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VARENTROPY ESTIMATORS APPLIED TO TEST OF FIT FOR INVERSE GAUSSIAN DISTRIBUTION

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Abstract. Recently, Alizadeh and Shafaei (2023) introduced some estimators for the varentropy of a continuous random variable. The present article applies these estimators and constructs some tests of fit for Inverse Gaussian distribution. Percentage points and type I error of the new tests are obtained and then power values of the proposed tests against various alternatives are computed. The results of a simulation study show that the tests have a good performance in terms of power. Finally, a real data set is used to illustrate the application of the proposed tests.

Keywords: Varentropy estimator, Inverse Gaussian distribution, Maximum likelihood estimates, Goodness-of-fit test, Percentage points, Monte Carlo simulation, Test power.
2020 MSC: Primary 62B10, 94A17, 62G10.

1. Introduction

The Inverse Gaussian distribution (IG) is a continuous probability distribution that is widely used in statistics and probability theory. Its density function is

$$f(x;\mu,\lambda)=\left(\frac{\lambda}{2\pi x^3}\right)^{1/2}\exp\left\{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right\},\qquad x>0,$$

where μ and λ are parameters. The mean and variance of this distribution are μ and μ^3/λ, respectively.

The Inverse Gaussian distribution has many important properties that make it useful in a wide variety of applications. For example, it is a conjugate prior distribution for the mean of a normal distribution with unknown variance, which means that it can be used to derive a posterior distribution for the mean of the normal distribution. It is also used in survival analysis to model the time until an event occurs, such as the failure of a machine or the death of a patient.

The Inverse Gaussian distribution has several important applications in finance, where it is used to model the distribution of stock prices, interest rates, and other financial variables. It is also used in engineering and physics to model the

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distribution of reaction times, particle sizes, and other physical variables. Its distinctive shape and important properties make it a valuable tool for modeling a wide range of phenomena in many different fields. For more details about applications, one can see Folks and Chhikara (1978), Chhikara and Folks (1989), Seshadri (1999), and their references.
Here are some of the applications of the Inverse Gaussian distribution:

(1) **Finance:** The Inverse Gaussian distribution is widely used in finance to model the distribution of stock prices, interest rates, and other financial variables. It is particularly useful for modeling the distribution of asset returns because it allows for skewness and kurtosis, which are common in financial data.
(2) **Survival analysis:** The Inverse Gaussian distribution is commonly used in survival analysis to model the time until an event occurs, such as the failure of a machine or the death of a patient. It is particularly useful for modeling data that exhibit a non-constant hazard rate, which means that the probability of an event occurring changes over time.
(3) **Reaction times:** The Inverse Gaussian distribution is used in psychology and neuroscience to model the distribution of reaction times in experiments. It is particularly useful for modeling data that exhibit a skewed distribution, such as reaction times that are faster on average than they are slow.
(4) **Particle size analysis:** The Inverse Gaussian distribution is used in engineering and physics to model the distribution of particle sizes in materials. It is particularly useful for modeling data that exhibit a skewed distribution, such as particle sizes that are smaller on average than they are larger.
(5) **Quality control:** The Inverse Gaussian distribution is used in quality control to model the distribution of defects in a manufacturing process. It is particularly useful for modeling data that exhibit a skewed distribution, such as defects that are less common on average than they are more common.

In summary, the Inverse Gaussian distribution has a wide range of applications in many different fields, including finance, survival analysis, psychology, engineering, physics, and quality control. Its ability to model skewed distributions and non-constant hazard rates makes it a valuable tool for modeling a wide range of phenomena. One can see for example, Folks and Chhikara (1978), Bardsley (1980), Chhikara and Folks (1989), Seshadri (1993,1999), Johnson et al. (1994), and Barndorff-Nielsen (1994). Therefore, constructing powerful goodness of fit tests for the IG distribution is an important issue. In this article, we develop some goodness of tests for the IG distribution using different varentropy estimators.
The goodness of fit tests are statistical tests used to evaluate whether a set of

data is consistent with a particular probability distribution. The basic idea behind the goodness of fit test is to compare the observed data with the expected values under a given distribution, and to determine whether the differences between the observed and expected values are statistically significant.
One of the well-known goodness of fit tests is the Kolmogorov-Smirnov test, which is used to test whether a set of data follows a particular continuous distribution. This test is used in many statistical software and is particularly useful for testing the fit of an IG distribution to a set of data.
The goodness of fit tests has important applications in many fields, including biology, engineering, finance, physics, and psychology. For example, in biology, goodness of fit tests can be used to test whether the distribution of a particular trait in a population follows a particular genetic model. In finance, goodness of fit tests can be used to test whether stock returns follow a particular distribution, such as the normal or lognormal distribution. In psychology, goodness of fit tests can be used to test whether a set of responses to a questionnaire follows a particular response model, such as the Rasch model.
Overall, goodness of fit tests are powerful tools for evaluating the fit of a particular distribution to a set of data, and can be used to make important inferences and predictions in a wide range of fields. They are essential for ensuring the validity and accuracy of statistical models and predictions. Goodness-of-fit testing for the inverse Gaussian distribution has been investigated by some authors including O'Reilly and Rueda (1992), Pavur et al. (1992), Gunes el al. (1997), Mergel (1999), Ducharme (2001), Mudholkar et al. (2001), Henze and Klar (2002), Mudholkar and Tian (2002), Vladimirescu and Tunaru (2003), Natarajan and Mudholkar (2004), Vexler et al. (2011), Al-Omari and Haq (2012), Best et al. (2012), Choi (2013), Lequesne and Regnault (2018), Alizadeh and Mohtashami (2020), Allison et al. (2022), and Alizadeh and Shafaei (2023). Moreover, the first time Vasicek (1976) applied the entropy of a continuous random variable and introduced a goodness of fit test for normal distribution based on entropy.
In Section 2, some estimators for the varentropy of a continuous random variable are presented and their properties are stated. In Section 3, new test statistics for testing a hypothesis that the sample comes from an Inverse Gaussian distribution based on the varentropy estimators are introduced. In Section 4, the results of our simulation studies are described. The percentage points and type I error of our test statistics are obtained for different sample sizes based on 100,000 sample values generated by a Monte Carlo experiment. Moreover, the power values of the proposed tests are computed and compared with the well-known Kolmogorov-Smirnov test. Simulation results indicate that the varentropy-based tests perform well against various alternatives and have distinctly higher power than the competing test. In Section 5, a real data set is used to illustrate the application of the proposed tests. Some concluding remarks are contained in Section 6.

2. Varentropy and its estimators

Shannon differential entropy (1948) of a continuous random variable is a well-known information measure that represents the expectation of the information content of an absolutely continuous random variable. The corresponding variance is termed varentropy and is used in various applications of information theory. Recent contributions on the varentropy can be found in various articles by Bobkov and Madiman (2011), Kontoyiannis and Verdu (2014), Arikan (2016), Peccarelli (2017), Buono and Longobardi (2020), Di Crescenzo et al. (2021), Di Crescenzo and Paolillo (2021), Maadani et al. (2021), Paolillo (2021), Zaid et al. (2022), Saha and Kayal (2023), and Sharma and Kundu (2023).
Suppose that a random variable X has distribution function $F(x)$ with continuous density function $f(x)$. The random variable

$$IC(X) = -\log f(X),$$

is often referred as the information content of X. The expectation of the information content of X, is termed differential entropy and is given by

$$H(X) = E\left[IC(X)\right] = -E\left[\log f(X)\right] = -\int_{-\infty}^{\infty} f(x)\log f(x)dx\,.$$

Intuitively, $H(X)$ measures the expected uncertainty contained in $f(x)$ about the predictability of an outcome of X.
The varentropy of continuous random variable X is defined by the variance of the information content of X is given by

$$\begin{aligned} V(X) &= Var\left[IC(X)\right] = Var\left[\log f(X)\right] = E\left[(IC(X))^2\right] - E^2\left[IC(X)\right] \\ &= E\left[(IC(X))^2\right] - \left[H(X)\right]^2 \\ &= \int_{-\infty}^{\infty} f(x)[\log f(x)]^2 dx - \left[\int_{-\infty}^{\infty} f(x)\log f(x)dx\right]^2 . \end{aligned}$$

The varentropy thus measures the variability in the information content of X. This measure has been investigated by some authors. Fradelizi et al. (2016) used it to find an optimal varentropy bound for log-concave distributions. Also, a sharp uniform bound on varentropy for log-concave distributions is presented by Madiman (2014). An alternative way to calculate a bound for varentropy is discussed in Goodarzi et al. (2017). Application of the varentropy in reliability theory is conducted by Di Crescenzo and Paolillo (2021). Also, the properties of the varentropy of order statistics are studied by Maadani et al. (2021).
Recently, the problem of estimating the varentropy of continuous random variables is investigated by Alizadeh and Shafaei (2023). They introduced some varentropy estimators and stated their properties. Also, they used a Monte Carlo simulation to compute the root mean square error (RMSE) of the estimators and then a comparison between the proposed estimators in terms of RMSE is performed. Their estimators are described as follows.
Suppose $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ are order statistics of a random sample

of size n from an unknown continuous distribution F with a probability density function f. We interest in estimate of varentropy $V(X)$ of an unknown continuous probability density function f.

2.1. **The first estimator.** The first estimator is based on the fact that $V(X)$ can be expressed as

$$V(X) = \int_0^1 \log^2\left\{\frac{d}{dp}F^{-1}(p)\right\}^{-1} dp - \left[\int_{-\infty}^{\infty} \log\left\{\frac{d}{dp}F^{-1}(p)\right\}^{-1} dp\right]^2 .$$

We replace the distribution function F by the empirical distribution function F_n, and use a difference operator instead of the differential operator and then estimate the derivative of $F^{-1}(p)$ by a function of the order statistics. Assuming that $X_1, \cdots, X_n$ is a random sample, the first estimator is given by

$$\begin{aligned} VV_{mn} &= \tfrac{1}{n}\sum_{i=1}^{n} \log^2\left(\tfrac{2m/n}{X_{(i+m)}-X_{(i-m)}}\right) - \left[\tfrac{1}{n}\sum_{i=1}^{n} \log\left(\tfrac{2m/n}{X_{(i+m)}-X_{(i-m)}}\right)\right]^2 \\ &= \tfrac{1}{n}\sum_{i=1}^{n} \log^2\left(X_{(i+m)} - X_{(i-m)}\right) - \left[\tfrac{1}{n}\sum_{i=1}^{n} \log\left(X_{(i+m)} - X_{(i-m)}\right)\right]^2 . \end{aligned}$$

where the window size m is a positive integer smaller than $n/2$, $X_{(i)} = X_{(1)}$ if $i < 1$, $X_{(i)} = X_{(n)}$ if $i > n$. According to Alizadeh and Shafaei (2023), $VV_{mn} \to V(X)$ in probability as $n \to \infty, m \to \infty$, $m/n \to 0$.

2.2. **The second estimator.** Alizadeh and Shafaei (2023) adjusted the weights of the first estimator, in order to take into account the fact that the differences are truncated around the smallest and the largest data points. (i.e., $X_{(i+m)} - X_{(i-m)}$ is replaced by $X_{(i+m)} - X_{(1)}$ when $i \leq m$ and $X_{(i+m)} - X_{(i-m)}$ is replaced by $X_{(n)} - X_{(i-m)}$ when $i \geq n - m + 1$). The second estimator is given by

$$VE_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log^2\left(\frac{c_i m/n}{X_{(i+m)} - X_{(i-m)}}\right) - \left[\frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{c_i m/n}{X_{(i+m)} - X_{(i-m)}}\right)\right]^2,$$

where

$$c_i = \begin{cases} 1 + \frac{i-1}{m}, & 1 \leq i \leq m, \\ 2, & m+1 \leq i \leq n-m, \\ 1 + \frac{n-i}{m}, & n-m+1 \leq i \leq n. \end{cases}$$

It is easy to show that $VE_{mn} \to V(X)$ as $n \to \infty, m \to \infty$, $m/n \to 0$.

2.3. **The third estimator.** The third estimator is as

$$VD = \int_{-\infty}^{\infty} \hat{f}(x)\left[\log \hat{f}(x)\right]^2 dx - \left[\int_{-\infty}^{\infty} \hat{f}(x)\log \hat{f}(x) dx\right]^2 ,$$

where $\hat{f}$ is the kernel density function estimation of f and is defined by

$$\hat{f}(x) = \frac{1}{nh}\sum_{j=1}^{n} k\left(\frac{x - X_j}{h}\right),$$

where h is a bandwidth and k is a kernel function which satisfies $\int_{-\infty}^{\infty} k(x)dx = 1$. Usually, k will be a symmetric probability density function.

2.4. **The fourth estimator.** Since

$$V(X) = E\left[(\log f(X))^2\right] - E^2\left[\log f(X)\right] ,$$

a simple estimator of varentropy can be obtained as

$$\frac{1}{n}\sum_{i=1}^{n}(\log f(X_i))^2 - \left[\frac{1}{n}\sum_{i=1}^{n}\log f(X_i)\right]^2 .$$

Therefore, Alizadeh and Shafaei (2023) proposed the following estimator.

$$VB_n = \frac{1}{n}\sum_{i=1}^{n}\left(\log \hat{f}(X_i)\right)^2 - \left[\frac{1}{n}\sum_{i=1}^{n}\log \hat{f}(X_i)\right]^2 ,$$

where

$$\hat{f}(X_i) = \frac{1}{nh}\sum_{j=1}^{n} k(\frac{X_i - X_j}{h}) ,$$

and the kernel function is chosen to be the standard normal density function and the bandwidth h is chosen to be the normal optimal smoothing formula, $h = 1.06sn^{-\frac{1}{5}}$, where s is the sample standard deviation. They showed that $VB_n \to V(X)\,,\ as\ n \to \infty$.

2.5. **The fifth estimator.** Based on a local linear model, Alizadeh and Shafaei (2023) obtained an estimator of varentropy as

$$VC_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log^2\left(\frac{\sum_{j=i-m}^{i+m}(X_{(j)}-\bar{X}_{(i)})(j-i)}{n\sum_{j=i-m}^{i+m}(X_{(j)}-\bar{X}_{(i)})^2}\right) - \left[\frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{\sum_{j=i-m}^{i+m}(X_{(j)}-\bar{X}_{(i)})(j-i)}{n\sum_{j=i-m}^{i+m}(X_{(j)}-\bar{X}_{(i)})^2}\right)\right]^2 ,$$

where

$$\bar{X}_{(i)} = \frac{1}{2m+1}\sum_{j=i-m}^{i+m} X_{(j)} .$$

This estimator is consistent, $VC_{mn} \to V(X)$ in probability as $n \to \infty\,, m \to \infty$, $m/n \to 0$.

2.6. **The sixth estimator.** Alizadeh and Shafaei (2023) proposed to estimate the varentropy $V(X)$ of an unknown continuous probability density function f by

$$\begin{aligned} VA_{mn} &= \frac{1}{n}\sum_{i=1}^{n}\log^2\left\{\frac{\hat{f}(X_{(i+m)})+\hat{f}(X_{(i-m)})}{2}\right\} - \left[\frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{\hat{f}(X_{(i+m)})+\hat{f}(X_{(i-m)})}{2}\right\}\right]^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}\log^2\left\{\hat{f}(X_{(i+m)})+\hat{f}(X_{(i-m)})\right\} - \left[\frac{1}{n}\sum_{i=1}^{n}\log\left\{\hat{f}(X_{(i+m)})+\hat{f}(X_{(i-m)})\right\}\right]^2 , \end{aligned}$$

where

$$\hat{f}(X_i) = \frac{1}{nh}\sum_{j=1}^{n} k(\frac{X_i - X_j}{h}),$$

and the kernel function is chosen to be the standard normal density function and the bandwidth h is chosen to be the normal optimal smoothing formula, $h = 1.06sn^{-\frac{1}{5}}$, where s is the sample standard deviation. Also $X_{(i)} = X_{(1)}$ if $i < 1$, $X_{(i)} = X_{(n)}$ if $i > n$. They proved that $VA_{mn} \to V(X)$ in probability as $n \to \infty, m \to \infty$, $m/n \to 0$.

2.7. **Comparison of the estimators.** Alizadeh and Shafaei (2023) performed a simulation study to analyze the behaviors of the proposed estimators of varentropy. They used the normal, exponential and uniform distributions and generated different sample sizes of these distributions and then obtained the root of mean squared errors (RMSEs) of the estimators. In varentropy estimation it is necessary to determine the value of m for given n. Based on the recommendation of Alizadeh and Shafaei (2023), we use the following heuristic formula:

$$m = \left[\sqrt{n} + 0.5\right].$$

The RMSE values and standard deviation of the six varentropy estimators at different sample size for normal, exponential and uniform distributions are reported by Alizadeh and Shafaei (2023). They observed that these estimators perform well under different distributions. Under normal distribution, the estimator VD has the best performance in terms of RMSE, and also the estimator VA has a better performance than the other estimators in terms of standard deviation. For the exponential population, again the estimators VD and VA have the least RMSE and SD than the competitors, respectively. Under uniform distribution, the estimator VA has a good performance in compared to the other estimators. Generally, they concluded that the estimators VD and VA behave better than the other estimators.

It should be noted that the derivation of the asymptotic properties of the considered estimators is substantially complicated. However, we investigate the asymptotic behavior of the estimators by simulation. Figures 1-3 show the empirical densities of the varentropy estimators generated with 10,000 samples of size $n = 1000$ from the standard normal, exponential and uniform distributions. It is evident that the limiting distribution of the estimators is a normal distribution with mean $V(X)$.

3. The proposed tests for IG distribution

In this section, we apply the varentropy estimators and propose some goodness of fit tests for the IG distribution. It should be mentioned that the results obtained in this article are not general and are specific to testing the validity of the IG distribution.

Given a random sample $X_1, \cdots, X_n$ from a continuous probability distribution F with a density $f(x)$ over a non-negative support, the hypothesis of interest is

$$H_0: \ f(x) = f_0(x;\mu,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right\}, \quad x > 0, \ \ for\ some\ (\mu,\lambda) \in \Theta,$$

where μ and λ are unspecified and $\Theta = R^+ \times R^+$. The alternative to H_0 is

$$H_1: \ f(x) \neq f_0(x;\mu,\lambda), \qquad for\ any\ (\mu,\lambda) \in \Theta.$$

Without loss of any generality, one can reduce the above problem of goodness-of-fit, to testing the hypothesis of uniformity on the unit interval, by means of the probability

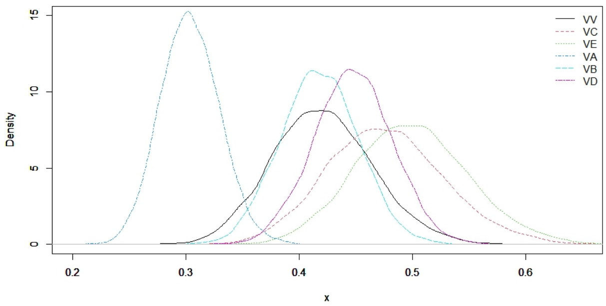

FIGURE 1. Estimated empirical densities of the varentropy estimators generated with 10,000 samples of size $n = 1000$ from the standard normal distribution.

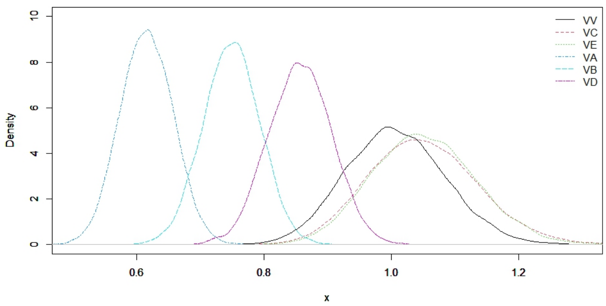

FIGURE 2. Estimated empirical densities of the varentropy estimators generated with 10,000 samples of size $n = 1000$ from the standard exponential distribution.

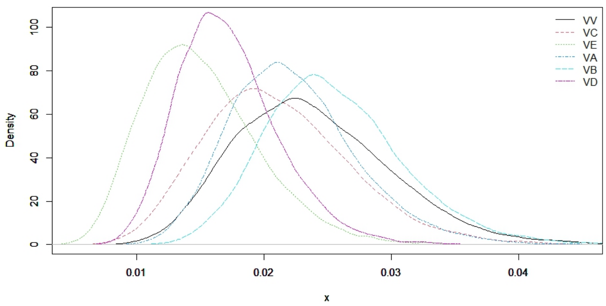

FIGURE 3. Estimated empirical densities of the varentropy estimators generated with 10,000 samples of size $n = 1000$ from the uniform distribution.

integral transformation $U = F_0(X)$. Therefore if $U_i = F_0(X_i)$, $i = 1, 2, \cdots, n$ be the transformed sample, the problem becomes the following testing uniformity.

$$H_0: \; f(u) = 1, \quad 0 < u < 1$$

against

$$H_1: \; f(u) \neq 1, \quad 0 < u < 1.$$

Now, we use the test proposed by Alizadeh and Shafaei (2023) for uniformity. Therefore, the proposed test statistics can be constructed as follows.
Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ denote the order statistics of the sample. Alizadeh and Shafaei (2023) showed that for an f concentrated on $[0, 1]$ one always has

$$V(X) \geq 0\,,$$

with the minimum value of $V(X)$, zero, being uniquely attained by the $U(0, 1)$ density. Based on this property, we construct our test of H_0. A consistent test of the hypothesis of uniformity is then given by

$$T_n = \hat{V}(X),$$

where $\hat{V}(X)$ is the sample estimate of $V(X)$. In the previous section, we investigated some estimators for varentropy $V(X)$, and now we use them here.
Since the mentioned estimators, $\hat{V}(X)$, converges in probability to $V(X)$ as $n \to \infty$, under the null hypothesis H_0, T_n converges in probability to 0 as $n \to \infty$ and under an alternative distribution on $[0, 1]$ with absolutely continuous density f, T_n converges in probability to a number larger than zero as $n \to \infty$.
Based on different varentropy estimators, we propose the following test statistics for the test of fit for IG distribution.

$$TV = VV_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log^2\left(U_{(i+m)} - U_{(i-m)}\right) - \left[\frac{1}{n}\sum_{i=1}^{n} \log\left(U_{(i+m)} - U_{(i-m)}\right)\right]^2;$$

$$TE = VE_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log^2\left(\frac{c_i m/n}{U_{(i+m)} - U_{(i-m)}}\right) - \left[\frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{c_i m/n}{U_{(i+m)} - U_{(i-m)}}\right)\right]^2;$$

$$TD = VD = \int_{-\infty}^{\infty} \hat{f}(u)\left[\log \hat{f}(u)\right]^2 du - \left[\int_{-\infty}^{\infty} \hat{f}(u)\log \hat{f}(u)du\right]^2;$$

$$TB = VB_n = \frac{1}{n}\sum_{i=1}^{n}\left(\log \hat{f}(U_i)\right)^2 - \left[\frac{1}{n}\sum_{i=1}^{n} \log \hat{f}(U_i)\right]^2;$$

$$TC = VC_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log^2\left(\frac{\sum\limits_{j=i-m}^{i+m}(U_{(j)} - \bar{U}_{(i)})(j-i)}{n\sum\limits_{j=i-m}^{i+m}(U_{(j)} - \bar{U}_{(i)})^2}\right) - \left[\frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{\sum\limits_{j=i-m}^{i+m}(U_{(j)} - \bar{U}_{(i)})(j-i)}{n\sum\limits_{j=i-m}^{i+m}(U_{(j)} - \bar{U}_{(i)})^2}\right)\right]^2;$$

$$TA = VA_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log^2\left\{\hat{f}(U_{(i+m)}) + \hat{f}(U_{(i-m)})\right\} - \left[\frac{1}{n}\sum_{i=1}^{n} \log\left\{\hat{f}(U_{(i+m)}) + \hat{f}(U_{(i-m)})\right\}\right]^2.$$

where $U_i = F_0(X_i; \hat{\mu}, \hat{\lambda})$, $i = 1, 2, \cdots, n$, denote the transformed sample. Also, $\hat{\mu}$ and $\hat{\lambda}$ are the maximum likelihood estimates of the parameters μ and λ, respectively.

$$\hat{\mu} = \bar{X} \quad ; \quad \hat{\lambda} = \frac{n}{\sum\limits_{i=1}^{n}\left(1/X_i - 1/\bar{X}\right)}.$$

Clearly, as $n \to \infty$, the maximum likelihood estimators $\hat{\mu}$ and $\hat{\lambda}$ tend to μ and λ. Let $U_{(1)} \leq U_{(2)} \leq \cdots \leq U_{(n)}$ denote the order statistics of the transformed sample. Since the empirical varentropy converges to the varentropy of U, i.e., $\hat{V}(U) \to V(U)$, the proposed tests are consistent. Under the null hypothesis H_0, the proposed test statistics converge in probability to 0 as $n \to \infty$ and under an alternative distribution H_1, they converge in probability to a number larger than zero as $n \to \infty$.
Guided by these properties, given any significance level α, and any finite sample size n, our test procedure is then defined by the critical region

$$T_n = \hat{V}(X) \geq C_{1-\alpha}$$

where $C_{1-\alpha}$ is set so that the test has the desired level α for given n. For specific α and n, the $C_{1-\alpha}$ can be obtained by Monte Carlo methods. For the proposed tests, we determine the $C_{1-\alpha}$ based on Monte Carlo simulation.

4. Percentage points and power study

In this section, we obtain the percentage points of the proposed test statistics by Monte Carlo methods and then power values of the tests are computed.

4.1. **Critical values and the actual sizes.** The distribution of the test statistics TV, TE, TD, TB, TC and TA under the null hypothesis cannot be evaluated analytically. Therefore, the critical values of the test statistics are computed by the Monte Carlo method.
For selected values of the sample size n, 100,000 samples of size n from IG distribution are generated. For each sample, the test statistics are computed. For level α, the percentage point $C_{1-\alpha}$ of the distribution of T_n is estimated by the $1-\alpha$ percentile of the empirical distribution function of T_n based on the observed 100,000 samples. These estimates are presented in Table 1.

TABLE 1. Percentage points of the proposed test statistics at level $\alpha = 0.05$

n	TV	TE	TD	TB	TC	TA
10	0.5012	0.3137	0.0970	0.1802	4587	0.0230
20	0.3450	0.2062	0.0830	0.1399	0.3035	0.0382
30	0.2561	0.1526	0.0730	0.1189	0.2246	0.0449
40	0.2066	0.1228	0.0666	0.1057	0.1814	0.0469
50	0.1744	0.1032	0.0613	0.0960	0.1529	0.0470
75	0.1316	0.0779	0.0526	0.0812	0.1153	0.0465
100	0.1112	0.0698	0.0474	0.0725	0.0999	0.0473

Moreover, Figure 4 shows the empirical density of the proposed test statistics for $n = 50$, with 100,000 simulated random samples. We can see that the test statistic TA has a smaller variance than the other test statistics.

Table 2. Type I error control of the tests for the nominal significance level $\alpha = 0.05$.

$IG(\mu,\lambda)$	n	D	TV	TE	TD	TB	TC	TA
$IG(0.5,0.5)$	10	0.0499	0.0479	0.0485	0.0485	0.0492	0.0482	0.0506
	20	0.0498	0.0507	0.0496	0.0498	0.0492	0.0493	0.0516
	30	0.0498	0.0500	0.0497	0.0502	0.0496	0.0490	0.0500
	50	0.0503	0.0496	0.0487	0.0493	0.0496	0.0499	0.0497
$IG(0.5,1)$	10	0.0380	0.0480	0.0466	0.0449	0.0462	0.0481	0.0449
	20	0.0393	0.0471	0.0432	0.0447	0.0460	0.0456	0.0420
	30	0.0399	0.0472	0.0461	0.0453	0.0468	0.0474	0.0427
	50	0.0368	0.0448	0.0451	0.0454	0.0474	0.0471	0.0445
$IG(0.5,2)$	10	0.0312	0.0452	0.0435	0.0403	0.0429	0.0452	0.0395
	20	0.0302	0.0461	0.0426	0.0425	0.0450	0.0459	0.0379
	30	0.0306	0.0454	0.0433	0.0433	0.0442	0.0451	0.0394
	50	0.0304	0.0415	0.0407	0.0430	0.0440	0.0420	0.0404
$IG(1,0.5)$	10	0.0666	0.0532	0.0549	0.0543	0.0537	0.0535	0.0568
	20	0.0666	0.0551	0.0567	0.0560	0.0553	0.0557	0.0601
	30	0.0683	0.0535	0.0554	0.0561	0.0551	0.0538	0.0584
	50	0.0674	0.0546	0.0573	0.0584	0.0574	0.0545	0.0591
$IG(1,1)$	10	0.0503	0.0491	0.0496	0.0501	0.0496	0.0496	0.0498
	20	0.0505	0.0493	0.0479	0.0489	0.0494	0.0486	0.0512
	30	0.0504	0.0504	0.0499	0.0503	0.0500	0.0502	0.0498
	50	0.0498	0.0518	0.0496	0.0507	0.0521	0.0508	0.0500
$IG(1,2)$	10	0.0393	0.0467	0.0465	0.0441	0.0458	0.0473	0.0442
	20	0.0397	0.0493	0.0454	0.0451	0.0470	0.0477	0.0414
	30	0.0381	0.0475	0.0453	0.0456	0.0465	0.0470	0.0431
	50	0.0368	0.0436	0.0426	0.0442	0.0456	0.0427	0.0439
$IG(2,0.5)$	10	0.0826	0.0560	0.0580	0.0587	0.0564	0.0566	0.0629
	20	0.0860	0.0601	0.0647	0.0651	0.0623	0.0608	0.0715
	30	0.0887	0.0624	0.0657	0.0635	0.0603	0.0625	0.0672
	50	0.0844	0.0610	0.0621	0.0640	0.0620	0.0595	0.0657
$IG(2,1)$	10	0.0651	0.0524	0.0542	0.0536	0.0531	0.0534	0.0560
	20	0.0655	0.0553	0.0553	0.0537	0.0527	0.0543	0.0597
	30	0.0678	0.0559	0.0571	0.0556	0.0542	0.0562	0.0590
	50	0.0668	0.0553	0.0565	0.0554	0.0536	0.0550	0.0568
$IG(2,2)$	10	0.0500	0.0491	0.0495	0.0489	0.0488	0.0490	0.0500
	20	0.0500	0.0519	0.0500	0.0497	0.0503	0.0512	0.0518
	30	0.0504	0.0495	0.0504	0.0499	0.0489	0.0490	0.0495
	50	0.0486	0.0500	0.0482	0.0494	0.0500	0.0489	0.0492

In Table 2 the estimated type I error control using the 0.05 percentiles of the tests is evaluated and reported ($\alpha = 0.05$). According to the results of Table 2, the value of the type I error increases with the increase of the value of μ/λ, so that if $\mu/\lambda \approx 1$, then α is close to the nominal value. It is evident that for all tests, when the parameters are equal the type I error are acceptable. Generally, we see that the

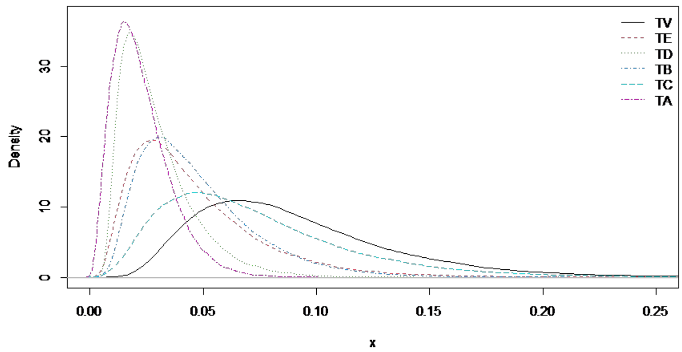

FIGURE 4. Estimated empirical densities of the proposed test statistics generated with 100,000 samples of size $n = 50$ from the Inverse Gaussian distribution.

actual sizes of the proposed tests are acceptable and therefore we can use these tests in practice.

4.2. **Power comparison.** The power of each test is studied by means of Monte Carlo simulations. In power comparison, we considered the following alternatives.

- the exponential distribution $Exp(\theta)$ with density $\theta \exp(-\theta x)$;
- the Weibull distribution with density $\theta x^{\theta-1} \exp\left(-x^{\theta}\right)$, denoted by $W(\theta)$;
- the gamma distribution with density $\Gamma(\theta)^{-1} x^{\theta-1} \exp\left(-x\right)$, denoted by $\Gamma(\theta)$;
- the half-normal HN distribution with density $\Gamma(2/\pi)^{1/2} \exp\left(-x^2/2\right)$;
- the lognormal distribution $LN(\theta)$ with density $(\theta x)^{-1}(2\pi)^{-1/2} \exp\left(-(\log x)^2/(2\theta^2)\right)$;
- the Pareto distribution $Pa(\theta)$ with density $\theta/x^{\theta+1}$;
- the uniform distribution U with density 1, $0 \leq x \leq 1$;
- the Beta distribution $Beta(\alpha, \beta)$ with density $x^{\alpha-1}(1-x)^{\beta-1} \Big/ Beta(\alpha, \beta)$, $0 \leq x \leq 1$;
- the modified extreme value $EV(\theta)$, with distribution function $1-\exp\left(\theta^{-1}(1-e^{x})\right)$;
- the linear increasing failure rate law $LF(\theta)$ with density $(1+\theta x) \exp\left(-x - \theta x^2/2\right)$;
- Dhillon's (1981) distribution with distribution function $1-\exp\left(-(\log(x+1))^{\theta+1}\right)$;

- Chen's (2000) distribution $CH(\theta)$, with distribution function $1-\exp\left(2\left(1-e^{x^{\theta}}\right)\right)$.

Also, we consider the popular and common Kolmogorov-Smirnov test (1933) which is used in practice and statistical software as the competitor test. The KS test statistic as follows. Let $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ are the order statistics based on the random sample $X_1, \cdots, X_n$.

$$D = \max(D^+, D^-),$$

where

$$D^+ = \max_{1\leq i\leq n}\left\{\frac{i}{n} - F_0(X_{(i)};\hat{\mu},\hat{\lambda})\right\}; \quad D^- = \max_{1\leq i\leq n}\left\{F_0(X_{(i)};\hat{\mu},\hat{\lambda}) - \frac{i-1}{n}\right\}.$$

Here, $F_0(x)$ is the cumulative distribution function of the IG distribution and $(\hat{\mu},\hat{\lambda})$ are the maximum likelihood estimates of the parameter (μ,λ).

The powers of the considered tests are computed by Monte Carlo simulation. Under each alternative, 100,000 samples of sizes 10, 20, 30 and 50 are generated. Then, the power of the corresponding test was estimated by the frequency of the event "the test statistic is larger than the critical point". The power estimates are presented in Tables 3–6. For each alternative, the bold type in these tables indicates the test achieving the maximal power.

TABLE 3. Monte Carlo power estimates of the tests for $n = 10$ and at the significance level $\alpha = 0.05$.

Alternative	D	TV	TE	TD	TB	TC	TA
$Exp(1)$	**0.3847**	0.2451	0.3159	0.3227	0.2963	0.2662	0.3633
$W(0.5)$	**0.7628**	0.6145	0.6713	0.6190	0.5679	0.6310	0.6850
$W(2)$	0.1609	0.1072	0.1488	0.1703	0.1574	0.1188	**0.1900**
$\Gamma(0.5)$	**0.7189**	0.5882	0.6485	0.6008	0.5567	0.6072	0.6681
$\Gamma(2)$	0.1563	0.0920	0.1259	0.1493	0.1375	0.1015	**0.1626**
HN	0.3917	0.2766	0.3528	0.3432	0.3155	0.2991	**0.3976**
$LN(0, 0.5)$	0.0376	0.0428	0.0443	0.0480	**0.0500**	0.0431	0.0453
$LN(0, 1)$	**0.0966**	0.0543	0.0669	0.0896	0.0866	0.0566	0.0877
$LN(0, 2)$	**0.4337**	0.2288	0.2933	0.2916	0.2527	0.2465	0.3352
$Pa(0.5)$	0.1932	0.1900	0.2235	0.1495	0.1315	0.1953	**0.2356**
$Pa(1)$	0.2962	0.3225	0.4008	0.2860	0.2623	0.3391	**0.4308**
$Pa(2)$	0.2679	0.3542	0.4281	0.2816	0.2590	0.3691	**0.4405**
U	0.4795	0.4876	0.5545	0.4143	0.3851	0.5024	**0.5603**
$CH(0.5)$	**0.7397**	0.6153	0.6715	0.6163	0.5692	0.6320	0.6859
$CH(1)$	0.4029	0.2866	0.3589	0.3426	0.3150	0.3079	**0.4039**
$CH(1.5)$	0.2593	0.1858	0.2507	0.2474	0.2261	0.2046	**0.2995**
$LF(2)$	0.3758	0.2625	0.3374	0.3351	0.3096	0.2869	**0.3874**
$LF(4)$	0.3561	0.2493	0.3238	0.3272	0.3036	0.2723	**0.3739**
$EV(0.5)$	0.4024	0.2875	0.3618	0.3462	0.3180	0.3112	**0.4084**
$EV(1.5)$	0.4064	0.3123	0.3912	0.3617	0.3343	0.3365	**0.4354**
$DL(1)$	**0.1460**	0.0758	0.1059	0.1401	0.1314	0.0838	0.1412
$DL(1.5)$	0.1101	0.0676	0.0914	0.1156	0.1102	0.0732	**0.1185**

Table 4. Monte Carlo power estimates of the tests for $n = 20$ and at the significance level $\alpha = 0.05$.

Alternative	D	TV	TE	TD	TB	TC	TA
$Exp(1)$	0.6309	0.4427	0.5670	0.5572	0.5247	0.4746	**0.6350**
$W(0.5)$	**0.9508**	0.8888	0.9268	0.8875	0.8648	0.9026	0.9313
$W(2)$	0.2955	0.1558	0.2754	0.2981	0.2711	0.1919	**0.3597**
$\Gamma(0.5)$	**0.9363**	0.8627	0.9094	0.8704	0.8477	0.8803	0.9196
$\Gamma(2)$	0.2676	0.1231	0.2220	0.2573	0.2363	0.1522	**0.2988**
HN	0.6538	0.4875	0.6261	0.5937	0.5595	0.5387	**0.6872**
$LN(0, 0.5)$	0.0365	0.0402	0.0404	0.0576	**0.0602**	0.0401	0.0480
$LN(0, 1)$	0.1247	0.0490	0.0820	0.1280	0.1213	0.0557	**0.1279**
$LN(0, 2)$	**0.6672**	0.4123	0.5393	0.5049	0.4581	0.4561	0.5941
$Pa(0.5)$	0.3582	0.4209	**0.5045**	0.2612	0.2245	0.4346	0.4340
$Pa(1)$	0.6121	0.6954	**0.8038**	0.5667	0.5255	0.7245	0.7609
$Pa(2)$	0.5456	0.7194	**0.8085**	0.5339	0.4910	0.7429	0.7500
U	0.7826	0.8385	**0.8909**	0.7089	0.6764	0.8518	0.8532
$CH(0.5)$	**0.9463**	0.8841	0.9242	0.8797	0.8586	0.8992	0.9290
$CH(1)$	0.6679	0.5115	0.6474	0.5983	0.5623	0.5612	**0.6993**
$CH(1.5)$	0.4761	0.3325	0.4846	0.4471	0.4105	0.3845	**0.5559**
$LF(2)$	0.6355	0.4600	0.6079	0.5886	0.5556	0.5135	**0.6738**
$LF(4)$	0.6094	0.4323	0.5840	0.5713	0.5394	0.4870	**0.6537**
$EV(0.5)$	0.6685	0.5111	0.6448	0.5978	0.5629	0.5586	**0.6943**
$EV(1.5)$	0.6856	0.5559	0.6901	0.6278	0.5928	0.6051	**0.7362**
$DL(1)$	0.2328	0.0880	0.1727	0.2341	0.2184	0.1112	**0.2510**
$DL(1.5)$	0.1784	0.0768	0.1443	0.1887	0.1745	0.0947	**0.2082**

From Tables 3-6, we observe that no single test can be said to perform the best against all alternatives. However, for almost alternatives the tests based on D, TE and TA statistics have the most power. Generally, three tests D, TE and TA have a good performance and power differences between these tests and the other tests are substantial. Also, we can see that against alternative $LN(0, 0.5)$, the test TB has the most power.
In the other hand, from Table 2, we found that the actual sizes of the proposed tests based on varentropy estimators were acceptable. Consequently, we can confidently recommend the proposed tests in practice. Since for small sample sizes, the test based on TA has a better performance than the other tests, we can generally conclude that the test TA has a good performance against almost alternatives. For large sample sizes, each of the tests D, TE and TA against different alternatives have the most power and usually the differences of power between these tests are small.
We can also see that the power values of the tests increase when the sample sizes increase.

TABLE 5. Monte Carlo power estimates of the tests for $n = 30$ and at the significance level $\alpha = 0.05$.

Alternative	D	TV	TE	TD	TB	TC	TA
$Exp(1)$	**0.7864**	0.5845	0.7305	0.7115	0.6794	0.6401	0.7758
$W(0.5)$	**0.9912**	0.9729	0.9847	0.9697	0.9612	0.9772	0.9836
$W(2)$	0.4159	0.2181	0.3907	0.4045	0.3689	0.2772	**0.4808**
$\Gamma(0.5)$	**0.9853**	0.9609	0.9787	0.9615	0.9529	0.9685	0.9768
$\Gamma(2)$	0.3742	0.1628	0.3085	0.3444	0.3153	0.2079	**0.3988**
HN	0.8151	0.6629	0.7936	0.7519	0.7204	0.7138	**0.8224**
$LN(0, 0.5)$	0.0369	0.0385	0.0442	0.0624	**0.0625**	0.0409	0.0553
$LN(0, 1)$	0.1497	0.0462	0.0994	0.1598	0.1499	0.0576	**0.1613**
$LN(0, 2)$	**0.8077**	0.5719	0.7026	0.6526	0.6078	0.6170	0.7313
$Pa(0.5)$	0.5167	0.6421	**0.7198**	0.3574	0.3015	0.6488	0.5293
$Pa(1)$	0.8137	0.9061	**0.9493**	0.7472	0.7082	0.9163	0.8759
$Pa(2)$	0.7469	0.9135	**0.9511**	0.7122	0.6720	0.9219	0.8680
U	0.9201	0.9649	**0.9799**	0.8641	0.8410	0.9680	0.9413
$CH(0.5)$	**0.9887**	0.9724	0.9844	0.9670	0.9589	0.9770	0.9817
$CH(1)$	0.8265	0.6910	0.8094	0.7541	0.7222	0.7366	**0.8271**
$CH(1.5)$	0.6476	0.4852	0.6549	0.5947	0.5547	0.5477	**0.6949**
$LF(2)$	0.7979	0.6311	0.7759	0.7445	0.7132	0.6852	**0.8112**
$LF(4)$	0.7772	0.5979	0.7503	0.7262	0.6942	0.6558	**0.7947**
$EV(0.5)$	0.8270	0.6890	0.8102	0.7546	0.7232	0.7356	**0.8276**
$EV(1.5)$	0.8458	0.7512	0.8551	0.7899	0.7593	0.7902	**0.8646**
$DL(1)$	0.3118	0.1058	0.2350	0.3068	0.2863	0.1451	**0.3318**
$DL(1.5)$	0.2474	0.0896	0.1952	0.2496	0.2294	0.1192	**0.2765**

5. An Illustrative Example

In this section, we illustrate how the proposed tests can be applied to test the goodness-of-fit for the IG distribution when the observations are available.

Example 5.1. *Folks and Chhikara (1989) considered the following dataset, consisting of 19 fracture toughness of MIG (metal inert gas) welds.*
54.4, 62.6, 63.2, 67.0, 70.2, 70.5, 70.6, 71.4, 71.8, 74.1, 74.1, 74.3, 78.8, 81.8, 83.0, 84.4, 85.3, 86.9, 87.3.
Folks and Chhikara (1989) concluded by using the KS statistic that the IG distribution is a reasonable fit. The histogram of the considered data set is presented in Figure 5.

Here, we apply the proposed tests to these data. First, the ML estimates of μ and λ are computed as

$$\hat{\mu} = \bar{X} = 74.3 \quad and \quad \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} \left(1/X_i - 1/\bar{X}\right)} = 4923.952\,.$$

Then, the transformed sample $u_i = F_0(x_i; \hat{\mu}, \hat{\lambda})$, $i = 1, 2, \cdots, n$, are obtained. Finally, based on the formula of the test statistics given in Section 3, the value of each test statistic is computed. Using the method described in Section 4.1, the critical

Table 6. Monte Carlo power estimates of the tests for $n = 50$ and at the significance level $\alpha = 0.05$.

Alternative	D	TV	TE	TD	TB	TC	TA
$Exp(1)$	**0.9324**	0.7963	0.8933	0.8748	0.8573	0.8364	0.9064
$W(0.5)$	**0.9997**	0.9986	0.9993	0.9977	0.9969	0.9990	0.9987
$W(2)$	0.6209	0.3507	0.5649	0.5643	0.5289	0.4273	**0.6305**
$\Gamma(0.5)$	**0.9993**	0.9972	0.9987	0.9966	0.9955	0.9978	0.9979
$\Gamma(2)$	**0.5477**	0.2443	0.4481	0.4849	0.4532	0.3131	0.5338
HN	**0.9523**	0.8719	0.9400	0.9094	0.8935	0.8995	0.9410
$LN(0, 0.5)$	0.0379	0.0330	0.0423	0.0685	**0.0710**	0.0357	0.0639
$LN(0, 1)$	0.1937	0.0465	0.1248	0.2093	0.1978	0.0660	**0.2132**
$LN(0, 2)$	**0.9358**	0.7840	0.8736	0.8311	0.7995	0.8167	0.8723
$Pa(0.5)$	0.7541	0.8925	**0.9255**	0.5219	0.4517	0.8892	0.6636
$Pa(1)$	0.9655	0.9950	**0.9980**	0.9265	0.9078	0.9952	0.9700
$Pa(2)$	0.9422	0.9949	**0.9974**	0.9040	0.8842	0.9952	0.9619
U	0.9918	0.9988	**0.9995**	0.9725	0.9658	0.9989	0.9900
$CH(0.5)$	**0.9997**	0.9986	0.9994	0.9974	0.9966	0.9988	0.9984
$CH(1)$	**0.9561**	0.8931	0.9469	0.9107	0.8957	0.9149	0.9400
$CH(1.5)$	**0.8536**	0.7198	0.8508	0.7832	0.7535	0.7703	0.8484
$LF(2)$	**0.9439**	0.8457	0.9277	0.9031	0.8875	0.8780	0.9320
$LF(4)$	**0.9321**	0.8181	0.9155	0.8946	0.8770	0.8610	0.9257
$EV(0.5)$	**0.9564**	0.8930	0.9487	0.9140	0.8983	0.9146	0.9433
$EV(1.5)$	**0.9673**	0.9303	0.9709	0.9354	0.9227	0.9467	0.9625
$DL(1)$	**0.4537**	0.1474	0.3390	0.4251	0.4032	0.2118	0.4520
$DL(1.5)$	0.3664	0.1182	0.2789	0.3516	0.3296	0.1714	**0.3817**

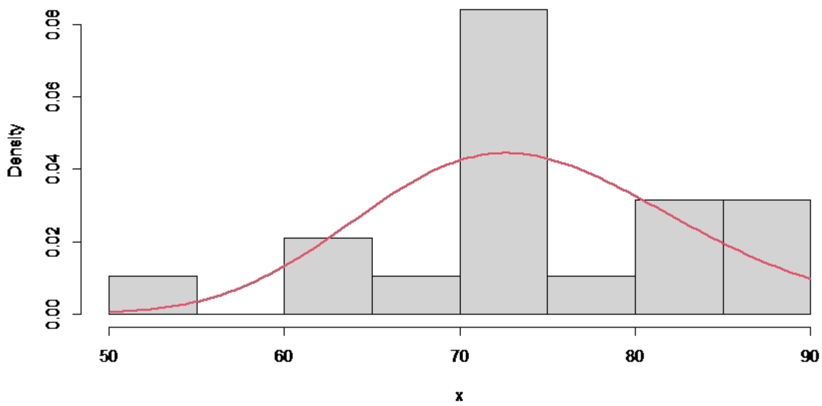

Figure 5. Histogram of data and a fitted IG density function.

value of each test for $n = 19$, *at the significance level 0.05 is obtained by Monte Carlo simulation. Results are summarized in Table 7.*

TABLE 7. The value of the test statistics and critical values at 5% level.

Test	*Value of the test statistic*	*Critical value*	*Decision*
D	*0.13339*	*0.21478*	*Not reject* H_0
TV	*0.20993*	*0.34779*	*Not reject* H_0
TE	*0.11087*	*0.20449*	*Not reject* H_0
TD	*0.02391*	*0.08349*	*Not reject* H_0
TB	*0.04531*	*0.14182*	*Not reject* H_0
TC	*0.20041*	*0.30509*	*Not reject* H_0
TA	*0.00840*	*0.03523*	*Not reject* H_0

Because the value of each test statistic is smaller than the corresponding critical value, the IG hypothesis is not rejected for these data at the significance level of 0.05. Therefore, we can conclude that the underlying distribution of these data is an IG distribution.

6. Conclusions

In this paper, we first discussed the varentropy estimators of a continuous random variable. We then constructed some new goodness of fit tests for IG distribution based on estimators of varentropy. The proposed test statistics are easy to compute and consistency and the other properties of the proposed test statistics have presented. We obtained the percentage points and type I error of the suggested tests. Power values of the proposed tests against various alternatives for different sample sizes were reported. Power simulations of the new tests based on the varentropy and the power comparisons with KS test showed that the varentropy-based tests are viable for testing the hypothesis of IG distribution. In fact, our power studies indicate that generally the proposed test TA has distinctly higher power than the other tests.

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Conflict of interest

The authors declare no conflict of interest.

Data Availability Statement

“Not applicable”.

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References

[1] Alizadeh H.N. and Mohtashami B., G.R. (2020), An updated review of goodness of fit tests based on entropy, Journal of The Iranian Statistical Society, 19, 175-204.

[2] Alizadeh H.N. and Shafaei, M.N. (2023), Varentropy estimators with applications in testing uniformity, Journal of Statistical Computation and Simulation, 93, 2582-2599.

[3] Alizadeh, H.N. and Shafaei, M.N. (2023), Monte Carlo comparison of goodness-of-fit tests for the Inverse Gaussian distribution based on empirical distribution function, Journal of Mahani Mathematical Research, 71-84.

[4] Allison, J. S., Betsch, S., Ebner, B., and Visagie, J. (2022), On testing the adequacy of the inverse Gaussian distribution, Mathematics, 10, 350.

[5] Al-Omari, A.I. and Haq, A. (2012), Goodness-of-fit testing for the inverse Gaussian distribution based on new entropy estimation using ranked set sampling and double ranked set sampling, Environmental Systems Research, 1, 1-10.

[6] Arikan, E. (2016), Varentropy decreases under the polar transform, IEEE Transactions on Information Theory, 62, 3390–400.

[7] Bardsley, W.E. (1980), Note on the use of the inverse Gaussian distribution for wind energy applications, Journal of Applied Meteorology, 19, 1126–1130.

[8] Barndorff-Nielsen, O.E. (1994), A note on electrical networks and the inverse Gaussian distribution, Advances in Applied Probability, 26, 63–67.

[9] Best, D.J., Rayner, J.C., and Thas, O. (2012), Comparison of some tests of fit for the inverse Gaussian distribution, Advances in Decision Sciences, 2012.

[10] Bobkov, S. and Madiman M. (2011), Concentration of the information in data with log-concave distributions, The Annals of Probability, 39, 1528–43.

[11] Buono, F. and Longobardi, M. (2020), Varentropy of past lifetimes, arXiv preprint arXiv:2008.07423.

[12] Chhikara, R.S. and Folks, J.L. (1977), The inverse Gaussian distribution as a lifetime model, Technometrics, 19, 461–468.

[13] Chhikara, Raj S. and Folks, J. Leroy (1989), The Inverse Gaussian Distribution: Theory, Methodology and Applications, New York, NY, USA: Marcel Dekker, Inc, ISBN 0-8247-7997-5.

[14] Choi, B. (2013), A Test of Fit for Inverse Gaussian Distribution Based on the Probability Integration Transformation, The Korean Journal of Applied Statistics, 26, 611-622.

[15] Dhillon, B.S. (1981), Lifetime Distributions, IEEE Transactions on Reliability, 30 457–459.

[16] Di Crescenzo, A. and Paolillo, L. (2021), Analysis and applications of the residual varentropy of random lifetimes, Probability in the Engineering and Informational Sciences, 35, 680-698.

[17] Di Crescenzo, A., Paolillo, L. and Suarez-Llorens A. (2021), Stochastic comparisons, differential entropy and varentropy for distributions induced by probability density functions, arXiv preprint arXiv:2103.11038.

[18] Ducharme, G.R. (2001). Goodness-of-fit tests for the inverse Gaussian and related distributions, Test, 10, 271-290.

[19] Folks, J.L. and Chhikara, R.S. (1978), The inverse Gaussian distribution and its statistical application-a review, Journal of the Royal Statistical Society, Series B, 40, 263–289.

[20] Folks, J.L. and Chhikara, R.S. (1989), In: The Inverse Gaussian Distribution, Theory, Methodology and Applications. Marcel Dekker, New York.

[21] Fradelizi, M., Madiman, M., and Wang, L. (2016), Optimal concentration of information content for logconcave densities, In C. Houdr´e, D. Mason, P. Reynaud-Bouret and J.

Rosinski (eds), High dimensional probability VII. Progress in Probability, vol. 71. Cham: Springer, pp. 45–60.
[22] Goodarzi, F., Amini, M., and Mohtashami Borzadaran, G.R. (2017), Characterizations of continous distributions through inequalities involving the expected values of selected functions, Applications of Mathematics, 62, 493–507.
[23] Gunes, H., Dietz, D. C., Auclair, P. F., and Moore, A. H. (1997), Modified goodness-of-fit tests for the inverse Gaussian distribution, Computational statistics and Data analysis, 24, 63-77.
[24] Henze, N. and Klar, B. (2002), Goodness-of-fit tests for the inverse Gaussian distribution based on the empirical Laplace transform, Annals of the Institute of Statistical Mathematics, 54, 425-444.
[25] Kolmogorov, A.N. (1933), Sulla Determinazione Empirica di une legge di Distribuzione. Giornale dell'Intituto Italiano degli Attuari, 4, 83-91.
[26] Kontoyiannis, I. and Verdu, S. (2014), Optimal lossless compression: Source varentropy and dispersion, IEEE Transactions in Information Theory, 60, 777–95.
[27] Lequesne, J. and Regnault, P. (2018), vsgoftest: An Package for Goodness-of-Fit Testing Based on Kullback-Leibler Divergence. arXiv preprint arXiv:1806.07244.
[28] Maadani, S., Mohtashami Borzadaran, G.R. and Rezaei Roknabadi, A.H. (2021), Varentropy of order statistics and some stochastic comparisons, Communications in Statistics - Theory and Methods, 51, 6447-6460.
[29] Madiman, M. and Wang, L. (2014), An optimal varentropy bound for log-concave distributions. In International Conference on Signal Processing and Communications (SPCOM). Bangalore: Indian Institute of Science, 1 p. doi:10.1109/SPCOM.2014.6983953.
[30] Mergel, V. (1999), Test of goodness of fit for the inverse-gaussian distribution, Mathematical Communications, 4, 191-195.
[31] Mudholkar, G. S., Natarajan, R. and Chaubey, Y. P. (2001), A goodness-of-fit test for the inverse Gaussian distribution using its independence characterization, Sankhya: The Indian Journal of Statistics, 63, 362-374.
[32] Mudholkar, G.S. and Tian, L. (2002), An entropy characterization of the inverse Gaussian distribution and related goodness-of-fit test, Journal of Statistical Planning and Inference, 102, 211-221.
[33] Natarajan, R. and Mudholkar, G.S. (2004), Moment-based goodness-of-fit tests for the inverse Gaussian distribution, Technometrics, 46, 339-347.
[34] O'Reilly, F.J. and Rueda, R. (1992), Goodness of fit for the inverse Gaussian distribution, Canadian Journal of Statistics, 20, 387-397.
[35] Paolillo, L. (2021), Properties and applications of pdf-related information measures and distributions, PhD Thesis, Universita degli Studi di Salerno.
[36] Pavur, R.J., Edgeman, R.L. and Scott, R.C. (1992), Quadratic statistics for the goodness-of-fit test of the inverse Gaussian distribution, IEEE Transactions on Reliability, 41, 118-123.
[37] Peccarelli, A.M. (2017), A comparison of variance and Renyi's entropy with application to machine learning. Northern Illinois University.
[38] Saha, S. and Kayal, S. (2023), Weighted (residual) varentropy with properties and applications, arXiv preprint, arXiv:2305.00852.
[39] Seshadri, V. (1999), In: The Inverse Gaussian Distribution: Statistical Theory and Applications, Springer, New York.
[40] Shannon, C.E. (1948), A mathematical theory of communications, Bell System Technical Journal, 27, 379–423; 623–656.
[41] Sharma, A. and Kundu, C. (2023), Varentropy of doubly truncated random variable. Probability in the Engineering and Informational Sciences, 37, 852-871.
[42] Vasicek, O. (1976), A test for normality based on sample entropy, Journal of the Royal Statistical Society Series B: Statistical Methodology, 38(1), 54-59.

[43] Vexler, A., Shan, G., Kim, S., Tsai, W.M., Tian, L. and Hutson, A.D. (2011), An empirical likelihood ratio based goodness-of-fit test for Inverse Gaussian distributions, Journal of Statistical Planning and Inference, 141, 2128-2140.

[44] Vladimirescu, I. and Tunaru, R. (2003), Estimation functions and uniformly most powerful tests for inverse Gaussian distribution, Commentationes Mathematicae Universitatis Carolinae, 44, 153-164.

[45] Zaid, E.O.A., Attwa, R.A.E.W., and Radwan, T. (2022), Some measures information for generalized and q-generalized extreme values and its properties, Fractals, 30, 2240246.

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