x, s), \\
\beta_{1m} &= U^{(10)}((m+\mu_1\Delta x), s), \\
\beta_{2m} &= U^{(10)}((m+\mu_2\Delta x), s), \\
\beta_{3m} &= U^{(10)}((m+\mu_3\Delta x), s), \\
\beta_{4m} &= U^{(10)}((m+\mu_4\Delta x), s), \\
\beta_{5m} &= U^{(10)}((m+\mu_5\Delta x), s), \\
\beta_{6m} &= U^{(10)}((m+\mu_6\Delta x), s), \\
\beta_{7m} &= U^{(10)}((m+\mu_7\Delta x), s), \\
\beta_{8m} &= U^{(10)}((m+\mu_8\Delta x), s),
\end{aligned}$$

where the values α_{1m}, α_{2m}, β_{1m}, β_{2m}, β_{3m}, β_{4m}, β_{5m}, β_{6m}, β_{7m} and β_{8m} are finite. Moreover, taking into account that

$$\begin{aligned}\int_{t_n}^{t_{n+1}} \mathbb{E}|U(x_m,s)-U_m^n|^2\,\mathrm{d}s &= \mathbb{E}\int_{t_n}^{t_{n+1}} |U(x_m,s)-U_m^n|^2\,\mathrm{d}s\\ &\le \sup_{s\in[t_n,t_{n+1}]} |U(x_m,s)-U(m\Delta x,n\Delta t)|^2\Delta t \le \Lambda\Delta t,\end{aligned}$$

where

$$\Lambda = \sup_{s\in[t_n,t_{n+1}]} |U(x_m,s)-U(m\Delta x,n\Delta t)|^2.$$

It follows that

$$\begin{aligned}\sup_m \mathbb{E}\left|e_m^{n+1}\right|^2 &\le 4\left(1+\frac{113}{70}\sigma+\frac{12769}{19600}\sigma^2+\eta^2\Delta t\right)\sup_m \mathbb{E}|e_m^n|^2\\ &+ 8\sup_m \mathbb{E}\left|-\alpha\int_{t_n}^{t_{n+1}}\left[-\frac{(\Delta x)^2}{12}(\alpha_{1m}+\alpha_{2m})\right]\mathrm{d}s\right|^2\\ &+ 8\sup_m \mathbb{E}\left|\lambda\int_{t_n}^{t_{n+1}}(\Delta x)^8\left(\frac{256}{496125}(\beta_{1m}+\beta_{2m})-\frac{81}{196000}(\beta_{3m}+\beta_{4m})\right.\right.\\ &\left.\left.+\frac{4}{70875}(\beta_{5m}+\beta_{6m})-\frac{1}{2268000}(\beta_{7m}+\beta_{8m})\right)\mathrm{d}s\right|^2+4\eta^2\Lambda\Delta t. \qquad (14)\end{aligned}$$

Then it sufficient to select the parameter δ such that the inequality

$$\frac{113}{70}\sigma+\frac{12769}{19600}\sigma^2+\eta^2\Delta t \le \delta^2\Delta t. \qquad (15)$$

holds for all m. Let

$$\begin{aligned}\Omega' &= \alpha\frac{(\Delta x)^2}{12}(\alpha_{1m}+\alpha_{2m}),\\ \Omega_3 &= \lambda(\Delta x)^8\left(\frac{256}{496125}(\beta_{1m}+\beta_{2m})-\frac{81}{196000}(\beta_{3m}+\beta_{4m})\right.\\ &\quad\left.+\frac{4}{70875}(\beta_{5m}+\beta_{6m})-\frac{1}{2268000}(\beta_{7m}+\beta_{8m})\right), \qquad \Omega_4 = 4\eta^2\Lambda.\end{aligned}$$

According to these notations, (14) can be rewritten as

$$\begin{aligned}\sup_m \mathbb{E}|e_m^{n+1}|^2 \le 4(1+\delta^2\Delta t)\sup_m \mathbb{E}|e_m^n|^2 + 8\sup_m \mathbb{E}|\Omega'|^2\Delta t\\ + 8\sup_m \mathbb{E}|\Omega_3|^2\Delta t + \Omega_4\Delta t,\end{aligned}$$

which implies

$$\sup_m \mathbb{E}|e_m^{n+1}|^2 \le 4(1+\delta^2\Delta t)\sup_m \mathbb{E}|e_m^n|^2 + \Omega_5\Delta t,$$

where

$$\Omega_5 = 8\sup_m \mathbb{E}|\Omega'|^2 + 8\sup_m \mathbb{E}|\Omega_3|^2 + \Omega_4.$$

Therefore

$$\begin{aligned}\mathbb{E}\|e^{n+1}\|^2 &\le 4(1+\delta^2\Delta t)\mathbb{E}\|e^n\|^2 + \Omega_5\Delta t\\ &\le \left(1+\delta^2\frac{t}{n+1}\right)^{n+1}\sum_{j=1}^{n}(4\Omega_5\Delta t)^j + \Omega_5\Delta t\\ &\le e^{\delta^2 t}\sum_{j=1}^{n}(4\Omega_5\Delta t)^j + \Omega_5\Delta t.\end{aligned}$$

From the last inequality, when the time stepsize $\Delta t \to 0$, we see that

$$\begin{aligned}\mathbb{E}\|e^{n+1}\|^2 &\le (n-1)e^{\delta^2 t}(4\Omega_5\Delta t)^2 + 4e^{\delta^2 t}\Omega_5\Delta t + \Omega_5\Delta t\\ &\le te^{\delta^2 t}(4\Omega_5)^2\Delta t + 4e^{\delta^2 t}\Omega_5\Delta t + \Omega_5\Delta t\\ &= (te^{\delta^2 t}(4\Omega_5)^2 + 4e^{\delta^2 t}\Omega_5 + \Omega_5)\Delta t. \qquad (16)\end{aligned}$$

This implies $\mathbb{E}\|e^{n+1}\|^2 \to 0$, when Δt tends to zero. □

In continuation, we investigate the convergence order of the stochastic difference scheme (3). Substituting the expressions Ω_5, Ω', Ω_3 and Ω_4 into (16) and using the fact that $t \in [0,1]$, we get

$$\begin{aligned}\mathbb{E}\|e^{n+1}\|^2 &\le (e^{\delta^2}(4\Omega_5)^2 + 4e^{\delta^2}\Omega_5 + \Omega_5)\Delta t\\ &\le \Big[16e^{\delta^2}(8\sup_m \mathbb{E}|\Omega'|^2 + 8\sup_m \mathbb{E}|\Omega_3|^2 + \Omega_4)^2\\ &\quad + 4e^{\delta^2}(8\sup_m \mathbb{E}|\Omega'|^2 + 8\sup_m \mathbb{E}|\Omega_3|^2 + \Omega_4)\\ &\quad + 8\sup_m \mathbb{E}|\Omega'|^2 + 8\sup_m \mathbb{E}|\Omega_3|^2 + \Omega_4\Big]\Delta t\\ &= \Bigg[16e^{\delta^2}\Bigg(8\sup_m \mathbb{E}\left|\alpha\frac{(\Delta x)^2}{12}(\alpha_{1m}+\alpha_{2m})\right|^2\\ &\quad + 8\sup_m \mathbb{E}\Bigg|\lambda(\Delta x)^8\bigg(\frac{256}{496125}(\beta_{1m}+\beta_{2m}) - \frac{81}{196000}(\beta_{3m}+\beta_{4m})\\ &\quad + \frac{4}{70875}(\beta_{5m}+\beta_{6m}) - \frac{1}{2268000}(\beta_{7m}+\beta_{8m})\bigg)\Bigg|^2 + 4\eta^2\Lambda\Bigg)^2\\ &\quad + 4e^{\delta^2}\Bigg(8\sup_m \mathbb{E}\left|\alpha\frac{(\Delta x)^2}{12}(\alpha_{1m}+\alpha_{2m})\right|^2\\ &\quad + 8\sup_m \mathbb{E}\Bigg|\lambda(\Delta x)^8\bigg(\frac{256}{496125}(\beta_{1m}+\beta_{2m}) - \frac{81}{196000}(\beta_{3m}+\beta_{4m})\\ &\quad + \frac{4}{70875}(\beta_{5m}+\beta_{6m}) - \frac{1}{2268000}(\beta_{7m}+\beta_{8m})\bigg)\Bigg|^2 + 4\eta^2\Lambda\Bigg)\end{aligned}$$

$$+ 8\sup_m \mathbb{E}\left|\alpha\frac{(\Delta x)^2}{12}(\alpha_{1m}+\alpha_{2m})\right|^2$$
$$+ 8\sup_m \mathbb{E}\bigg|\lambda(\Delta x)^8\bigg(\frac{256}{496125}(\beta_{1m}+\beta_{2m})-\frac{81}{196000}(\beta_{3m}+\beta_{4m})$$
$$+\frac{4}{70875}(\beta_{5m}+\beta_{6m})-\frac{1}{2268000}(\beta_{7m}+\beta_{8m})\bigg)\bigg|^2+4\eta^2\Lambda\bigg]\Delta t.$$

Using the following inequality:

$$\mathbb{E}|W+X+Y+Z|^2 \leq 8\mathbb{E}|W|^2+8\mathbb{E}|X|^2+4\mathbb{E}|Y|^2+2\mathbb{E}|Z|^2,$$

we have

$$\mathbb{E}\|e^{n+1}\|^2 \leq \bigg[16e^{\delta^2}\bigg\{8\frac{\alpha^2(\Delta x)^4}{12^2}\sup_m \mathbb{E}|\alpha_{1m}+\alpha_{2m}|^2$$
$$+8\lambda^2(\Delta x)^{16}\bigg[\sup_m\bigg(8\bigg(\frac{256}{496125}\bigg)^2\mathbb{E}|\beta_{1m}+\beta_{2m}|^2$$
$$+8\bigg(\frac{81}{196000}\bigg)^2\mathbb{E}|\beta_{3m}+\beta_{4m}|^2+4\bigg(\frac{4}{70875}\bigg)^2\mathbb{E}|\beta_{5m}+\beta_{6m}|^2$$
$$+2\bigg(\frac{1}{2268000}\bigg)^2\mathbb{E}|\beta_{7m}+\beta_{8m}|^2\bigg)\bigg]+4\eta^2\Lambda\bigg\}^2$$
$$+4e^{\delta^2}\bigg\{8\frac{\alpha^2(\Delta x)^4}{12^2}\sup_m \mathbb{E}|\alpha_{1m}+\alpha_{2m}|^2$$
$$+8\lambda^2(\Delta x)^{16}\bigg[\sup_m\bigg(8\bigg(\frac{256}{496125}\bigg)^2\mathbb{E}|\beta_{1m}+\beta_{2m}|^2$$
$$+8\bigg(\frac{81}{196000}\bigg)^2\mathbb{E}|\beta_{3m}+\beta_{4m}|^2+4\bigg(\frac{4}{70875}\bigg)^2\mathbb{E}|\beta_{5m}+\beta_{6m}|^2$$
$$+2\bigg(\frac{1}{2268000}\bigg)^2\mathbb{E}|\beta_{7m}+\beta_{8m}|^2\bigg)\bigg]+4\eta^2\Lambda\bigg\}$$
$$+8\frac{\alpha^2(\Delta x)^4}{12^2}\sup_m \mathbb{E}|\alpha_{1m}+\alpha_{2m}|^2$$
$$+8\lambda^2(\Delta x)^{16}\bigg[\sup_m\bigg(8\bigg(\frac{256}{496125}\bigg)^2\mathbb{E}|\beta_{1m}+\beta_{2m}|^2$$
$$+8\bigg(\frac{81}{196000}\bigg)^2\mathbb{E}|\beta_{3m}+\beta_{4m}|^2+4\bigg(\frac{4}{70875}\bigg)^2\mathbb{E}|\beta_{5m}+\beta_{6m}|^2$$
$$+2\bigg(\frac{1}{2268000}\bigg)^2\mathbb{E}|\beta_{7m}+\beta_{8m}|^2\bigg)\bigg]+4\eta^2\Lambda\bigg]\Delta t$$
$$\leq\bigg[\frac{4\times16\times8^2}{12^4}e^{\delta^2}\alpha^4(\Delta x)^8(\sup_m \mathbb{E}|\alpha_{1m}+\alpha_{2m}|^2)^2$$

$$4\times 16\lambda^4(\Delta x)^{32}e^{\delta^2}\Bigg(\sup_m\Bigg(8\left(\frac{256}{496125}\right)^2\mathbb{E}|\beta_{1m}+\beta_{2m}|^2$$
$$+8\left(\frac{81}{196000}\right)^2\mathbb{E}|\beta_{3m}+\beta_{4m}|^2+4\left(\frac{4}{70875}\right)^2\mathbb{E}|\beta_{5m}+\beta_{6m}|^2$$
$$+2\left(\frac{1}{2268000}\right)^2\mathbb{E}|\beta_{7m}+\beta_{8m}|^2\Bigg)\Bigg)+2\times 16^2\eta^4\Lambda^2e^{\delta^2}$$
$$+\frac{4\times 8}{12^2}\alpha^2e^{\delta^2}(\Delta x)^4\sup_m\mathbb{E}|\alpha_{1m}+\alpha_{2m}|^2$$
$$+4\times 8\lambda^2e^{\delta^2}(\Delta x)^{16}\Bigg(\sup_m\Bigg(8\left(\frac{256}{496125}\right)^2\mathbb{E}|\beta_{1m}+\beta_{2m}|^2$$
$$+8\left(\frac{81}{196000}\right)^2\mathbb{E}|\beta_{3m}+\beta_{4m}|^2+4\left(\frac{4}{70875}\right)^2\mathbb{E}|\beta_{5m}+\beta_{6m}|^2$$
$$+2\left(\frac{1}{2268000}\right)^2\mathbb{E}|\beta_{7m}+\beta_{8m}|^2\Bigg)\Bigg)+4^2\eta^2e^{\delta^2}\Lambda$$
$$+\frac{8}{12^2}\alpha^2(\Delta x)^4\sup_m\mathbb{E}|\alpha_{1m}+\alpha_{2m}|^2$$
$$+8\lambda^2(\Delta x)^{16}\Bigg(\sup_m\Bigg(8\left(\frac{256}{496125}\right)^2\mathbb{E}|\beta_{1m}+\beta_{2m}|^2$$
$$+8\left(\frac{81}{196000}\right)^2\mathbb{E}|\beta_{3m}+\beta_{4m}|^2+4\left(\frac{4}{70875}\right)^2\mathbb{E}|\beta_{5m}+\beta_{6m}|^2$$
$$+2\left(\frac{1}{2268000}\right)^2\mathbb{E}|\beta_{7m}+\beta_{8m}|^2\Bigg)\Bigg)+4\eta^2\Lambda\Bigg]\Delta t$$
$$=\Bigg[\left\{\frac{16}{81}e^{\delta^2}\alpha^4(\sup_m\mathbb{E}|\alpha_{1m}+\alpha_{2m}|^2)^2\right\}(\Delta x)^8$$
$$+\Bigg\{64\lambda^4e^{\delta^2}\Bigg(\sup_m\Bigg(8\left(\frac{256}{496125}\right)^2\mathbb{E}|\beta_{1m}+\beta_{2m}|^2$$
$$+8\left(\frac{81}{196000}\right)^2\mathbb{E}|\beta_{3m}+\beta_{4m}|^2+4\left(\frac{4}{70875}\right)^2\mathbb{E}|\beta_{5m}+\beta_{6m}|^2$$
$$+2\left(\frac{1}{2268000}\right)^2\mathbb{E}|\beta_{7m}+\beta_{8m}|^2\Bigg)\Bigg)\Bigg\}(\Delta x)^{32}$$
$$+512e^{\delta^2}\eta^4\Lambda^2+\left\{\frac{2}{9}e^{\delta^2}\alpha^2\sup_m\mathbb{E}|\alpha_{1m}+\alpha_{2m}|^2\right\}(\Delta x)^4$$
$$+\Bigg\{32\lambda^2e^{\delta^2}\Bigg(\sup_m\Bigg(8\left(\frac{256}{496125}\right)^2\mathbb{E}|\beta_{1m}+\beta_{2m}|^2$$
$$+8\left(\frac{81}{196000}\right)^2\mathbb{E}|\beta_{3m}+\beta_{4m}|^2+4\left(\frac{4}{70875}\right)^2\mathbb{E}|\beta_{5m}+\beta_{6m}|^2$$

$$+ 2\left(\frac{1}{2268000}\right)^2 \mathbb{E}|\beta_{7m} + \beta_{8m}|^2\Big)\Big)\Big\}(\Delta x)^{16} + 16\eta^2 e^{\delta^2}\Lambda$$

$$\left\{\frac{1}{18}\alpha^2 \sup_m \mathbb{E}|\alpha_{1m} + \alpha_{2m}|^2\right\}(\Delta x)^4$$

$$+ \Big\{8\lambda^2\Big(\sup_m \Big(8\left(\frac{256}{496125}\right)^2 \mathbb{E}|\beta_{1m} + \beta_{2m}|^2$$

$$+ 8\left(\frac{81}{196000}\right)^2 \mathbb{E}|\beta_{3m} + \beta_{4m}|^2 + 4\left(\frac{4}{70875}\right)^2 \mathbb{E}|\beta_{5m} + \beta_{6m}|^2$$

$$+ 2\left(\frac{1}{2268000}\right)^2 \mathbb{E}|\beta_{7m} + \beta_{8m}|^2\Big)\Big)\Big\}(\Delta x)^{16} + 4\eta^2\Lambda\Big]\Delta t.$$

By letting

$$K_1 = \left(\frac{2}{9}e^{\delta^2} + \frac{1}{18}\right)\alpha^2 \sup_m \mathbb{E}|\alpha_{1m} + \alpha_{2m}|^2,$$

$$K_2 = \frac{16}{81}\alpha^4 e^{\delta^2}(\sup_m \mathbb{E}|\alpha_{1m} + \alpha_{2m}|^2)^2,$$

$$K_3 = (32e^{\delta^2} + 8)\lambda^2 \sup_m \Big(8\left(\frac{256}{496125}\right)^2 \mathbb{E}|\beta_{1m} + \beta_{2m}|^2 + 8\left(\frac{81}{196000}\right)^2 \mathbb{E}|\beta_{3m} + \beta_{4m}|^2 + 4\left(\frac{4}{70875}\right)^2 \mathbb{E}|\beta_{5m} + \beta_{6m}|^2 + 2\left(\frac{1}{2268000}\right)^2 \mathbb{E}|\beta_{7m} + \beta_{8m}|^2\Big),$$

$$K_4 = 64\lambda^4 e^{\delta^2} \sup_m \Big(8\left(\frac{256}{496125}\right)^2 \mathbb{E}|\beta_{1m} + \beta_{2m}|^2 + 8\left(\frac{81}{196000}\right)^2 \mathbb{E}|\beta_{3m} + \beta_{4m}|^2 + 4\left(\frac{4}{70875}\right)^2 \mathbb{E}|\beta_{5m} + \beta_{6m}|^2 + 2\left(\frac{1}{2268000}\right)^2 \mathbb{E}|\beta_{7m} + \beta_{8m}|^2\Big),$$

$$K_5 = 512e^{\delta^2}\eta^4\Lambda^2 + 16e^{\delta^2}\eta^2\Lambda + 4\eta^2\Lambda,$$

we see that

$$\mathbb{E}\|e_{n+1}\|^2 \leq \big(K_1(\Delta x)^4 + K_2(\Delta x)^8 + K_3(\Delta x)^{16} + K_4(\Delta x)^{32} + K_5\big)\Delta t. \tag{17}$$

Hence,

$$\mathbb{E}\|e_{n+1}\|^2 = O(\Delta t(\Delta x)^4) + O(\Delta t).$$

Here, it is worth mentioning that the values $K_1, K_2 \to 0$ as $\alpha \to 0$. Therefore, the inequality (17) can be written as

$$\mathbb{E}\|e_{n+1}\|^2 = O(\Delta t(\Delta x)^{16}) + O(\Delta t).$$

4. **Numerical experiments**

In this part, theoretical results obtained in the previous section are examined by two test problems. In this paper, we used MATLAB 2015 on Intel(R) CORE(TM) i3 CPU with 8 GB RAM and 64-bit system (Windows 10) for numerical simulations. In order to generate the Brownian motion increments the *randn* command of MATLAB is used.

Example 4.1. *Consider the SPDE as follows*

$$U_t(x,t) = \lambda U_{xx}(x,t) + \eta\, U(x,t)\dot{B}(t), \quad x \in [0,1], \quad t \in [0,1], \tag{18}$$

with initial conditions and boundary conditions

$$U(x,0) = \exp\left(-\frac{(x-0.2)^2}{\lambda}\right),$$
$$U(0,t) = \frac{1}{\sqrt{4t+1}}\exp\left(-\frac{0.04}{\lambda(4t+1)}\right),$$
$$U(1,t) = \frac{1}{\sqrt{4t+1}}\exp\left(-\frac{0.64}{\lambda(4t+1)}\right).$$

One may check that in the absence of a stochastic term, the analytical solution problem can be expressed as

$$U(x,t) = \frac{1}{\sqrt{4t+1}}\exp\left(-\frac{(x-0.2)^2}{\lambda(4t+1)}\right).$$

In Figure 1, the numerical results of the SFD scheme (3) *are depicted along with the exact solution for the different values of parameters* $\lambda = 0.005$, $\eta = -2$, $\Delta x = 0.01$, $\Delta t = 0.0063$, *and* $\lambda = 0.001$, $\eta = 2$, $\Delta x = 0.005$, $\Delta t = 0.008$.

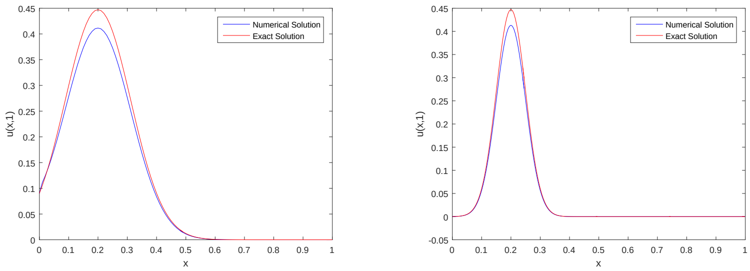
Let $\Delta x = \frac{1}{M}$ *and* $\Delta t = \frac{1}{N}$, *where* M *and* N *are positive constants. If* $\lambda = 0.0005$, $\eta = 1.5$ *and* $M = 400$, *then for the stability condition, we require* $\Delta t < 0.0044$ *or* $N > 228$. *In Table 1, the absolute errors of the numerical scheme* (3) *with* $\lambda = 0.001$, $\eta = 1.5$ *and* $\Delta x = \Delta t = 0.01$ *have been compared with the stochastic difference scheme in [2]. In Figure 2, the instability and stability SFD scheme for different values of* N *are investigated.*

In the proof of Theorem 3.4, it is assumed that

$$\frac{113}{70}\sigma + \frac{12769}{19600}\sigma^2 + \eta^2\Delta t \leq \delta^2\Delta t.$$

In Table 2 the least value of δ^2 *for some value of* N *are given. In Table 3, we observe that the results for* $\mathbb{E}(u(0.2,1))$ *are destroyed when* $N \leq 228$, *whereas, for* $N > 228$, *the stability results are obtained.*

In Table 4, the CPU times (execution time in seconds) of the numerical scheme (3) *for the values given in Fig. 1 have been compared with the mentioned difference scheme in [2]. It can be seen that the proposed stochastic difference scheme costs less CPU times than the cited stochastic difference scheme in [2].*



(a) (b)

FIGURE 1. Comparison between deterministic and stochastic numerical solution of (18) using the stochastic method with $\lambda = 0.005$, $\eta = -2$, $\Delta x = 0.01$, $\Delta t = 0.0063$ (a) and with $\lambda = 0.001$, $\eta = 2$, $\Delta x = 0.005$, $\Delta t = 0.008$ (b).

TABLE 1. Absolute errors for the numerical scheme (3) and the proposed scheme in [2] with $\lambda = 0.001$, $\eta = 1.5$ and $\Delta x = \Delta t = 0.01$.

x	*Present scheme*	*Scheme of [2]*
0.1	4.8×10^{-5}	4.3×10^{-3}
0.2	3.44×10^{-4}	3.68×10^{-2}
0.3	4.8×10^{-5}	4.3×10^{-3}
0.4	1.1282×10^{-7}	3.4287×10^{-5}
0.5	4.2326×10^{-11}	9.7327×10^{-9}
0.6	3.8176×10^{-16}	6.7303×10^{-14}
0.7	8.2260×10^{-23}	2.3412×10^{-20}
0.8	7.6570×10^{-33}	7.7082×10^{-28}
0.9	4.8105×10^{-41}	3.4080×10^{-36}
1	3.4864×10^{-78}	6.4864×10^{-70}

TABLE 2. Stability of the SFD scheme (3) for the different values of δ^2.

N	227	250	300	1000
δ^2	149.7606	148.0707	145.2911	135.5623
N	3000	5000	10000	20000
δ^2	132.7827	132.2268	131.8098	131.6013

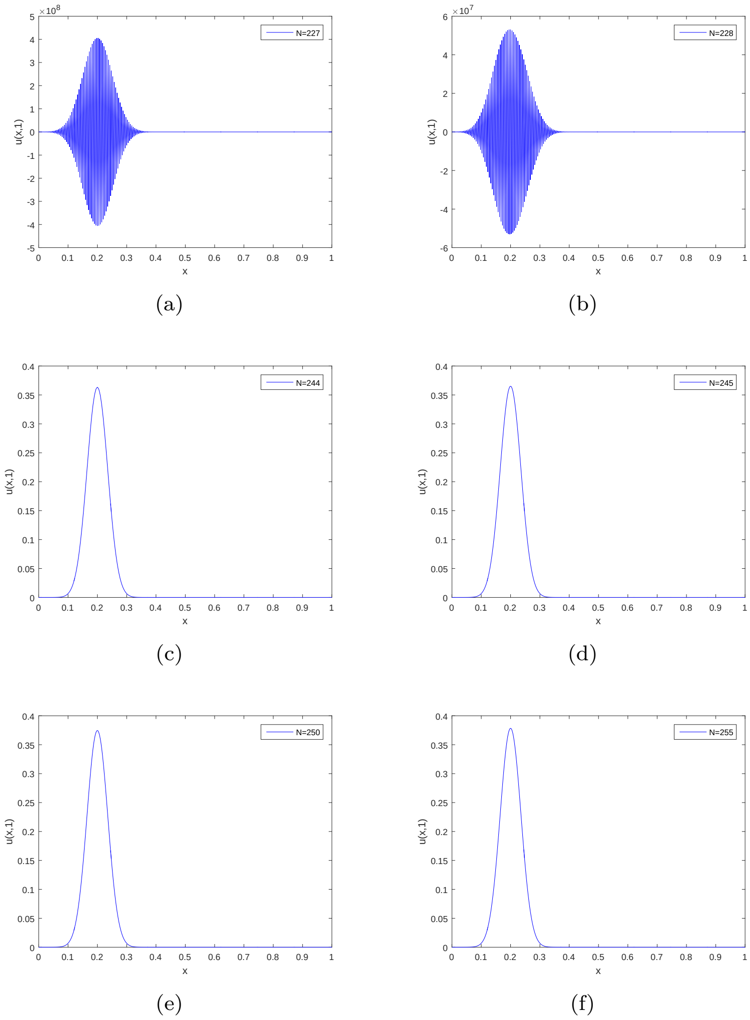


FIGURE 2. Representation of conditional convergence, $u(x,1)$ for different values of N.

This is due to the fact that the scheme (3) *is an explicit scheme whereas, the mentioned scheme in [2] is an implicit scheme.*

Example 4.2. *Consider the following SPDE*

$$U_t(x,t) + \alpha U_x(x,t) = \lambda U_{xx}(x,t) + \eta\, U(x,t)\dot{B}(t), \quad x \in [0,1], \quad t \in [0,1], \tag{19}$$

Table 3. Representation of stability condition and $\mathbb{E}(u(0.2,1))$ for different values of N.

N	$\mathbb{E}(u(0.2,1))$	$\mathbb{E}((u(0.2,1))^2$
226	$2.2770\ e+09$	$5.1848\ e+18$
228	$5.2946\ e+07$	$2.8033\ e+15$
240	0.6877	0.4730
245	0.3651	0.1333
250	0.3746	0.1404
300	0.3841	0.1475

Table 4. Comparison CPU times of the proposed scheme with the stochastic difference scheme in [2].

Δx	0.01	0.005
Δt	0.0063	0.008
λ	0.005	0.001
η	-2	2
CPU time (present scheme)	2.2113	3.3440
CPU time scheme of [2]	30.3994	76.7870

subject to the initial condition

$$U(x,0)=\exp\left(-\frac{(x-0.5)^2}{\lambda}\right),\quad x\in[0,1],$$

with the boundary conditions

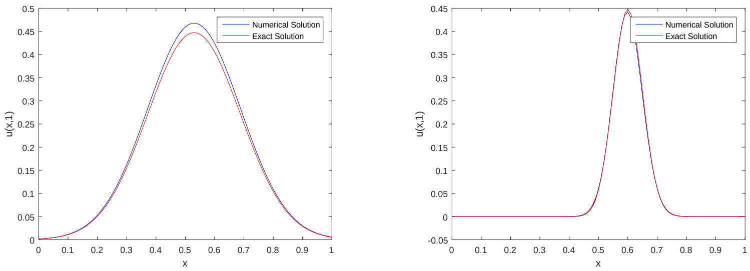
$$U(0,t)=\frac{1}{\sqrt{4t+1}}\exp\left(-\frac{(-0.5-\alpha t)^2}{\lambda(4t+1)}\right),\quad t\in[0,1],$$
$$U(1,t)=\frac{1}{\sqrt{4t+1}}\exp\left(-\frac{(0.5-\alpha t)^2}{\lambda(4t+1)}\right),\quad t\in[0,1].$$

It is easy to check that in the deterministic case, the exact solution is given by

$$U(x,t)=\frac{1}{\sqrt{4t+1}}\exp\left(-\frac{(x-0.5-\alpha t)^2}{\lambda(4t+1)}\right).$$

In Figure 3 the numerical solution of the SPDE (19) *by using the SFD scheme* (3) *for specific values of the parameters λ, α, η and the various of Δx and Δt are presented.*

To account of the stability condition of Theorem 3.4, we require $\frac{5\gamma}{8}\le\sigma\le\frac{72}{205}$, as a result, if $\lambda=0.01$, $\alpha=0.03$, $\eta=1.5$ and $M=100$, then for the stability or convergence condition, we must have $\Delta t\le 0.0035$ or $N\ge 285$.



(a) (b)

FIGURE 3. Comparsion between deterministic and stochastic numerical solution of (18) using the SFD scheme with $\lambda = 0.01$, $\alpha = 0.03$, $\eta = 2$, $\Delta x = 0.01$, $\Delta t = 0.002$ (a) and with $\lambda = 0.001$, $\alpha = 0.1$, $\eta = -2$, $\Delta x = 0.01$, $\Delta t = 0.001$ (b).

In the proof of Theorem 3.4, we assumed that

$$\frac{113}{70}\sigma + \frac{12769}{19600}\sigma^2 + \eta^2\Delta t \leq \delta^2\Delta t,$$

and for different values of N, we obtained the least value of δ^2 in Table 5. From Table 6, we observe that the values $\mathbb{E}(u(0.5,1))$ when $N \leq 285$ are destroyed, whereas for $N > 285$, the stability results are obtained.

TABLE 5. Stability of the SFD scheme (3) for different values of δ^2.

N	280	300	500	1000
δ^2	186.9457	185.3946	176.7082	170.1934
N	3000	5000	10000	20000
δ^2	165.8502	164.9815	164.3301	164.0043

TABLE 6. Representation of stability condition and $\mathbb{E}(u(0.5,1))$ for different values of N.

N	$\mathbb{E}(u(0.5,1))$	$\mathbb{E}((u(0.5,1))^2$
280	$7.5573\ e+17$	$5.7112\ e+35$
285	$-6.4713\ e+14$	$4.1878\ e+29$
310	0.4311	0.1858
320	0.3946	0.1557
350	0.4064	0.1652
360	0.4223	0.1783

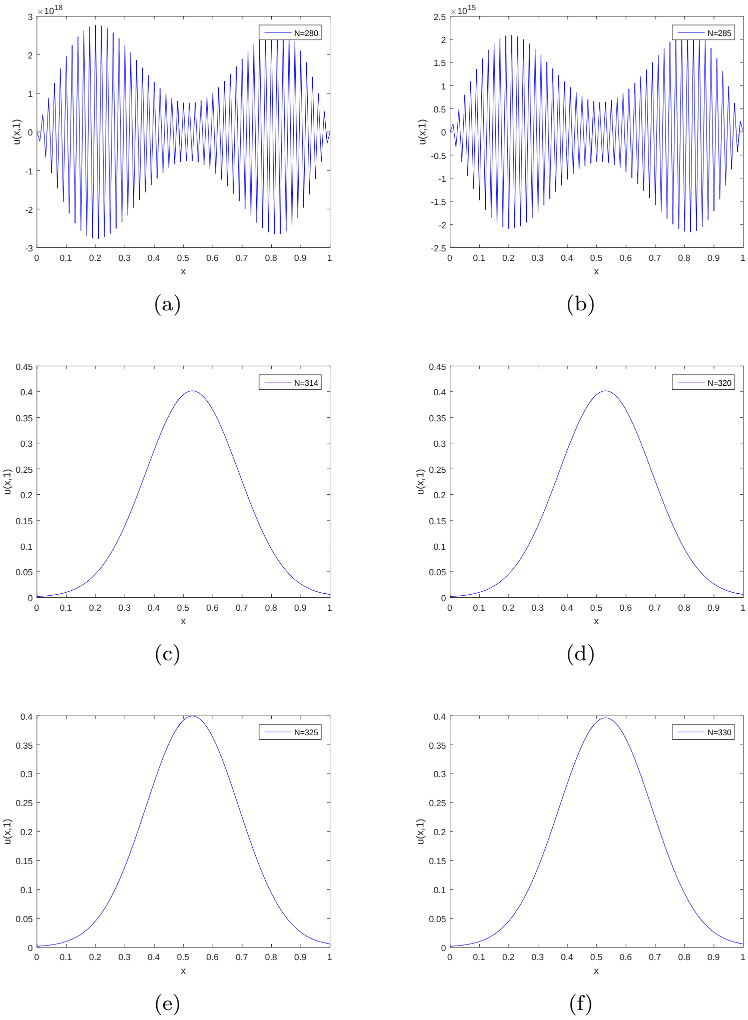


FIGURE 4. Representation of conditional convergence, $u(x, 1)$ for different values of N.

In Table 7, we reported the CPU times for the values are given in Fig. 3 with respect to the proposed scheme (3)*, and the presented scheme in [10].*

Note that the stochastic difference scheme in [10] is an implicit scheme whereas, the scheme (3) *is an explicit scheme.*

TABLE 7. Comparison CPU times of the proposed scheme with the stochastic difference scheme in [10].

Δx	0.001	0.01
Δt	0.002	0.001
α	0.03	0.1
λ	0.01	0.001
η	2	-2
CPU time (present scheme)	6.8362	13.5789
CPU time scheme of [10]	45.3094	90.4412

5. Conclusion and future work

In this manuscript, a stochastic finite difference scheme is suggested for the numerical solution of a class SPDEs. In continuation, we have studied the consistency, stability, and convergence of the proposed scheme. The numerical experiments show that the introduced scheme is efficient for a family of SPDEs. Future work focus on constructing superior nonstandard stochastic finite difference scheme for the SPDEs.

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