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# BAYESIAN INFERENCE ON RELIABILITY PARAMETER WITH NON-IDENTICAL-COMPONENT STRENGTHS FOR RAYLEIGH DISTRIBUTION

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Abstract. In this paper, we delve into Bayesian inference related to multi-component stress-strength parameters, focusing on non-identical component strengths within a two-parameter Rayleigh distribution under the progressive first failure censoring scheme. We explore various scenarios: the general case, and instances where the common location parameter is either unknown or known. For each scenario, point and interval estimates are derived using methods including the MCMC method, Lindley's approximation, exact Bayes estimates, and HPD credible intervals. The efficacy of these methods is evaluated using a Monte Carlo simulation and their practical applications are demonstrated with a real data set.

*Keywords*: Multi-component stress-strength reliability, Lindley's approximation, MCMC method, First failure progressive censored.
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## 1. Introduction

In reliability theory, the inference on the stress-strength parameter $R = P(Y < X)$ holds significant interest for researchers. Here, $Y$ and $X$ denote stress and strength variables, respectively. A recent study by [4] considered the inference for $P(Y < X)$ within the context of non-identical component strengths based on the Rayleigh distribution. A multi-component system comprises multiple components. Such a system has one overarching stress component and $k$ independent and identical strength components. A system's reliability is ensured if at least $s$ out of the $k$ strength components surpass its stress. This model was initially proposed by [3] as:

$$R_{s,k} = \sum_{p=s}^{k} \binom{k}{p} \int_{-\infty}^{\infty} \big(1 - F_X(y)\big)^p \big(F_X(y)\big)^{k-p} dF_Y(y).$$

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In this model, the strength variables $(X_1, \ldots, X_k)$ are independent and identically distributed (i.i.d) with cumulative distribution function $F_X(\cdot)$. Meanwhile, the stress variable $Y$ has a distribution represented by $F_Y(\cdot)$. Various researchers have explored this model. For instance, [8] investigated the estimation of $R_{s,k}$ for the Kumaraswamy distribution using progressively censored samples. Similarly, [9] evaluated both Bayesian and classical estimations of $R_{s,k}$ under adaptive hybrid progressive censored data for the Weibull distribution. A subsequent model developed by [11] extended multi-component reliability to comprise $\mathbf{k} = (k_1, k_2, \ldots, k_m)$ components as:

$$\begin{aligned} R_{\mathbf{s},\mathbf{k}} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \left( \prod_{i=1}^{m} \binom{k_i}{p_i} \right) \int_{-\infty}^{\infty} \prod_{i=1}^{m} \Big( & \big(1 - F_i(y)\big)^{p_i} \\ & \times \big(F_i(y)\big)^{k_i - p_i} \Big) dF_Y(y). \end{aligned} \tag{1}$$

In this configuration, $k_i$ components belong to type $i$, where $i = 1, \ldots, m$, and $F_i(\cdot)$ denotes the cumulative distribution function of strengths for components of the $i$-th type. It's posited that a shared stress $Y$ with distribution $F_Y(\cdot)$, impacts all components. The system is deemed reliable as long as at least $\mathbf{s} = (s_1, \ldots, s_m)$ out of $k$ strength components surpasses the stress level. Recent research by [10] contemplated this model for the modified Weibull extension distribution under progressive censoring. Furthermore, [12] delved into this model for the modified Kumaraswamy distribution, focusing on progressive first-failure censored samples. This paper narrows its scope to systems with two component types, represented as $\mathbf{k} = (k_1, k_2)$ and $\mathbf{s} = (s_1, s_2)$, primarily due to computational intricacies. This particular model was initially analyzed by [15] for upper record values within the Kumaraswamy generalized distributions family. Additionally, [5] examined the reliability estimation of a stress-strength model featuring nonidentical component strengths under a generalized progressive hybrid censoring scheme. In conclusion, the multi-component reliability parameter encompassing two non-identical strength components can be extracted from equation (1) as:

$$\begin{aligned} R_{\mathbf{s},\mathbf{k}} = & \sum_{p_1=s_1}^{k_1} \sum_{p_2=s_2}^{k_2} \binom{k_1}{p_1} \binom{k_2}{p_2} \int_{-\infty}^{\infty} \Big( \big(1 - F_1(y)\big)^{p_1} \big(F_1(y)\big)^{k_1 - p_1} \Big) \\ & \times \Big( \big(1 - F_2(y)\big)^{p_2} \big(F_2(y)\big)^{k_2 - p_2} \Big) \, dF_Y(y). \end{aligned} \tag{2}$$

The versatility of the model is evident from its generality. By adjusting parameters, it can be simplified to represent multi-component stress-strength with a singular strength variable, or even just the stress-strength parameter. This can be achieved by setting $k_2 = 0$ and $k_1 = 1$, $k_2 = 0$ respectively. This adaptability of the model has been noted in studies such as [11], where the model was applied to scenarios where strength and stress variables adhere to a bathtub-shaped distribution, paired with adaptive Type-II hybrid progressive censoring samples.

When considering censoring schemes, Type-I and Type-II are foundational. Their amalgamation gives rise to the hybrid scheme. A major limitation of these methods is their inability to eliminate active units during the test phase. This drawback paved the way for the introduction of the progressive censoring scheme, a topic elaborately discussed in the seminal monographs by [1]. A novel approach, termed first failure censoring, was proposed by [2]. By synergizing the elements of both progressive and first failure censoring schemes, [18] introduced the progressive first failure censoring scheme. This approach has since garnered significant attention in the reliability literature, as evidenced by works like [13], [16], and [17]. The Progressive First Failure Censoring Scheme (PFFC) can be elucidated as follows: Suppose there are $N$ groups in the experiment, with each group containing $w$ items. During the testing phase, $R_1$ groups are randomly removed from the test, along with the group containing the first failure unit at the time of its first failure. Similarly, $R_2$ groups are randomly removed from the test when the second failure unit fails, and so on for $R_n$ groups and the $n$-th failure unit, occurring at the $n$-th failure time. In this scheme, the PFFC sample is represented as $\{T_{1:n:N:w}, \ldots, T_{n:n:N:w}\}$, and the progressive censoring scheme is $\{R_1, \ldots, R_n\}$, subject to the constraint that $R_1 + \cdots + R_n + n = N$. In the subsequent discussion, the PFFC sample is denoted as $\{T_1, \ldots, T_n\}$. The joint Probability Density Function (pdf) for the failure times $T_1 < \cdots < T_n$ under study, characterized by a continuous pdf denoted as $f(\cdot)$ and a Cumulative Distribution Function (cdf) denoted as $F(\cdot)$, is provided as follows:

$$f(t_1, \ldots, t_n) \propto \prod_{i=1}^{n} f(t_i)\big(1 - F(t_i)\big)^{w(R_i+1)-1}, \quad 0 < t_1 < \cdots < t_n < \infty.$$

We present a schematic representation of this scheme in Figure 1. It is evident that the Progressive First Failure Censoring Scheme (PFFC) can be transformed into First Failure Censoring and Progressive Censoring by setting $R_1 = \cdots = R_n = 0$ and $w = 1$, respectively. Furthermore, it can be simplified to Type-II Censoring by setting $R_1 = \cdots = R_{n-1} = 0, R_n = N - n$, and $w = 1$. Finally, it becomes equivalent to complete data when $R_1 = \cdots = R_n = 0$, and $w = 1$. The Two-Parameter Rayleigh (tR) distribution, characterized by scale and location parameters $\lambda$ and $\mu$, respectively, has the following pdf and cdf:

$$f(x) = 2\lambda(x - \mu)e^{-\lambda(x-\mu)^2}, \ x > \mu, \lambda, \mu > 0, \tag{3}$$

$$F(x) = 1 - e^{-\lambda(x-\mu)^2}, \ x > \mu, \lambda, \mu > 0. \tag{4}$$

The Bayesian approach is a statistical procedure that allows the systematic incorporation of prior knowledge about the model and model parameters, the

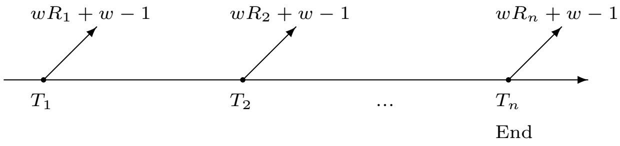


FIGURE 1. Schematic representation of progressive first failure scheme.

appropriate weighting of experimental data, and the use of probabilistic models for the modeling of sources of experimental error. In fact, this approach allows researchers to take into account data as well as prior beliefs to calculate the probability that an alternative is superior. In this paper, we derive Bayesian inference for $R_{\mathbf{s},\mathbf{k}}$ based on the PFFC sample when $X$ and $Y$ are two independent random variables following the tR distribution.

The remainder of this paper is organized as follows: In Section 2, we explore Bayesian inference for $R_{\mathbf{s},\mathbf{k}}$ in various scenarios, including cases where all parameters are unknown, the common location parameter is unknown, and when it is known. To achieve this, we utilize the Markov Chain Monte Carlo (MCMC) method, Lindley's approximation, exact Bayes estimation, and Highest Posterior Density (HPD) intervals for $R_{\mathbf{s},\mathbf{k}}$. Section 3 presents the results of simulations and data analysis. Finally, in Section 4, we draw conclusions based on the findings of this study.

## 2. Inference on $R_{\mathbf{s},\mathbf{k}}$ in general case

If $X_1 \sim tR(\lambda_1, \mu_1)$, $X_2 \sim tR(\lambda_2, \mu_2)$ and $Y \sim tR(\lambda, \mu)$ are independent random variables, then the multi-component stress-strength parameter, $R_{\mathbf{s},\mathbf{k}}$ can be obtained, from (3) and (4), as

$$\begin{aligned} R_{\mathbf{s},\mathbf{k}} = \sum_{p_1=s_1}^{k_1} \sum_{p_2=s_2}^{k_2} \binom{k_1}{p_1}\binom{k_2}{p_2} \int_{\mu}^{\infty} & e^{-\lambda_1 (y-\mu_1)^2 p_1} (1 - e^{-\lambda_1 (y-\mu_1)^2})^{k_1 - p_1} \\ & \times e^{-\lambda_2 (y-\mu_2)^2 p_2} (1 - e^{-\lambda_2 (y-\mu_2)^2})^{k_2 - p_2} 2\lambda(y-\mu) e^{-\lambda(y-\mu)^2} dy. \end{aligned}$$

We construct the likelihood function by the following samples:

$$Y = \overset{\text{Stress}}{\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}} \quad \text{and} \quad \overset{\text{Strength}}{X_1 = \begin{pmatrix} U_{11} & \dots & U_{1k_1} \\ \vdots & \ddots & \vdots \\ U_{n1} & \dots & U_{nk_1} \end{pmatrix}, X_2 = \begin{pmatrix} V_{11} & \dots & V_{1k_2} \\ \vdots & \ddots & \vdots \\ V_{n1} & \dots & V_{nk_2} \end{pmatrix}},$$

where $\{Y_1, \dots, Y_n\}$ is PFFC sample from $tR(\lambda, \mu)$ with $\{N, n, w, S_1, \dots, S_n\}$, $\{U_{i1}, \dots, U_{ik_1}\}$ and $\{V_{i1}, \dots, V_{ik_2}\}$, $i = 1, \dots, n$, are two PFFC samples from $tR(\lambda_1, \mu)$ and $tR(\lambda_2, \mu)$ with the censoring scheme $\{K_1, k_1, w_1, R_1, \dots, R_{k_1}\}$ and $\{K_2, k_2, w_2, Q_1, \dots, Q_{k_2}\}$, respectively. The likelihood function of $\lambda_1$, $\lambda_2$,

$\lambda$, $\mu$, $\mu_1$ and $\mu_2$ can be obtained by

$$L(\lambda_1,\lambda_2,\lambda,\mu,\mu_1,\mu_2|\text{data}) \propto \prod_{i=1}^{n}\Big(\prod_{j_1=1}^{k_1} f_1(u_{ij_1})\big(1-F_1(u_{ij_1})\big)^{w_1(R_{j_1}+1)-1}\Big)$$
$$\times\Big(\prod_{j_2=1}^{k_2} f_2(v_{ij_2})\big(1-F_2(v_{ij_2})\big)^{w_2(Q_{j_2}+1)-1}\Big) f(y_i)\big(1-F(y_i)\big)^{w(S_i+1)-1}.$$

To obtain censored data from $X_1$, $X_2$, and $Y$ the following procedure is employed. Initially, in accordance with the method outlined in Section 1, certain elements from the $Y$ vector are censored. For any data point in $Y$ that has been censored, the corresponding row in the $X_1$ and $X_2$ matrices is removed. Subsequently, within the remaining matrices of $X_1$ and $X_2$, the censoring scheme is applied to each row. Using this approach, a PFFC sample of size $n$ for $Y$ is obtained. Additionally, for the $i$-th observation in the $Y$ vector, we acquire PFFC samples $\{U_{i1},\ldots,U_{ik_1}\}$ and $\{V_{i1},\ldots,V_{ik_2}\}$ from the $X_1$ and $X_2$ matrices. These samples are independent, and it is evident that the structure of the likelihood function can be described as previously mentioned.

In this section, Bayesian inference for $R_{\mathbf{s},\mathbf{k}}$ is examined under squared error loss functions, assuming that $\lambda_1$, $\lambda_2$, $\lambda$, $\mu$, $\mu_1$, and $\mu_2$ are independent random variables. Based on the observed censoring samples, the joint posterior density function is as follows:

$$\begin{aligned}\pi(\lambda_1,\lambda_2,\lambda,\mu,\mu_1,\mu_2|\text{data}) &\propto L(\text{data}|\lambda_1,\lambda_2,\lambda,\mu_1,\mu_2,\mu)\\ &\quad\times\pi_1(\lambda_1)\pi_2(\lambda_2)\pi_3(\lambda)\pi_4(\mu)\pi_5(\mu_1)\pi_6(\mu_2)\end{aligned}\tag{5}$$

where

$$\begin{aligned}\pi_1(\lambda_1) &\propto \lambda_1^{a_1-1}e^{-b_1\lambda_1},\ \lambda_1,a_1,b_1>0,\\ \pi_2(\lambda_2) &\propto \lambda_2^{a_2-1}e^{-b_2\lambda_2},\ \lambda_2,a_2,b_2>0,\\ \pi_3(\lambda) &\propto \lambda^{a_3-1}e^{-b_3\lambda},\ \lambda,a_3,b_3>0,\\ \pi_4(\mu) &\propto 1,\ \ \mu\in(0,t),\ \text{where } t \text{ can be any finite arbitrary value},\\ \pi_5(\mu_1) &\propto 1,\ \mu_1\in(0,t_1),\ \text{where } t_1 \text{ can be any finite arbitrary value},\\ \pi_6(\mu_2) &\propto 1,\ \mu_2\in(0,t_2),\ \text{where } t_2 \text{ can be any finite arbitrary value}.\end{aligned}$$

As evident from equation (5), it is not feasible to obtain the Bayes estimate in a closed form. Therefore, we resort to approximating it using the Markov Chain Monte Carlo (MCMC) method. From the joint posterior density function, we can derive the posterior pdfs of $\lambda_1$, $\lambda_2$, $\lambda$, $\mu$, $\mu_1$, and $\mu_2$ as follows:

$$\lambda_1|\mu_1, \text{data} \sim \Gamma\big(nk + a_1, b_1 + w_1 \sum_{i=1}^{n}\sum_{j_1=1}^{k_1}(u_{ij_1} - \mu_1)^2(R_{j_1} + 1)\big),$$

$$\lambda_2|\mu_2, \text{data} \sim \Gamma\big(nk_2 + a_2, b_2 + w_2 \sum_{i=1}^{n}\sum_{j_2=1}^{k_2}(v_{ij_2} - \mu_2)^2(Q_{j_2} + 1)\big),$$

$$\lambda|\mu, \text{data} \sim \Gamma\big(n + a_3, b_3 + w \sum_{i=1}^{n}(y_i - \mu)^2(S_i + 1)\big),$$

$$\pi(\mu|\lambda, \text{data}) \propto \big(\prod_{i=1}^{n}(y_i - \mu)\big)e^{-\lambda w \sum\limits_{i=1}^{n}(y_i-\mu)^2(S_i+1)},$$

$$\pi(\mu_1|\lambda_1, \text{data}) \propto \big(\prod_{i=1}^{n}\prod_{j_1=1}^{k_1}(u_{ij_1} - \mu_1)\big)e^{-\lambda_1 w_1 \sum\limits_{i=1}^{n}\sum\limits_{j_1=1}^{k_1}(u_{ij_1}-\mu_1)^2(R_{j_1}+1)},$$

$$\pi(\mu_2|\lambda_2, \text{data}) \propto \big(\prod_{i=1}^{n}\prod_{j_2=1}^{k_2}(v_{ij_1} - \mu_2)\big)e^{-\lambda_2 w_2 \sum\limits_{i=1}^{n}\sum\limits_{j_2=1}^{k_2}(v_{ij_2}-\mu_2)^2(Q_{j_2}+1)}.$$

It is worth noting that generating samples from the posterior pdfs of $\mu_1$, $\mu_2$, and $\mu$ should be accomplished using the Metropolis-Hastings method, given that these pdfs are not known. To facilitate this, we propose the Gibbs sampling algorithm, which is elaborated in detail in Appendix A. Consequently, the Bayes' estimate of $R_{\mathbf{s},\mathbf{k}}$ under the squared error loss functions can be expressed as:

$$\widehat{R}^{MB}_{\mathbf{s},\mathbf{k}} = \frac{1}{T_b}\sum_{t=1}^{T_b} R_{(t)\mathbf{s},\mathbf{k}}. \tag{6}$$

Also, the $100(1-\gamma)\%$ HPD credible interval of $R_{\mathbf{s},\mathbf{k}}$ can be constructed, using the method of [6] as follows. Order $R_{(1)\mathbf{s},\mathbf{k}}, \ldots, R_{(T_b)\mathbf{s},\mathbf{k}}$ as $R_{\big((1)\mathbf{s},\mathbf{k}\big)} < \cdots < R_{\big((T_b)\mathbf{s},\mathbf{k}\big)}$ and construct all the $100(1-\gamma)\%$ confidence intervals of $R$, as:

$$\Big(R_{\big((1)\mathbf{s},\mathbf{k}\big)}, R_{\big(([T(1-\gamma)])\mathbf{s},\mathbf{k}\big)}\Big), \ldots, \Big(R_{\big(([T\gamma])\mathbf{s},\mathbf{k}\big)}, R_{\big(([T])\mathbf{s},\mathbf{k}\big)}\Big),$$

where $[T]$ symbolizes the largest integer less than or equal to $T$. The HPD credible interval of $R_{\mathbf{s},\mathbf{k}}$ is the shortest length interval.

*Remark* 2.1. Inference on $R_{\mathbf{s},\mathbf{k}}$ with unknown common $\mu$

If $X_1 \sim tR(\lambda_1, \mu)$, $X_2 \sim tR(\lambda_2, \mu)$ and $Y \sim tR(\lambda, \mu)$ are independent random variables, then the multi-component stress-strength parameter, $R_{\mathbf{s},\mathbf{k}}$ can be obtained, from (3) and (4), as

$$
\begin{aligned}
R_{\mathbf{s},\mathbf{k}} = &\sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\int_{\mu}^{\infty} e^{-\lambda_1(y-\mu)^2 p_1}(1-e^{-\lambda_1(y-\mu)^2})^{k_1-p_1}\\
&\times e^{-\lambda_2(y-\mu)^2 p_2}(1-e^{-\lambda_2(y-\mu)^2})^{k_2-p_2} 2\lambda(y-\mu)e^{-\lambda(y-\mu)^2}dy \; t=e^{-(y-\mu)^2}\\
&= \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\lambda\int_0^1 t^{\lambda_1 p_1+\lambda_2 p_2+\lambda-1}\\
&\times (1-t^{\lambda_1})^{k_1-p_1}(1-t^{\lambda_2})^{k_2-p_2}dt\\
&= \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2}\\
&\times (-1)^{q_1+q_2}\lambda\int_0^1 t^{\lambda_1(p_1+q_1)+\lambda_2(p_2+q_2)+\lambda-1}dt
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2}\\
&\times \frac{(-1)^{q_1+q_2}\lambda}{\lambda_1(p_1+q_1)+\lambda_2(p_2+q_2)+\lambda}.
\end{aligned} \tag{7}
$$

Indeed, as observed, it is not possible to evaluate the Bayes estimate of $R_{\mathbf{s},\mathbf{k}}$ in a closed form. Therefore, we need to approximate it using two methods:

**MCMC method:** The approach for utilizing the MCMC method is quite analogous to what has been previously outlined, so we will not delve into further detail here.

**Lindley's approximation:** One of the most practical techniques for deriving the Bayes estimate is introduced by [14]. We are aware that the Bayes estimation of $u(\theta)$, under the squared error loss function, should be computed as follows:

$$
\mathbb{E}(u(\theta)|\text{data}) = \frac{\int u(\theta)e^{Q(\theta)}d\theta}{\int e^{Q(\theta)}d\theta}, \tag{8}
$$

where $Q(\theta)=\rho(\theta)+\ell(\Theta)$, $\rho(\theta)$ and $\ell(\theta)$ are logarithm of the prior density of $\theta$ and log-likelihood function, respectively. By [14], we can approximate the equation (8) as

$$
\begin{aligned}
\mathbb{E}(u(\theta)|\text{data}) = u &+ \frac{1}{2}\sum_i\sum_j (u_{ij}+2u_i\rho_j)\sigma_{ij}\\
&+ \frac{1}{2}\sum_i\sum_j\sum_k\sum_p \ell_{ijk}\sigma_{ij}\sigma_{kp}u_p\Big|_{\theta=\widehat{\theta}},
\end{aligned} \tag{9}
$$

where $\theta = (\theta_1, \ldots, \theta_m)$, $i, j, k, p = 1, \ldots, m$, $\widehat{\theta}$ is the Maximum Likelihood Estimates (MLEs) of $\theta$, $u = u(\theta)$, $u_i = \frac{\partial u}{\partial \theta_i}$, $u_{ij} = \frac{\partial^2 u}{\partial \theta_i \partial \theta_j}$, $\ell_{ijk} = \frac{\partial^3 \ell}{\partial \theta_i \partial \theta_j \partial \theta_k}$, $\rho_j = \frac{\partial \rho}{\partial \theta_j}$, and $\sigma_{ij} = (i, j)$-th element in the inverse of matrix $[-\ell_{ij}]$ all evaluated at the MLE of the parameters. So, by simplifying the equation (9), in some steps, in the case of four parameters $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$, we have

$$\mathbb{E}(u(\theta)|\text{data}) = u + \sum_{j=1}^{4} u_j d_j + d_5 + d_6 + \frac{1}{2} \sum_{j=1}^{4} \sum_{i=1}^{4} A_j u_i \sigma_{ji},$$

where

$$d_j = \sum_{i=1}^{4} \rho_i \sigma_{ji},\ j = 1, 2, 3, 4, \qquad d_5 = \sum_{\substack{i=1 \\ i<p}}^{3} \sum_{p=1}^{4} u_{ip} \sigma_{ip}, \qquad d_6 = \frac{1}{2} \sum_{i=1}^{4} u_{ii} \sigma_{ii},$$
$$A_j = \sum_{i=1}^{4} \ell_{iij} \sigma_{ii} + \sum_{\substack{i=1 \\ i<p}}^{3} \sum_{p=1}^{4} \ell_{jip} \sigma_{ip},\ j = 1, \cdots, 4.$$

In our case $(\theta_1, \theta_2, \theta_3, \theta_4) \equiv (\lambda_1, \lambda_2, \lambda, \mu)$ and $u \equiv u(\lambda_1, \lambda_2, \lambda, \mu) = R_{\mathbf{s},\mathbf{k}}$, we have

$$\rho_1 = \frac{a_1 - 1}{\lambda_1} - b_1,\ \rho_2 = \frac{a_2 - 1}{\lambda_2} - b_2,\ \rho_3 = \frac{a_3 - 1}{\lambda} - b_3,\ \rho_4 = 0,$$
$$\ell_{11} = -\frac{nk_1}{\lambda_1^2} \quad \ell_{12} = 0, \quad \ell_{13} = 0, \quad \ell_{22} = -\frac{nk_2}{\lambda_2^2}, \quad \ell_{23} = 0, \quad \ell_{33} = -\frac{n}{\lambda^2},$$
$$\ell_{14} = 2w_1 \sum_{i=1}^{n} \sum_{j_1=1}^{k_1} (u_{ij_1} - \mu)(R_{j_1} + 1), \quad \ell_{24} = 2w_2 \sum_{i=1}^{n} \sum_{j_2=1}^{k_2} (v_{ij_2} - \mu)(Q_{j_2} + 1),$$
$$\ell_{34} = 2w \sum_{i=1}^{n} (y_i - \mu)(S_i + 1),$$
$$\ell_{44} = -\sum_{i=1}^{n} \sum_{j_1=1}^{k_1} \frac{1}{(u_{ij_1} - \mu)^2} - \sum_{i=1}^{n} \sum_{j_2=2}^{k_2} \frac{1}{(v_{ij_1} - \mu)^2} - \sum_{i=1}^{n} \frac{1}{(y_i - \mu)^2}$$
$$- 2n\lambda_1 w_1 \sum_{j_1=1}^{k_1} (R_{j_1} + 1) - 2n\lambda_2 w_2 \sum_{j_2=1}^{k_2} (Q_{j_2} + 1) - 2\lambda w \sum_{i=1}^{n} (S_i + 1).$$

Using $\ell_{ij}, i,j=1,2,3,4$, we can obtain $\sigma_{ij}, i,j=1,2,3,4$ and

$$\ell_{111}=\frac{2nk_1}{\lambda_1^3}\quad \ell_{222}=\frac{2nk_2}{\lambda_2^3},\quad \ell_{333}=\frac{2n}{\lambda^3},\quad \ell_{144}=-2nw_1\sum_{j_1=1}^{k_1}(R_{j_1}+1),$$
$$\ell_{244}=-2nw_2\sum_{j_2=1}^{k_2}(Q_{j_2}+1),\quad \ell_{344}=-2w\sum_{i=1}^{n}(S_i+1),$$
$$\ell_{444}=-\sum_{i=1}^{n}\sum_{j_1=1}^{k_1}\frac{1}{(u_{ij_1}-\mu)^2}-\sum_{i=1}^{n}\sum_{j_2=2}^{k_2}\frac{1}{(v_{ij_1}-\mu)^2}-\sum_{i=1}^{n}\frac{1}{(y_i-\mu)^2}.$$

and other $\ell_{ijk}=0$. Moreover, $u_4=u_{i4}=0$, $i=1,2,3,4$ and we have

$$u_1=\sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2}$$
$$\times\frac{(-1)^{q_1+q_2+1}\lambda(p_1+q_1)}{\big(\lambda_1(p_1+q_1)+\lambda_2(p_2+q_2)+\lambda\big)^2},$$
$$u_2=\sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2}$$
$$\times\frac{(-1)^{q_1+q_2+1}\lambda(p_2+q_2)}{\big(\lambda_1(p_1+q_1)+\lambda_2(p_2+q_2)+\lambda\big)^2},$$

$$
\begin{aligned}
u_3 &= \sum_{p_1=s_1}^{k_1} \sum_{p_2=s_2}^{k_2} \sum_{q_1=0}^{k_1-p_1} \sum_{q_2=0}^{k_2-p_2} \binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2} \\
&\quad \times \frac{(-1)^{q_1+q_2}\big(\lambda_1(p_1+q_1)+\lambda_2(p_2+q_2)\big)}{\big(\lambda_1(p_1+q_1)+\lambda_2(p_2+q_2)+\lambda\big)^2}, \\
u_{11} &= \sum_{p_1=s_1}^{k_1} \sum_{p_2=s_2}^{k_2} \sum_{q_1=0}^{k_1-p_1} \sum_{q_2=0}^{k_2-p_2} \binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2} \\
&\quad \times \frac{2(-1)^{q_1+q_2}\lambda(p_1+q_1)^2}{\big(\lambda_1(p_1+q_1)+\lambda_2(p_2+q_2)+\lambda\big)^3}, \\
u_{22} &= \sum_{p_1=s_1}^{k_1} \sum_{p_2=s_2}^{k_2} \sum_{q_1=0}^{k_1-p_1} \sum_{q_2=0}^{k_2-p_2} \binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2} \\
&\quad \times \frac{2(-1)^{q_1+q_2}\lambda(p_2+q_2)^2}{\big(\lambda_1(p_1+q_1)+\lambda_2(p_2+q_2)+\lambda\big)^3}, \\
u_{33} &= \sum_{p_1=s_1}^{k_1} \sum_{p_2=s_2}^{k_2} \sum_{q_1=0}^{k_1-p_1} \sum_{q_2=0}^{k_2-p_2} \binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2} \\
&\quad \times \frac{2(-1)^{q_1+q_2+1}\big(\lambda_1(p_1+q_1)+\lambda_2(p_2+q_2)\big)}{\big(\lambda_1(p_1+q_1)+\lambda_2(p_2+q_2)+\lambda\big)^3}, \\
u_{12} &= \sum_{p_1=s_1}^{k_1} \sum_{p_2=s_2}^{k_2} \sum_{q_1=0}^{k_1-p_1} \sum_{q_2=0}^{k_2-p_2} \binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2} \\
&\quad \times \frac{2(-1)^{q_1+q_2}\lambda(p_1+q_1)(p_2+q_2)}{\big(\lambda_1(p_1+q_1)+\lambda_2(p_2+q_2)+\lambda\big)^3}, \\
u_{13} &= \sum_{p_1=s_1}^{k_1} \sum_{p_2=s_2}^{k_2} \sum_{q_1=0}^{k_1-p_1} \sum_{q_2=0}^{k_2-p_2} \binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2} \\
&\quad \times \frac{(-1)^{q_1+q_2+1}(p_1+q_1)\big(\lambda_1(p_1+q_1)+\lambda_2(p_2+q_2)-\lambda\big)}{\big(\lambda_1(p_1+q_1)+\lambda_2(p_2+q_2)+\lambda\big)^3}, \\
u_{23} &= \sum_{p_1=s_1}^{k_1} \sum_{p_2=s_2}^{k_2} \sum_{q_1=0}^{k_1-p_1} \sum_{q_2=0}^{k_2-p_2} \binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2} \\
&\quad \times \frac{(-1)^{q_1+q_2+1}(p_2+q_2)\big(\lambda_1(p_1+q_1)+\lambda_2(p_2+q_2)-\lambda\big)}{\big(\lambda_1(p_1+q_1)+\lambda_2(p_2+q_2)+\lambda\big)^3}.
\end{aligned}
$$

Consequently, under the squared error loss function, the Bayes estimate of $R_{\mathbf{s},\mathbf{k}}$ is

$$\widehat{R}^{Lin}_{\mathbf{s},\mathbf{k}} = R_{\mathbf{s},\mathbf{k}} + \sum_{j=1}^{3} u_j d_j + d_5 + d_6 + \frac{1}{2}\sum_{j=1}^{4}\sum_{i=1}^{3} A_j u_i \sigma_{ji}. \tag{10}$$

It is worth noting that all parameters should be computed using their MLEs, denoted as $(\widehat{\lambda}_1, \widehat{\lambda}_2, \widehat{\lambda}, \widehat{\mu})$.

*Remark* 2.2. Inference on $R_{\mathbf{s},\mathbf{k}}$ with known common $\mu$

Similar to Remark 2.1, when the common parameter $\mu$ is known, and assuming that $\lambda_1$, $\lambda_2$, and $\lambda$ follow independent gamma distributions as prior distributions, the joint posterior density function of $\lambda_1$, $\lambda_2$, and $\lambda$ can be derived as follows:

$$\begin{aligned}\pi(\lambda_1, \lambda_2, \lambda|\mu, \text{data}) &= A\lambda_1^{nk_1+a_1-1}\lambda_2^{nk_2+a_2-1}\lambda^{n+a_3-1}\\ &\times e^{-\lambda_1 w_1 \sum_{i=1}^{n}\sum_{j_1=1}^{k_1}(u_{ij_1}-\mu)^2(R_{j_1}+1)-\lambda_2 w_2\sum_{i=1}^{n}\sum_{j_2=1}^{k_2}(v_{ij_2}-\mu)^2(Q_{j_2}+1)}\\ &\times e^{-\lambda w\sum_{i=1}^{n}(y_i-\mu)^2(S_i+1)},\end{aligned} \tag{11}$$

where

$$\begin{aligned}A = &\Big(b_1 + w_1\sum_{i=1}^{n}\sum_{j_1=1}^{k_1}(u_{ij_1}-\mu_{(t-1)})^2(R_{j_1}+1)\Big)^{nk_1+a_1}\\ &\times\Big(b_2 + w_2\sum_{i=1}^{n}\sum_{j_2=1}^{k_2}(v_{ij_2}-\mu_{(t-1)})^2(Q_{j_2}+1)\Big)^{nk_2+1}\\ &\times\Big(b_3 + w\sum_{i=1}^{n}(y_i-\mu_{(t-1)})^2(S_i+1)\Big)^{n+a_3}\Big\{\Gamma(nk_1+a_1)\Gamma(nk_2+a_2)\Gamma(n+a_3)\Big\}^{-1}\end{aligned}$$

By solving the following triple integral, the Bayesian estimation of $R_{\mathbf{s},\mathbf{k}}$, under the squared error loss function can be obtained. So,

$$\begin{aligned}\widehat{R}^{B}_{\mathbf{s},\mathbf{k}} &= \int_0^\infty\int_0^\infty\int_0^\infty R_{\mathbf{s},\mathbf{k}}\pi(\lambda_1, \lambda_2, \lambda|\mu, \text{data})d\lambda_1 d\lambda_2 d\lambda\\ &= \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2}(-1)^{q_1+q_2}\\ &\times\int_0^\infty\int_0^\infty\int_0^\infty\frac{\lambda}{\lambda_1(p_1+q_1)+\lambda_2(p_2+q_2)+\lambda}\\ &\times\pi(\lambda_1, \lambda_2, \lambda|\mu, \text{data})d\lambda_1 d\lambda_2 d\lambda.\end{aligned} \tag{12}$$

Now by applying $\pi(\lambda_1,\lambda_2,\lambda|\mu,\text{data})$ from (11) in the equation (12) and utilizing the idea of [11], part of the triple integral can be solved as follows:

$$M = \begin{cases} A_1 & |w_1|<1, |w_2|<1, \\ \\ A_2 & w_1<-1, w_2<-1, \\ \\ A_3 & |w_1|<1, w_2<-1, \\ \\ A_4 & w_1<-1, |w_2|<1, \end{cases}$$

where

$$\begin{aligned}
A_1 = &\ \tfrac{\nu_3(1-w_1)^{\nu_1}(1-w_2)^{\nu_2}}{\nu_1+\nu_2+\nu_3}\ F_1(\nu_1+\nu_2+\nu_3,\nu_1,\nu_2,1+\nu_1+\nu_2+\nu_3;w_1,w_2),\\
A_2 = &\ \tfrac{\nu_3}{\nu_1+\nu_2+\nu_3}\ F_1(1,\nu_1,\nu_2,1+\nu_1+\nu_2+\nu_3;\tfrac{w_1}{w_1-1},\tfrac{w_2}{w_2-1}),\\
A_3 = &\ \tfrac{\nu_3(1-w_1)}{\nu_1+\nu_2+\nu_3}\ F_1(1,\nu_3+1,\nu_2,1+\nu_1+\nu_2+\nu_3;w_1,\tfrac{w_1-w_2}{1-w_2}),\\
A_4 = &\ \tfrac{\nu_3(1-w_2)}{\nu_1+\nu_2+\nu_3}\ F_1(1,\nu_1,\nu_3+1,1+\nu_1+\nu_2+\nu_3;\tfrac{w_2-w_1}{1-w_1},w_2).
\end{aligned}$$

Also, in this representation, we have

$$\nu_1=nk_1+a_1,\quad \nu_2=nk_2+a_2,\quad \nu_3=n+a_3,$$

$$w_1 = 1-\frac{b_1+w_1\sum\limits_{i=1}^{n}\sum\limits_{j_1=1}^{k_1}(u_{ij_1}-\mu_{(t-1)})^2(R_{j_1}+1)}{(p_1+q_1)\big(b_3+w\sum\limits_{i=1}^{n}(y_i-\mu_{(t-1)})^2(S_i+1)\big)},$$

$$w_2 = 1-\frac{b_2+w_2\sum\limits_{i=1}^{n}\sum\limits_{j_2=1}^{k_2}(v_{ij_2}-\mu_{(t-1)})^2(Q_{j_2}+1)}{(p_2+q_2)\big(b_3+w\sum\limits_{i=1}^{n}(y_i-\mu_{(t-1)})^2(S_i+1)\big)}.$$

Also,

$$F_1(\alpha,\beta,\beta',\gamma;x,y)=\frac{1}{B(\alpha,\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tx)^{-\beta}(1-ty)^{-\beta'}dt.$$

The function $F_1(\alpha,\beta,\beta',\gamma;x,y)$ is a hypergeometric series and can be readily evaluated using standard software. Consequently, the Bayes estimate of $R_{\mathbf{s},\mathbf{k}}$ can be obtained as follows:

(13)

$$\widehat{R}^B_{\mathbf{s},\mathbf{k}}=\sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\sum_{q_1=0}^{k_1-p_1}\sum_{q_2=0}^{k_2-p_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\binom{k_1-p_1}{q_1}\binom{k_2-p_2}{q_2}(-1)^{q_1+q_2}\times M.$$

As we need to solve a numerical integral to obtain the Bayes estimate in equation (13), similar to previous sections when the parameter $\mu$ is known, we employ the MCMC Bayes estimation method and calculate the HPD credible intervals.

## 3. Simulation study and data analysis

3.1. **Numerical experiment and discussion.** In this section, we perform a comparison of different estimates using Monte Carlo simulations. The criteria for comparing point estimates is the mean square errors (MSEs), while for interval estimates, we consider average confidence lengths (AL) and coverage percentages (CP). Various censoring schemes, different parameter values, and hyper-parameters are employed in our simulation studies. Our results are based on 2000 repetitions, and the number of repetitions in the Gibbs sampling algorithm is set to $T = 3000$. The significance level for obtaining HPD credible intervals is set at 0.95. The different censoring schemes used in obtaining the results are detailed in Table 1.

TABLE 1. Different censoring schemes.

| $(k_l, K_l, w_l)$ | | C.S. | $(n, N, w)$ | | C.S. |
|---|---|---|---|---|---|
| | $R_1$ | (0,0,0,0,5) | | $S_1$ | (0,0,0,0,5) |
| (5,10,2) | $R_2$ | (5,0,0,0,0) | (5,10,2) | $S_2$ | (5,0,0,0,0) |
| | $R_3$ | (1,1,1,1,1) | | $S_3$ | (1,1,1,1,1) |
| | $R_4$ | $(0^{*9},10)$ | | $S_4$ | $(0^{*9},10)$ |
| (10,20,3) | $R_5$ | $(10,0^{*9})$ | (10,20,3) | $S_5$ | $(10,0^{*9})$ |
| | $R_6$ | $(1^{*10})$ | | $S_6$ | $(1^{*10})$ |

Now, let's consider different scenarios. First Case (General Case): Assuming $(\lambda_1, \lambda_2, \lambda, \mu_1, \mu_2, \mu) = (1.5, 0.5, 0.4, 2, 1, 1.5)$, we conduct simulation studies and compare different Bayes estimates under two priors: Prior 1 ($a_j = 0,\ b_j = 0,\ j = 1, 2, 3$) and Prior 2 ($a_j = 1,\ b_j = 0.4,\ j = 1, 2, 3$). We apply equation (6) to obtain the Bayes estimates of $R_{\mathbf{s},\mathbf{k}}$. The simulation results are presented in Table 2.

Second Case (Common Parameter $\mu$ Unknown): Assuming $(\lambda_1, \lambda_2, \lambda, \mu) = (0.75, 1.5, 0.5, 2)$, we conduct simulation studies and compare different Bayes estimates under two priors: Prior 3 ($a_j = 0,\ b_j = 0,\ j = 1, 2, 3$) and Prior 4 ($a_j = 1,\ b_j = 0.5,\ j = 1, 2, 3$). We apply Lindley's approximation (equation (10)) to obtain the Bayes estimates of $R_{\mathbf{s},\mathbf{k}}$. The simulation results are presented in Table 3.

Third Case (Common Parameter $\mu$ Known): Assuming $(\lambda_1, \lambda_2, \lambda, \mu) = (1, 0.5, 0.45, 2.5)$, we conduct simulation studies and compare different Bayes estimates under two priors: Prior 5 ($a_j = 0,\ b_j = 0,\ j = 1, 2, 3$) and Prior 6 ($a_j = 1,\ b_j = 0.75,\ j = 1, 2, 3$). We apply equation (13) to obtain the Bayes estimates of $R_{\mathbf{s},\mathbf{k}}$. The simulation results are presented in Table 4.

Tables 2-4 reveal that informative priors (priors 2, 4, and 6) perform the best based on the MSE values. Additionally, in the first case, Bayes estimates obtained by the MCMC method outperform those obtained by Lindley's approximation. It is also observed that among the different intervals, HPD intervals based on informative priors (priors 2, 4, and 6) have the best performance in terms of AL and CP values. Furthermore, we have some general observations from Tables 2-4:

- As $n$ increases for fixed $\mathbf{s}$ and $\mathbf{k}$, MSEs and ALs decrease while CPs increase. This may be attributed to the fact that with an increase in $n$, the number of failures increases, leading to more information being gathered and thus improving the performance of the estimates.
- By increasing $k$ for fixed $\mathbf{s}$ and $\mathbf{n}$, MSEs and ALs decrease while CPs increase. This could be due to the same reason as mentioned above, as more failures lead to better estimation performance.

TABLE 2. Simulation results in general case.

| $(k_1, k_2, n, s_1, s_2)$ | C.S | MCMC | | | | | |
|---|---|---|---|---|---|---|---|
| | | Prior 1 | | | Prior 2 | | |
| | | MSE | AL | CP | MSE | AL | CP |
| (5,5,5,2,2) | $(R_1, R_1, S_1)$ | 0.0531 | 0.6152 | 0.941 | 0.0475 | 0.5843 | 0.944 |
| | $(R_2, R_2, S_2)$ | 0.0542 | 0.6142 | 0.940 | 0.0485 | 0.5714 | 0.945 |
| | $(R_3, R_3, S_3)$ | 0.0550 | 0.6167 | 0.941 | 0.0467 | 0.5981 | 0.945 |
| (5,5,10,2,2) | $(R_1, R_1, S_4)$ | 0.0445 | 0.5237 | 0.945 | 0.0409 | 0.4851 | 0.949 |
| | $(R_2, R_2, S_5)$ | 0.0428 | 0.5034 | 0.946 | 0.0401 | 0.4777 | 0.948 |
| | $(R_3, R_3, S_6)$ | 0.0433 | 0.5367 | 0.946 | 0.0400 | 0.4950 | 0.949 |
| (10,10,5,2,2) | $(R_4, R_4, S_1)$ | 0.0395 | 0.4574 | 0.948 | 0.0358 | 0.4218 | 0.949 |
| | $(R_5, R_5, S_2)$ | 0.0384 | 0.4496 | 0.949 | 0.0347 | 0.4128 | 0.950 |
| | $(R_6, R_6, S_3)$ | 0.0378 | 0.4480 | 0.949 | 0.0338 | 0.4119 | 0.950 |
| (10,10,10,2,2) | $(R_4, R_4, S_4)$ | 0.0335 | 0.4019 | 0.950 | 0.0281 | 0.3636 | 0.952 |
| | $(R_5, R_5, S_5)$ | 0.0315 | 0.3951 | 0.950 | 0.0275 | 0.3815 | 0.951 |
| | $(R_6, R_6, S_6)$ | 0.0323 | 0.4185 | 0.951 | 0.0267 | 0.3625 | 0.951 |
| (5,5,5,4,4) | $(R_1, R_1, S_1)$ | 0.0524 | 0.6235 | 0.940 | 0.0459 | 0.5618 | 0.945 |
| | $(R_2, R_2, S_2)$ | 0.0534 | 0.6185 | 0.941 | 0.0475 | 0.5596 | 0.944 |
| | $(R_3, R_3, S_3)$ | 0.0539 | 0.6060 | 0.940 | 0.0486 | 0.5758 | 0.944 |
| (5,5,10,4,4) | $(R_1, R_1, S_4)$ | 0.0438 | 0.5218 | 0.946 | 0.0395 | 0.4681 | 0.949 |
| | $(R_2, R_2, S_5)$ | 0.0445 | 0.5395 | 0.945 | 0.0390 | 0.4625 | 0.948 |
| | $(R_3, R_3, S_6)$ | 0.0429 | 0.5308 | 0.945 | 0.0387 | 0.4663 | 0.949 |
| (10,10,5,4,4) | $(R_4, R_4, S_1)$ | 0.0368 | 0.4750 | 0.948 | 0.0348 | 0.4309 | 0.950 |
| | $(R_5, R_5, S_2)$ | 0.0374 | 0.4763 | 0.947 | 0.0339 | 0.4325 | 0.949 |
| | $(R_6, R_6, S_3)$ | 0.0389 | 0.4609 | 0.949 | 0.0322 | 0.4285 | 0.949 |
| (10,10,10,4,4) | $(R_4, R_4, S_4)$ | 0.0310 | 0.3951 | 0.950 | 0.0251 | 0.3324 | 0.950 |
| | $(R_5, R_5, S_5)$ | 0.0325 | 0.3851 | 0.951 | 0.0264 | 0.3374 | 0.951 |
| | $(R_6, R_6, S_6)$ | 0.0336 | 0.3962 | 0.951 | 0.0275 | 0.3418 | 0.952 |

3.2. **Real data analysis.** In this section, we analyze a real dataset for illustrative purposes. Recently, [7] conducted a comparison of wind speed data from two districts on the Aegean coast of Turkey to illustrate the stress-strength model. Such a comparison is essential for researchers involved in installing wind turbines. In this study, we also use NASA's POWER satellite data for Fethiye and Datça stations are located on the Aegean coast of Turkey. Given the high wind energy potential in this region, it is valuable to assess wind capacities using only satellite data without any physical investment. Data from NASA's POWER source can be directly accessed using its data access viewer (https://power.larc.nasa.gov/data-access-viewer/). For this study, we utilize wind speed observations (m/s) at a 10 m height on two hourly basis in January 2023. The data from the Fethiye station, recorded from 12 a.m. to 12 p.m. (every two hours) each day of January 2023, is considered the first type of

TABLE 3. Simulation results when common parameter $\mu$ is unknown.

| $(k_1, k_2, n, s_1, s_2)$ | C.S | MCMC | | | | | | Lindley | |
|---|---|---|---|---|---|---|---|---|---|
| | | Prior 3 | | | Prior 4 | | | Prior 3 | Prior 4 |
| | | MSE | AL | CP | MSE | AL | CP | MSE | MSE |
| (5,5,5,2,2) | $(R_1, R_1, S_1)$ | 0.0484 | 0.5291 | 0.941 | 0.0437 | 0.5050 | 0.945 | 0.0523 | 0.0503 |
| | $(R_2, R_2, S_2)$ | 0.0496 | 0.5314 | 0.940 | 0.0427 | 0.5196 | 0.946 | 0.0518 | 0.0509 |
| | $(R_3, R_3, S_3)$ | 0.0467 | 0.5237 | 0.940 | 0.0441 | 0.4967 | 0.945 | 0.0520 | 0.0514 |
| (5,5,10,2,2) | $(R_1, R_1, S_4)$ | 0.0412 | 0.4328 | 0.947 | 0.0393 | 0.3919 | 0.949 | 0.0467 | 0.0452 |
| | $(R_2, R_2, S_5)$ | 0.0409 | 0.4219 | 0.946 | 0.0389 | 0.4082 | 0.948 | 0.0470 | 0.0442 |
| | $(R_3, R_3, S_6)$ | 0.0417 | 0.4209 | 0.947 | 0.0375 | 0.3967 | 0.949 | 0.0481 | 0.0439 |
| (10,10,5,2,2) | $(R_4, R_4, S_1)$ | 0.0370 | 0.5037 | 0.949 | 0.0358 | 0.4747 | 0.950 | 0.0411 | 0.0391 |
| | $(R_5, R_5, S_2)$ | 0.0381 | 0.5029 | 0.948 | 0.0349 | 0.4695 | 0.951 | 0.0420 | 0.0399 |
| | $(R_6, R_6, S_3)$ | 0.0379 | 0.4967 | 0.949 | 0.0367 | 0.4782 | 0.950 | 0.0415 | 0.0381 |
| (10,10,10,2,2) | $(R_4, R_4, S_4)$ | 0.0313 | 0.3799 | 0.950 | 0.0275 | 0.3595 | 0.953 | 0.0378 | 0.0356 |
| | $(R_5, R_5, S_5)$ | 0.0303 | 0.3826 | 0.951 | 0.0269 | 0.3447 | 0.952 | 0.0385 | 0.0367 |
| | $(R_6, R_6, S_6)$ | 0.0315 | 0.3819 | 0.951 | 0.0270 | 0.3426 | 0.952 | 0.0390 | 0.0349 |
| (5,5,5,4,4) | $(R_1, R_1, S_1)$ | 0.0475 | 0.5344 | 0.940 | 0.0423 | 0.4927 | 0.944 | 0.0537 | 0.0496 |
| | $(R_2, R_2, S_2)$ | 0.0485 | 0.5319 | 0.941 | 0.0418 | 0.5067 | 0.945 | 0.0529 | 0.0509 |
| | $(R_3, R_3, S_3)$ | 0.0466 | 0.5263 | 0.940 | 0.0435 | 0.4996 | 0.945 | 0.0544 | 0.0499 |
| (5,5,10,4,4) | $(R_1, R_1, S_4)$ | 0.0408 | 0.4228 | 0.947 | 0.0369 | 0.4019 | 0.949 | 0.0485 | 0.0466 |
| | $(R_2, R_2, S_5)$ | 0.0412 | 0.4310 | 0.948 | 0.0374 | 0.4067 | 0.950 | 0.0496 | 0.0453 |
| | $(R_3, R_3, S_6)$ | 0.0400 | 0.4375 | 0.947 | 0.0360 | 0.3994 | 0.949 | 0.0477 | 0.0449 |
| (10,10,5,4,4) | $(R_4, R_4, S_1)$ | 0.0385 | 0.5037 | 0.950 | 0.0349 | 0.4674 | 0.950 | 0.0409 | 0.0388 |
| | $(R_5, R_5, S_2)$ | 0.0377 | 0.4960 | 0.949 | 0.0356 | 0.4628 | 0.950 | 0.0417 | 0.0396 |
| | $(R_6, R_6, S_3)$ | 0.0396 | 0.4970 | 0.949 | 0.0337 | 0.4646 | 0.951 | 0.0419 | 0.0377 |
| (10,10,10,4,4) | $(R_4, R_4, S_4)$ | 0.0303 | 0.3790 | 0.951 | 0.0286 | 0.3419 | 0.953 | 0.0369 | 0.0344 |
| | $(R_5, R_5, S_5)$ | 0.0315 | 0.3619 | 0.950 | 0.0296 | 0.3510 | 0.952 | 0.0359 | 0.0340 |
| | $(R_6, R_6, S_6)$ | 0.0309 | 0.3648 | 0.950 | 0.0288 | 0.3449 | 0.952 | 0.0377 | 0.0337 |

TABLE 4. Simulation results when common parameter $\mu$ is known.

| $(k_1, k_2, n, s_1, s_2)$ | C.S | MCMC | | | | Exact | |
|---|---|---|---|---|---|---|---|
| | | Prior 5 | | Prior 6 | | Prior 5 | Prior 6 |
| | | AL | CP | AL | CP | MSE | MSE |
| (5,5,5,2,2) | $(R_1, R_1, S_1)$ | 0.4957 | 0.940 | 0.4528 | 0.945 | 0.0443 | 0.0415 |
| | $(R_2, R_2, S_2)$ | 0.5019 | 0.941 | 0.4417 | 0.945 | 0.0438 | 0.0407 |
| | $(R_3, R_3, S_3)$ | 0.4936 | 0.940 | 0.4328 | 0.946 | 0.0429 | 0.0423 |
| (5,5,10,2,2) | $(R_1, R_1, S_4)$ | 0.4325 | 0.946 | 0.3975 | 0.949 | 0.0385 | 0.0343 |
| | $(R_2, R_2, S_5)$ | 0.4284 | 0.947 | 0.3888 | 0.948 | 0.0395 | 0.0338 |
| | $(R_3, R_3, S_6)$ | 0.4222 | 0.946 | 0.3812 | 0.949 | 0.0374 | 0.0330 |
| (10,10,5,2,2) | $(R_4, R_4, S_1)$ | 0.4019 | 0.949 | 0.3684 | 0.950 | 0.0352 | 0.0329 |
| | $(R_5, R_5, S_2)$ | 0.4163 | 0.948 | 0.3519 | 0.950 | 0.0349 | 0.0319 |
| | $(R_6, R_6, S_3)$ | 0.4028 | 0.949 | 0.3618 | 0.949 | 0.0364 | 0.0300 |
| (10,10,10,2,2) | $(R_4, R_4, S_4)$ | 0.3625 | 0.950 | 0.3028 | 0.952 | 0.0319 | 0.0285 |
| | $(R_5, R_5, S_5)$ | 0.3592 | 0.950 | 0.3019 | 0.952 | 0.0308 | 0.0279 |
| | $(R_6, R_6, S_6)$ | 0.3519 | 0.951 | 0.3067 | 0.953 | 0.0323 | 0.0267 |
| (5,5,5,4,4) | $(R_1, R_1, S_1)$ | 0.5019 | 0.941 | 0.4628 | 0.946 | 0.0437 | 0.0423 |
| | $(R_2, R_2, S_2)$ | 0.4918 | 0.940 | 0.4527 | 0.947 | 0.0455 | 0.0433 |
| | $(R_3, R_3, S_3)$ | 0.5067 | 0.942 | 0.4592 | 0.946 | 0.0446 | 0.0415 |
| (5,5,10,4,4) | $(R_1, R_1, S_4)$ | 0.4281 | 0.945 | 0.3517 | 0.948 | 0.0377 | 0.0334 |
| | $(R_2, R_2, S_5)$ | 0.4235 | 0.945 | 0.3469 | 0.948 | 0.0367 | 0.0328 |
| | $(R_3, R_3, S_6)$ | 0.4197 | 0.946 | 0.3684 | 0.949 | 0.0388 | 0.0367 |
| (10,10,5,4,4) | $(R_4, R_4, S_1)$ | 0.3974 | 0.948 | 0.3281 | 0.950 | 0.0320 | 0.0308 |
| | $(R_5, R_5, S_2)$ | 0.4028 | 0.948 | 0.3319 | 0.951 | 0.0335 | 0.0300 |
| | $(R_6, R_6, S_3)$ | 0.4095 | 0.949 | 0.3274 | 0.950 | 0.0328 | 0.0289 |
| (10,10,10,4,4) | $(R_4, R_4, S_4)$ | 0.3418 | 0.950 | 0.2749 | 0.951 | 0.0285 | 0.0259 |
| | $(R_5, R_5, S_5)$ | 0.3320 | 0.951 | 0.2817 | 0.952 | 0.0295 | 0.0246 |
| | $(R_6, R_6, S_6)$ | 0.3417 | 0.950 | 0.2899 | 0.951 | 0.0276 | 0.0230 |

strength data $X_1$. Data recorded from 12 p.m. to 12 a.m. (every two hours) is considered the second type of strength data $X_2$. The daily average wind speed data from the Datça station is regarded as stress data $Y$. Consequently, we have $k_1 = k_2 = 6$ and $n = 31$ for our model.

First, we fit the tR distribution to the three datasets separately and obtain the following results:

For $X_1$, $\widehat{\lambda}_1 = 0.1691$, $\widehat{\mu}_1 = 0.0998$, and the p-value= 0.1240.
For $X_2$, $\widehat{\lambda}_2 = 0.1636$, $\widehat{\mu}_2 = 0.1012$, and the p-value= 0.1398.
For $Y$, $\widehat{\lambda} = 0.0366$, $\widehat{\mu} = 0.5070$, and the p-value= 0.7119.

Based on the p-values, we conclude that the tR distribution provides suitable fits for the $X_1$, $X_2$, and $Y$ datasets. The estimated parameters for these datasets indicate that only the general case can be considered for their analysis. We provide the empirical distribution functions and PP plots for these three datasets in Figure 2. For the complete dataset, with $\mathbf{s} = (2, 2)$ and $\mathbf{k} = (6, 6)$,

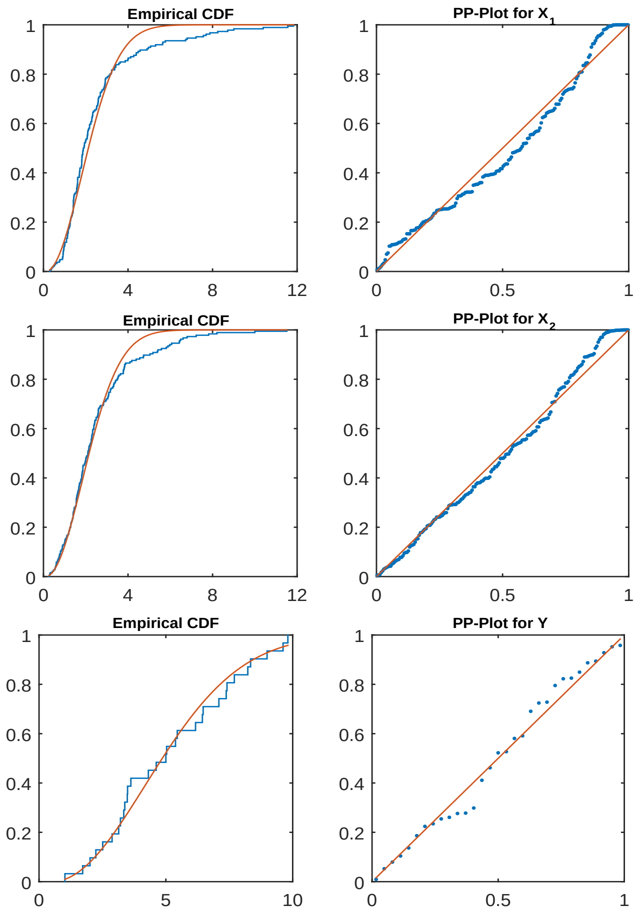


FIGURE 2. Empirical distribution function (left) and the PP-plot (right) for $X_1$ (first row), for $X_2$ (middle row), and for $Y$ (third row).

using non-informative priors, we obtain $\widehat{R}^{MC}_{\mathbf{s},\mathbf{k}}$ as 0.1412 and the corresponding 95% HPD interval as $(0.0815, 0.2098)$. Now, we generate two different censoring the progressive scheme as follows:

$$\begin{aligned}\text{Scheme 1: } & R^{(1)} = R^{(2)} = (0^{*6}),\ S = (2^{*8}, 0^{*7}),\\ & (\mathbf{k} = (6,6), \mathbf{s} = (2,2), w = w_1 = w_2 = 1).\\ \text{Scheme 2: } & R^{(1)} = R^{(2)} = (1^{*3}),\ S = (2^{*8}, 0^{*7}),\\ & (\mathbf{k} = (3,3), \mathbf{s} = (2,2), w = w_1 = w_2 = 1).\end{aligned}$$

For Scheme 1, using non-informative priors, we obtain $\widehat{R}^{MC}_{\mathbf{s},\mathbf{k}}$ as 0.1596 and the corresponding 95% HPD interval as $(0.0915, 0.2215)$. For Scheme 2, also with non-informative priors, we obtain $\widehat{R}^{MC}_{\mathbf{s},\mathbf{k}}$ as 0.0158, and the corresponding 95% HPD interval as $(0.0006, 0.0351)$. Upon comparing point and interval estimates, we observe that Scheme 1 performs better than Scheme 2, as anticipated. Furthermore, it is noticed that since all the estimates of $R_{\mathbf{s},\mathbf{k}}$ are less than 0.5, the Datça district should be given special attention for further investigations into wind energy power plant investments based on the considered scenario.

## 4. Conclusion

In this paper, Bayesian inference on $R_{\mathbf{s},\mathbf{k}}$ with non-identical component strengths in the presence of the PFFC scheme for the tR distribution has been thoroughly examined. This problem is quite general, and it encompasses various scenarios, including:

- The $R_{\mathbf{s},\mathbf{k}}$ parameter can be transformed into the $R_{s,k}$ parameter when $\mathbf{k} = (k, 0)$, or the $R = P(X < Y)$ parameter when $\mathbf{k} = (1, 0)$.
- The PFFC scheme can be adapted into different censoring schemes, such as first failure censoring when $R_1 = \cdots = R_n = 0$, progressive censoring when $w = 1$, Type II censoring when $R_1 = \cdots = R_{n-1} = 0, R_n = N - n, w = 1$, and complete sample case when $R_1 = \cdots = R_n = 0, w = 1$.
- The two-parameter Rayleigh distribution can be reduced to the Rayleigh distribution when $\mu = 0$.

As demonstrated, by addressing this problem, we can automatically tackle several related problems and scenarios. The study presents different estimates considering various scenarios, including cases where location parameters are different, the same, unknown, or known. Through Monte Carlo simulation studies, different estimates are compared. The results highlight that informative priors outperform non-informative ones in both point and interval estimates. Additionally, Bayes estimates obtained using the MCMC method perform better than those obtained using Lindley's approximation. Moreover, an increase in the number of failures leads to the gathering of more information, subsequently improving the accuracy of estimates. This comprehensive examination offers valuable insights into Bayesian inference in the presence of non-identical component strengths and progressive censoring schemes, providing a foundation for addressing similar problems in various contexts.

## Appendix A

The algorithm of Gibbs sampling is as follows:

(1) Start with initial values $(\lambda_{1(0)}, \lambda_{2(0)}, \lambda_{(0)})$, $\mu_{1(0)}$, $\mu_{2(0)}$ and $\mu_{(0)}$.
(2) Set $t = 1$.
(3) Generate $\mu_{1(t)}$ from $\pi(\mu_1|\lambda_{1(t-1)}, \text{data})$, using Metropolis-Hastings method, with $N(\mu_{1(t-1)}, 1)$ as proposal distribution.
(4) Generate $\mu_{2(t)}$ from $\pi(\mu_2|\lambda_{2(t-1)}, \text{data})$, using Metropolis-Hastings method, with $N(\mu_{2(t-1)}, 1)$ as proposal distribution.
(5) Generate $\mu_{(t)}$ from $\pi(\mu|\lambda_{(t-1)}, \text{data})$, using Metropolis-Hastings method, with $N(\mu_{(t-1)}, 1)$ as proposal distribution.
(6) Generate $\lambda_{1(t)}$ from $\Gamma\big(nk+a_1, b_1+w_1 \sum_{i=1}^{n}\sum_{j_1=1}^{k_1}(u_{ij_1}-\mu_{1(t-1)})^2(R_{j_1}+1)\big)$.
(7) Generate $\lambda_{2(t)}$ from $\Gamma\big(nk_2+a_2, b_2+w_2 \sum_{i=1}^{n}\sum_{j_2=1}^{k_2}(v_{ij_2}-\mu_{2(t-1)})^2(Q_{j_2}+1)\big)$.
(8) Generate $\lambda_{(t)}$ from $\Gamma\big(n + a_3, b_3 + w \sum_{i=1}^{n}(y_i - \mu_{(t-1)})^2(S_i + 1)\big)$.
(9) Evaluate

$$
\begin{aligned}
R_{(t)\mathbf{s},\mathbf{k}} = & \sum_{p_1=s_1}^{k_1}\sum_{p_2=s_2}^{k_2}\binom{k_1}{p_1}\binom{k_2}{p_2}\int_{\mu}^{\infty} e^{-\lambda_{1(t)}(y-\mu_{1(t)})^2 p_1} \\
& \times (1 - e^{-\lambda_{1(t)}(y-\mu_{1(t)})^2})^{k_1-p_1} e^{-\lambda_{2(t)}(y-\mu_{2(t)})^2 p_2} \\
& \times (1 - e^{-\lambda_{2(t)}(y-\mu_{2(t)})^2})^{k_2-p_2} 2\lambda(y-\mu_{(t)}) e^{-\lambda_{(t)}(y-\mu_{(t)})^2} dy.
\end{aligned}
$$

(10) Set $t = t + 1$.
(11) Repeat $T_b$ times, steps 3-10.

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