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GRADIENT RICCI BOURGUIGNON SOLITONS ON PERFECT FLUID SPACE-TIMES

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Abstract. The main purpose of the present paper is about characterizing properties of the perfect fluid space-time admits the gradient Ricci-Bourguignon soliton. This gives some results about stability of the energy momentum tensor and also under some conditions pursues that a perfect fluid space-time is Ricci symmetric. As an special case, when a perfect fluid space-time equipped with the Ricci-Bourguignon soliton which has Ricci biconformal vector field, we show that the metric of this space is Einstein.

Keywords: Perfect fluid space-time, Gradient Ricci soliton, Gradient Ricci Bourguignon soliton, GRW space-time.
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1. Introduction

The importance of studying Lorentzian manifolds in physics is that we may model a gravitational field by some Lorentzian metric defined on a 4-dimensional manifold M. Actually, in the General Theory of Relativity, the gravitational field is the space-time curvature and its source is the energy-momentum tensor. This space-time is modeled as a connected 4-dimensional semi-Riemannian manifold endowed with a Lorentzian metric g of signature $(-,+,+,+)$. One of the most significant Lorentzian manifolds is the perfect fluid space-time, a Lorentzian manifold of dimension $n > 3$ with Ricci tensor satisfying in the following equation

$$Ric = h_1 g + h_2 \eta \otimes \eta, \tag{1}$$

where h_1 and h_2 are smooth functions on M and η is a 1-form that is metrically equivalent to the unit time-like vector field ρ and $g(X, \rho) = \eta(X)$, for all $X \in \chi(M)$ and $g(\rho, \rho) = -1$. Perfect fluids have been studied for many purposes of view, see ([2, 11, 16, 21, 22]).

Let M^n be a Lorentzian manifold, it is called a generalized Robertson-Walker space-time (GRW for short introduced in [1], [2]), if its metric in the local coordinate satisfies:

$$ds^2 = -dt^2 + q(t)^2 g^*_{ij}(x_2, ..., x_n) dx^i dx^j, \quad i, j = 2, ..., n,$$

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that is a warped product $I \times_q M^*$, and q is a positive smooth function on I, M^* is an $(n-1)$-dimensional Riemannian manifold and if M^* be 3-dimensional with constant curvature, then we call it as a Robertson-Walker space-time (RW). Every RW space-time is a perfect fluid space-time [26]. For more details, we refer to ([5,14,24,25,31,33]). Hence it is interesting to know under which condition a perfect fluid space-time is also a RW space-time or more importantly GRW. Chaubey [13], obtained the specific base condition that every perfect fluid space-time can be a GRW. He considered an η-Ricci soliton on a perfect fluid space-time, and under some conditions showed that every perfect fluid space-time admitting a gradient η-Ricci soliton can be a GRW space-time. Recently, Roy et al. [28] found some geometrical properties of perfect fluid space-time with torse forming vector field ξ endowed with conformal Ricci-Yamabe soliton. Also, De et al. [15] obtained some conditions when a perfect fluid space-time is Ricci recurrent. For more study on the above mentioned soliton, we cite ([4,7,18,20,22,23]). Mantica et al. [24] work on perfect fluid space-time endowed with harmonic generalized curvature tensor, and showed that under certain conditions it is a GRW space-time. So it is remarkable how different kinds of solitons on a perfect fluid space-time, characterize these manifolds.

In this paper, we consider a perfect fluid space-time with a gradient Ricci-Bourguignon soliton (GRBS for short). Consider a family of metrics $g(t)$ on a Riemannian manifold M^n evolve by the Ricci-Bourguignon flow (RB), which means that $g(t)$ satisfies

$$\frac{\partial g}{\partial t} = -2(Ric - \mu R g), \tag{2}$$

here Ric is the Ricci tensor, R is the scalar curvature and $\mu \in \mathbb{R}$ is an arbitrary constant [6]. The short time existence of this flow on any closed n-dimensional manifold starting with an arbitrary initial metric g for $\mu < \dfrac{1}{2(n-1)}$ have been studied in [8].

Definition 1.1 ([27]). Let M^n be a semi-Riemannian manifold with a smooth vector field X such that:

$$Ric + \frac{1}{2}\mathcal{L}_X g + \lambda g + \mu R g = 0, \tag{3}$$

here $\mathcal{L}_X g$ is the Lie derivative of the metric g along X; $\lambda, \mu \in \mathbb{R}$ are constants and R is the scalar curvature. M^n is called a Ricci-Bourguignon soliton.

Moreover, if for some smooth function $f : M^n \longrightarrow \mathbb{R}$, $X = \nabla f$, M^n is said to be gradient Ricci-Bourguignon soliton (GRBS). For a GRBS equation (3) becomes

$$Ric + Hess f + \lambda g + \mu R g = 0, \tag{4}$$

here Hess $= \nabla^2$ denotes the Hessian. If $\mu = 0$, then equation (2) reduces to Ricci flow which is an intrinsic geometric flow on a pseudo-Riemannian manifold (see [23]). Depending on λ the RB soliton is expanding, steady, or shrinking whenever $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$, respectively. Heretofore, some interesting results have been obtained from taking X as a conformal vector field which leads the manifold to be isometric to an Euclidean sphere [17]. Also, conditions that are needed for X to be Killing (i.e. $\mathcal{L}_X g = 0$) has been discussed in that paper. In [9, 10] various classification and rigidity result for a GRBS have been studied. Recently, some compactness conditions for the Finsler manifold admitting RB soliton have been investigated in [3].

Some characterization results for a perfect fluid space-time that admits Ricci-Bourguignon soliton were obtained in [30]. Motivated by all of the above mentioned ideas, we study the impact of considering the GRBS on a perfect fluid space-time, and present our results as follows:
First, we state some preliminaries on the perfect fluid space-time, which may be needed in our main Theorems, and then in Section 3, we peruse the geometrical properties of GRBS on the perfect fluid space-time. In the last Section, we suppose a Ricci bi-conformal vector field on perfect fluid space-time, which has RB soliton and study the properties of this space-time.

2. **Basic results about perfect fluid space-time**

This section includes important preliminaries about perfect fluid space-time. This manifold is described by the energy-momentum tensor, which could change the nature of space-time [32]. A perfect fluid space-time is characterized by an energy-momentum tensor T of the following form:

$$T(Y, Z) = pg(Y, Z) + (\sigma + p)\eta(Y)\eta(Z), \tag{5}$$

where p and σ represent the isotropic pressure and energy-density of the perfect fluid space-time, respectively. Here $p+\sigma \neq 0$ and $\sigma > 0$. Considering Einstein's field equation without cosmological constant, we conclude

$$2Ric - Rg = 2\kappa T, \tag{6}$$

here T is the energy momentum tensor and κ is a constant. If M^n be a perfect fluid space-time (1), using (5) and (6), we get

$$h_1 = \frac{\kappa(p - \sigma)}{2 - n}, \quad h_2 = \kappa(p + \sigma). \tag{7}$$

From (1), we obtain

$$R = nh_1 - h_2, \tag{8}$$

and

$$QY = h_1 Y + h_2 \eta(Y)\rho, \;\; \forall Y \in \chi(M), \tag{9}$$

where $\mathrm{Ric}(Y, Z) = g(QY, Z)$, $\forall Y, Z \in \chi(M)$. If R is a constant, then $nY(h_1) = Y(h_2)$ and the vice versa. We express the following lemma of [2] which will be need in our proofs.

Lemma 2.1. *For all perfect fluid space-times, one has*

$$(i) \qquad (\nabla_Y \eta)(\rho) = 0,$$
$$(ii) \qquad \eta(\nabla_Y \rho) = g(\nabla_Y \rho, \rho) = 0, \ \ \forall Y \in \chi(M).$$

3. GRBS on perfect fluid space-time

In this section we suppose that the perfect fluid space-time M^n has constant scalar curvature. Our first and most important result is about the condition that makes the manifold to be Ricci flat.

Proposition 3.1. *For a perfect fluid space-time admits a GRBS, we have*

$$R(Y, Z)Df = (\nabla_Y Q)(Z) - (\nabla_Z Q)(Y) + \mu[(\nabla_Z R)Y - (\nabla_Y R)Z], \tag{10}$$

for all $Y, Z \in \chi(M)$.

Proof. The method of proof is just like Proposition 3.1 in [13], so we briefly state some main steps. From (4), we can write

$$\nabla_Y Df + QY + \lambda Y + \mu RY = 0. \tag{11}$$

Taking covariant derivative of above along Z, we lead

$$\nabla_Z \nabla_Y Df + (\nabla_Z Q)(Y) + Q(\nabla_Z Y) + \lambda(\nabla_Z Y) + \mu(\nabla_Z R)Y + \mu R(\nabla_Z Y) = 0. \tag{12}$$

Relocating Y by Z in above equation, using $R(Y, Z)Df = \nabla_Y \nabla_Z Df - \nabla_Z \nabla_Y Df - \nabla_{[Y,Z]} Df$ and (11), we obtain (10). □

Since we consider R as a constant, from (10), we have

$$R(Y, Z)Df = (\nabla_Y Q)(Z) - (\nabla_Z Q)(Y).$$

Now, contracting the above equation along the vector field Y, we obtain

$$Ric(Z, Df) = 0. \tag{13}$$

On the other hand by (9), we get

$$(\nabla_Z Q)(Y) = h_2[(\nabla_Z \eta)(Y)\rho + \eta(Y)(\nabla_Z \rho)] + Z(h_1)Y + Z(h_2)\eta(Y)\rho. \tag{14}$$

Using (14) in (10), we find

$$\begin{aligned} R(Y, Z)Df &= h_2[(\nabla_Y \eta)(Z)\rho - (\nabla_Z \eta)(Y)\rho + \eta(Z)(\nabla_Y \rho) - \eta(Y)(\nabla_Z \rho)] \\ &\quad + Y(h_1)Z - Z(h_1)Y + Y(h_2)\eta(Z)\rho - Z(h_2)\eta(Y)\rho. \end{aligned}$$

Contracting this over Y gives

$$Ric(Z, Df) = h_2[(\nabla_\rho \eta)(Z) - (\nabla_Z \eta)\rho + \eta(Z) div\rho] + \rho(h_2)\eta(Z) + Z(h_2) + (1-n)Z(h_1).$$

This equation together with (13), leads to

$$-h_2[(\nabla_\rho \eta)(Z)-(\nabla_Z \eta)\rho+\eta(Z)div\rho] = \rho(h_2)\eta(Z)+Z(h_2)+(1-n)Z(h_1). \tag{15}$$

If $g(\nabla_Y \rho, Z) + g(Y, \nabla_Z \rho) = 0$, that means ρ is a Killing vector field, then due to lemma 2.1 and equation (7), we get

$$\nabla_\rho \rho = div\rho = \rho(h_2) = \rho(h_1) = 0. \tag{16}$$

Now we take a Killing vector field for the velocity field of space-time. So (15) and (16) leading us to the next equation

$$(1-n)Z(h_1) + Z(h_2) = 0. \tag{17}$$

Based on these assumptions we get:

Theorem 3.2. *If the velocity vector field of a perfect fluid space-time M^n $(n \geq 4)$ equipped with a GRBS is Killing, and the scalar curvature is constant, then either*
(i) $(n-1)p + (n-3)\sigma = 0$, or
(ii) $Ric = C\eta \otimes \eta$.
Moreover if $h_1 = h_2 \neq 0$, then ∇f is pointwise collinear with ρ.

Proof. By (1), we know

$$Ric(\rho, Df) = (h_1 - h_2)\rho(f). \tag{18}$$

Using (13), after considering $Z = \rho$, we get

$$(h_1 - h_2)\rho(f) = 0. \tag{19}$$

This shows that either $h_1 = h_2 (\neq 0)$ or $\rho(f) = 0$. If $h_1 = h_2$, then (7) gives

$$(n-1)p + (n-3)\sigma = 0. \tag{20}$$

Moreover, our hypothesis together with (17) conclude that h_1, h_2 are constants. From (1) and (13), we get

$$h_1 Z(f) + h_2 \rho(f)\eta(Z) = 0. \tag{21}$$

This equation implies that

$$Df + \rho(f)\rho = 0. \tag{22}$$

The above equation implies that ∇f is pointwise collinear with ρ.
Now, we assume that $\rho(f) = 0$ and $h_1 \neq h_2$. We know $Df \neq 0$ in general. Based on the fact that $\rho(f) = 0$, thus $g(\rho, Df) = 0$. Therefore, taking covariant derivative along the vector field X, and using (11), we conclude

$$g(\nabla_Y \rho, Df) - (h_1 - h_2)\eta(Y) - \lambda\eta(Y) - \mu R\eta(Y) = 0.$$

Let $Y = \rho$ in the above equation. Using (16), we find

$$(h_1 - h_2) + \lambda + \mu R = 0. \tag{23}$$

On the other hand by (21) we get $h_1 Z(f) = 0$ which implies that $h_1 = 0$. Indeed $C = h_2$ is a constant, so $Ric = C\eta \otimes \eta$ and by (7), $p = \sigma$. This finishes the proof. □

In this theorem, when $n = 4$, we have $\sigma + 3p = 0$, which means that the perfect fluid space-time represents radiation era. From second equation (ii) and Lemma 2.1, we infer that M^n has parallel Ricci tensor. Thus M^n is a Yang pure space (YPS). This space is so useful in the gravitational and gauge theory (see [19]).
The next result is obtained with just using (20). Actually, for 4-dimensional perfect fluid space-time by (20), we have zero-active mass condition $3p+\sigma = 0$ which named by Görnits [21].

Corollary 3.3. *Let M^4 be a perfect fluid space-time admitting a GRBS. Also consider that the velocity vector field of M is Killing and $\rho(f) \neq 0$, then $3p+\sigma = 0$.*

Corollary 3.4. *For a perfect fluid space-time M^n with GRBS, if the velocity vector field ρ is Killing, and $h_1 = h_2$ on M, then the energy momentum tensor is Killing along ρ.*

Proof. Since $h_1 = h_2$, therefore from (1), we have

$$Ric = h_1(g + \eta \otimes \eta),$$

moreover, by (8),(6), and (7), we get

$$T = -\frac{h_1}{2\kappa}[(n-3)g - 2\eta \otimes \eta]. \tag{24}$$

Taking Lie derivative of above equation along the vector field ρ, infers

$$(\mathcal{L}_\rho T)(Y, Z) = \frac{h_1}{\kappa}[(\mathcal{L}_\rho \eta)(Y)\eta(Z) + (\mathcal{L}_\rho \eta)(Z)\eta(Y)].$$

Also, the Lie derivative of $\eta(X) = g(X, \rho)$ along the vector field ρ gives $(\mathcal{L}_\rho \eta)(Y) = 0$. So we see

$$(\mathcal{L}_\rho T)(Y, Z) = 0.$$

This completes the proof. □

Corollary 3.5. *For a perfect fluid space-time M^n which admits GRBS, if $h_1 = h_2$ and the velocity vector field ρ is Killing, then the energy momentum tensor is cyclic parallel.*

Proof. Taking covariant derivative of (24), gives

$$(\nabla_X T)(Y, Z) = \frac{h_1}{\kappa}[(\nabla_X \eta)(Y)\eta(Z) + \eta(Y)(\nabla_X \eta)(Z)],$$

hence

$$\begin{aligned}(\nabla_X T)(Y, Z) + (\nabla_Y T)(Z, X) + (\nabla_Z T)(X, Y) \quad = \quad & \frac{h_1}{\kappa}\Big[\eta(X)[(\nabla_Y \eta)(Z) + (\nabla_Z \eta)(Y)] \\ & +\eta(Y)[(\nabla_X \eta)(Z) + (\nabla_Z \eta)(X)] \\ & +\eta(Z)[(\nabla_Y \eta)(X) + (\nabla_X \eta)(Y)]\Big].\end{aligned}$$

Since ρ is Killing, we get

$$(\nabla_X \eta)(Y) + (\nabla_Y \eta)(X) = 0.$$

Which yields

$$(\nabla_X T)(Y, Z) + (\nabla_Y T)(Z, X) + (\nabla_Z T)(X, Y) = 0$$

this completes the proof. □

Corollary 3.6. *With the same conditions as the above corollary for a perfect fluid space-time, we obtain*

$$\mathcal{L}_\rho \mathrm{Ric} = 0,$$

$$(\nabla_X Ric)(Y, Z) + (\nabla_Y Ric)(Z, X) + (\nabla_Z Ric)(X, Y) = 0,$$

in addition with considering $h_1 = h_2$, and $\mathcal{L}_\rho g = 0$, we know the velocity vector field is Ricci inheritance, and the Ricci tensor is cyclic parallel.

Proof. Based on the fact that R and κ are constants, by (6), we know

$$\mathcal{L}_\rho \mathrm{Ric} = \kappa \mathcal{L}_\rho T, \qquad \nabla_X \mathrm{Ric} = \kappa \nabla_X T.$$

Hence, by Corollary 3.4 and Corollary 3.5, we obtain the result. □

Theorem 3.7. *For a perfect fluid space-time M^n with the Lorentzian metric of structure of GRBS, if ρ is Killing, and $\rho(f) \neq 0$, then M^n admits a unit time-like tors forming vector field $\nabla\rho = \varphi(\eta \otimes \rho)$, and the Einstein potential function f satisfies in the following partial differential equation*

$$\frac{\partial^2 f}{\partial t^2} = \kappa(p + \sigma), \tag{25}$$

and it is determined by $f = c_1 t^2 + c_2$, where $c_1 = \dfrac{\kappa(p+\sigma)}{2} \neq 0$.

Proof. Let $h_1 = h_2$, then under this consideration (22) is satisfied. Differentiating equation (22) covariantly along the vector field X, and using (11), we find

$$X(\rho(f))\rho + \rho(f)\nabla_X \rho = -QX - \lambda X - \mu R X. \tag{26}$$

Letting $X = \rho$ in (26), and taking inner product of the foregoing equation with ρ, we get

$$h_2 - h_1 - \lambda - \mu R = \rho(\rho(f)). \tag{27}$$

We consider an orthonormal base on the perfect fluid space-time and contract (26) along X, then we get

$$\rho(\rho(f)) = -n\lambda - (n\mu + 1)R. \tag{28}$$

Equation (8) together with (27) and (28), yields

$$\lambda + \mu R + h_1 = 0. \tag{29}$$

From (8) and (7), we know

$$R(2 - n) = \kappa((n-1)p - \sigma). \tag{30}$$

Plugging (29) and (30) into (28), we conclude

$$\rho(\rho(f)) = \kappa(p+\sigma). \tag{31}$$

Supposing $\rho = \dfrac{\partial}{\partial t}$, (31) becomes

$$\frac{\partial^2 f}{\partial t^2} = \kappa(p+\sigma). \tag{32}$$

Considering $f = c_1 t^2 + c_2$, where $c_1 \neq 0$ and c_2 are independent of t. This f satisfies in the equation (32) with $c_1 = \dfrac{\kappa(p+\sigma)}{2}$. The inner product of equation (26) with ρ leads to

$$(h_2 - h_1 - \lambda - \mu R)\eta(X) = X(\rho(f)).$$

Using this in (26), and making use of (8), (7), and (29), with the same method as Theorem 4.6 in [13], we obtain

$$\nabla\rho = \varphi(\eta \otimes \rho),$$

where $\varphi = \dfrac{2h_2}{\rho(f)}$. This completes the proof. □

This theorem specifies the structure of the gradient of the time-like vector field.

Now we assume that the perfect fluid space-time admits a GRBS with gradient vector field ∇f, and $\rho(f) = 0$, then by Theorem 3.2, we infer

$$Ric = C\eta \otimes \eta, \tag{33}$$

where $h_2 = C \neq 0$. Taking the covariant derivative of (33), gives

$$(\nabla_X Ric)(Y,Z) = C[(\nabla_X \eta)(Y)\eta(Z) + \eta(Y)(\nabla_X \eta)(Z)]. \tag{34}$$

We consider a Ricci recurrent perfect fluid space-time (i.e. the Ricci tensor Ric satisfies $(\nabla_X Ric)(Y,Z) = A(X)Ric(Y,Z)$, for every $X,Y,Z \in \chi(M)$ and 1-form A). From (34), we have

$$C[(\nabla_X \eta)(Y)\eta(Z) + \eta(Y)(\nabla_X \eta)(Z)] = A(X)Ric(Y,Z).$$

Let $Z = \rho$, so we get

$$\nabla_X \rho = RA(X)\rho,$$

this shows $g(\nabla_X \rho, \rho) = RA(X)g(\rho,\rho) = 0$, and since $R \neq 0$, then $A(X) = 0$. Consequently $\nabla\rho = 0$. By these relations it can be easily concluded:

Corollary 3.8. *For a Ricci recurrent perfect fluid space-time with Killing velocity vector field equipped with a GRBS and $\rho(f) = 0$, we conclude that it is Ricci symmetric, and its velocity vector field ρ is parallel.*

As a final result we assume a perfect fluid space-time as a pseudo Ricci symmetric space with a GRBS.

Definition 3.9 ([29])**.** A semi-Riemannian manifold with a 1-form A such that the non-vanishing Ricci tensor satisfies the following estimate

$$(\nabla_X Ric)(Y,Z) = 2A(X)Ric(Y,Z) + A(Y)Ric(X,Z) + A(Z)Ric(X,Y), \tag{35}$$

for every $X, Y, Z \in \chi(M)$ called as a pseudo Ricci symmetric space-time.

By this definition it is clear that

Corollary 3.10. *For a pseudo Ricci symmetric space-time M^n which admits a GRBS, if ρ is Killing and $\rho(f) = 0$, then M^n is Ricci symmetric and ρ is parallel.*

Proof. Let $Z = \rho$, then (35) together with (34), becomes

$$R(\nabla_X \eta)(Y) = 2A(X)\eta(Y) + A(Y)\eta(X) - A(\rho)\eta(X)\eta(Y). \tag{36}$$

Now, replacing X with ρ in the above equation, we get

$$A(Y) = 3A(\rho)\eta(Y).$$

Since $\eta(\rho) = -1$, then $A(\rho) = 0$, and $A(Y) = 0$. Also, Y is arbitrary, from (35), we know $\nabla Ric = 0$, and by (36), $\nabla\rho = 0$, which means that the pesudo Ricci symmetric space-time M^n is Ricci symmetric, and the velocity vector field ρ is parallel. □

4. Perfect fluid space-time with Ricci biconformal vector field

Bi-conformal vector field on a Riemannian manifold studied in [12]. Similarly we can define bi-conformal vector fields for semi-Riemannian manifolds as follows:

Definition 4.1. A vector field X on a semi-Riemannian manifold is named Ricci bi-conformal vector field if it satisfies:

$$(\mathcal{L}_X g)(U,V) = h_1 g(U,V) + h_2 \mathrm{Ric}(U,V), \tag{37}$$

and

$$(\mathcal{L}_X Ric)(U,V) = h_1 \mathrm{Ric}(U,V) + h_2 g(U,V), \tag{38}$$

for some arbitrary non-zero smooth function h_1, h_2.

If we consider that a perfect fluid space-time equipped with a biconformal vector field, then we conclude:

Theorem 4.2. *For the perfect fluid space-time M^n endowed with a RBS, its metric is Einstein if it has a Ricci bi-conformal vector field with constants h_1, h_2 and its scalar curvature is constant.*

Proof. From (3), we obtain

$$Ric(U,V) + \frac{1}{2}\mathcal{L}_X g(U,V) + \lambda g(U,V) + \mu R g(U,V) = 0.$$

Substituting (37) in above, we get

$$(2+h_2)Ric(U,V)+(h_1+2\lambda+2\mu R)g(U,V)=0,$$

so,

$$\mathrm{Ric}(U,V)=\frac{-(h_1+2\lambda+2\mu R)}{(2+h_2)}g(U,V). \tag{39}$$

Therefore, if h_1, h_2 and R are constant, then the metric g is Einstein. □

5. Conclusion

6. Author Contributions

All authors have read and agreed to the published version of the manuscript.

7. Data Availability Statement

All data generated or analysed during this study are included in this published article.

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10. Conflict of interest

The authors declare no conflict of interest.

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