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SOME RESULTS ON DRAZIN-DAGGER MATRICES, RECIPROCAL MATRICES, AND CONJUGATE EP MATRICES

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Abstract. In this paper, a class of matrices, namely, *Drazin-dagger matrices*, in which the Drazin inverse and the Moore-Penrose inverse commute, is introduced. Also, some properties of the generalized inverses of these matrices, are investigated. Moreover, some results about the Moore-Penrose inverse, the Drazin inverse and the numerical range of some reciprocal matrices are obtained. In particular, the relations between reciprocal matrices, Drazin-Dagger matrices and star order are established. Also, some properties of the generalized inverses of the conjugate EP matrices are studied. To illustrate the results, some numerical examples are also given.

Keywords: Moore-Penrose inverse, Drazin inverse, reciprocal matrices, numerical range, conjugate EP matrices.
2020 MSC: 15A09, 15A18, 15A60.

1. Introduction and preliminaries

Let $\mathbb{M}_{m\times n}(\mathbb{F})$ be the set of all $m \times n$ matrices with entries in the field $\mathbb{F}$, where $\mathbb{F}$ is the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. For the case that $m = n$, the algebra $\mathbb{M}_{n\times n}(\mathbb{F})$ is denoted by $\mathbb{M}_n(\mathbb{F})$. For $A \in \mathbb{M}_{m\times n}(\mathbb{C})$, the Moore-Penrose inverse of A is the unique matrix $X \in \mathbb{M}_{n\times m}(\mathbb{C})$ that satisfies the following properties:

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA. \tag{1}$$

The Moore-Penrose inverse of A is denoted by $A^\dagger$, and it is known that $(A^\dagger)^\dagger = A$. For more information, see [2, 5, 15]. Next, we state some other known generalized inverses of a square complex matrix $A \in \mathbb{M}_n(\mathbb{C})$ with index k. Note that the index of A, $\text{ind}(A)$, is the smallest nonnegative integer m such that $\text{rank}(A^m) = \text{rank}(A^{m+1})$. The Drazin inverse of A, denoted by A^D, see [2] or [15], is the unique matrix $X \in \mathbb{M}_n(\mathbb{C})$ satisfying

$$XAX = X, \quad AX = XA, \quad XA^{m+1} = A^m. \tag{2}$$

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In 2014, Malik and Tome in [11] defined the DMP (Drazin Moore-Penrose) inverse of A as the unique matrix $X \in \mathbb{M}_n(\mathbb{C})$ which satisfies

$$XAX = X, \qquad XA = A^D A, \qquad A^m X = A^m A^\dagger. \tag{3}$$

The DMP inverse of A is denoted by $A^{D,\dagger}$, and it is known that $A^{D,\dagger} = A^D AA^\dagger$. The MPD (Moore-Penrose Drazin) inverse $A^{\dagger,D}$ of A is defined, see [11], as

$$A^{\dagger,D} = A^\dagger AA^D. \tag{4}$$

For $A \in \mathbb{M}_n(\mathbb{C})$, the core-nilpotent decomposition of A is $A = A_1 + A_2$ in which $A_1, A_2 \in \mathbb{M}_n(\mathbb{C})$, $\text{rank}(A_1) = \text{rank}(A_1^2)$, A_2 is nilpotent, and $A_1A_2 = A_2A_1 = 0$ [15]. Moreover, $A_1 = AA^DA$. In 2018, Mehdipour and Salemi introduced the CMP (Core-Moore-Penrose) inverse of A by using the core part A_1. The CMP inverse of A, see [14], is a matrix $X \in \mathbb{M}_n(\mathbb{C})$ such that satisfies the following properties:

$$XAX = X, \quad AXA = A_1, \quad AX = A_1A^\dagger, \quad XA = A^\dagger A_1. \tag{5}$$

This matrix X is unique and denoted by $A^{c\dagger}$. It is known that $A^{c\dagger} = A^\dagger AA^DAA^\dagger$. In 2020, Mosić in [16] introduced a new class of matrices, namely, the Drazin-Star matrices, as $A^{D,*} = A^DAA^*$. A matrix $A \in \mathbb{M}_n(\mathbb{C})$ is a normal matrix if $AA^* = A^*A$, and A is called an EP (Equal Projection) matrix if $AA^\dagger = A^\dagger A$. Malik, Rueda and Thome in [10] introduced m-EP matrices, and Mehdipour and Salemi in [14] introduced core-EP matrices. A matrix $A \in \mathbb{M}_n(\mathbb{C})$ with $\text{ind}(A) = m$, is called an m-EP matrix if $A^mA^\dagger = A^\dagger A^m$, and it is said to be a core-EP matrix if $A^\dagger A_1 = A_1A^\dagger$. Moreover, A is called SD (star-dagger) if $A^*A^\dagger = A^\dagger A^*$. The Moore-Penrose and Drazin inverses of matrices have many applications in various fields such as engineering, statistics and other sciences. In particular, they can be used in graph theory, differential equations, and Markov chains; one can see [3, 7, 18, 20] for the mentioned applications.

A matrix $A = (a_{ij}) \in \mathbb{M}_n(\mathbb{R})$ is called a reciprocal matrix if $a_{ij} > 0$ and $a_{ij} = \dfrac{1}{a_{ji}}$ for all $i, j = 1, 2, \ldots, n$. We know that the reciprocal matrices have important applications in the AHP method, the Analytic Hierarchy Process Method which is a tool in the analysis of decision-making, designed to assist decision makers in solving complex problems involving a larger number of decision-makers, as well as numerous criteria; for more information, see [19].

Let $A \in \mathbb{M}_n(\mathbb{C})$. The numerical range or the field of values of A is defined as:

$$W(A) = \{x^*Ax \ : \ x \in \mathbb{C}^n, \ x^*x = 1\},$$

which is useful in understanding matrices and has many applications in numerical analysis, differential equations, systems theory, etc; see [6, 9]. $W(A)$ is a compact and convex set in $\mathbb{C}$ and contains the spectrum of A, $\sigma(A)$. Recall that, the open right half plane of $\mathbb{C}$ is denoted by $RHP := \{z \in \mathbb{C} : Re\ z > 0\}$. The following properties are several fundamental facts about the numerical range of complex matrices:

Proposition 1.1. *[9, Chapter 1] Let $A \in \mathbb{M}_n(\mathbb{C})$. Then the following assertions are true:*

1. $W(A^T) = W(A)$, *and* $W(\overline{A}) = \overline{W(A)} := \{\overline{z} : z \in W(A)\}$;
2. $W(\alpha A + \beta I) = \alpha W(A) + \beta$, *where* $\alpha, \beta \in \mathbb{C}$;
3. $W(U^* A U) = W(A)$, *where* $U \in \mathbb{M}_n(\mathbb{C})$ *is unitary;*
4. $W(A) \subset RHP$ *if and only if* $A + A^*$ *is positive definite, where a matrix* $X \in \mathbb{M}_n(\mathbb{C})$ *is positive definite if* $x^* X x > 0$ *for all nonzero* $x \in \mathbb{C}^n$;
5. $W(A) \subset \mathbb{R}$ *if and only if* A *is Hermitian.*

In this paper, we will study some algebraic properties of the Drazin-Dagger matrices, reciprocal matrices and con-EP matrices. For this, in Section 2, by using the Drazin inverse and the Moore-Penrose inverse, we introduce a class of matrices, namely, *Drazin-Dagger matrices* (shortly, DD matrices), and we study some algebraic properties of these matrices. In Section 3, we consider the reciprocal matrices and we give some results about their well-known generalized inverses and their numerical range. In Section 4, we consider the conjugate EP matrices and we give some results for the generalized inverses of these matrices and by considering the star order, we obtain some other results. To illustrate the main results in the paper, we give some numerical examples.

2. Drazin-Dagger matrices

In this section, by using the Drazin inverse and the Moore-Penrose inverse, we introduce a class of matrices in which the Drazin inverse and the Moore-Penrose inverse commute. Also, we obtain some results for these matrices.

Definition 2.1. A matrix $A \in \mathbb{M}_n(\mathbb{C})$ is called a DD (Drazin-Dagger) matrix, if

$$A^D A^\dagger = A^\dagger A^D.$$

In the following theorem, we give some properties of DD matrices.

Theorem 2.2. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be a DD matrix with $ind(A) = m$. Then the following assertions are true:*

1. $A^{D,\dagger} A^D = A^{\dagger,D} A^\dagger$;
2. $A^{c\dagger} = A^D A^\dagger A^2 A^\dagger = A^\dagger A^2 A^\dagger A^D$;
3. $A^\dagger A^m = A^D A^m$;
4. $A^D A^\dagger A^D = A^\dagger A^D A^\dagger$;
5. $A^D A^\dagger A^{m+1} A^{D,\dagger} = A^\dagger A^m A^\dagger$;
6. $A^* A (A^D)^2 A^\dagger A = A^* A^D$.

Proof. To prove the assertion (i), by (1), (2), (3) and (4), we have

$$\begin{aligned} A^{D,\dagger}A^D &= A^DAA^\dagger A^D \\ &= A^DAA^DA^\dagger \\ &= A^DA^\dagger \\ &= A^\dagger A^D \\ &= A^\dagger AA^\dagger A^D \\ &= A^\dagger AA^DA^\dagger \\ &= A^{\dagger,D}A^\dagger. \end{aligned}$$

So, the result in (i) holds. To prove the first equality in (ii), by (2), we have

$$\begin{aligned} A^{c\dagger} &= A^\dagger AA^DAA^\dagger \\ &= A^\dagger A^DAAA^\dagger \\ &= A^DA^\dagger A^2A^\dagger; \end{aligned}$$

as required. The second equality can be verified in the same manner as in the proof of the first equality, and so, the proof of (ii) is complete. To prove the assertion (iii), by (2) and (1), we have

$$\begin{aligned} A^\dagger A^m &= A^\dagger A^DA^{m+1} \\ &= A^DA^\dagger A^{m+1} \\ &= A^DAA^DA^\dagger A^{m+1} \\ &= A^DA^DAA^\dagger AA^m \\ &= A^DA^DA^{m+1} \\ &= A^DA^m; \end{aligned}$$

as required. To prove the assertion (iv), by (1) and (2), we have

$$\begin{aligned} A^DA^\dagger A^D &= A^DA^\dagger AA^\dagger A^D \\ &= A^\dagger A^DAA^DA^\dagger \\ &= A^\dagger A^DA^\dagger. \end{aligned}$$

This completes the proof of (iv). To prove the assertion (v), by (3) and (2), we have

$$\begin{aligned} A^DA^\dagger A^{m+1}A^{D,\dagger} &= A^DA^\dagger A^{m+1}A^\dagger \\ &= A^\dagger A^DA^{m+1}A^\dagger \\ &= A^\dagger A^mA^\dagger. \end{aligned}$$

So, the result in (v) holds. Finally, to prove the assertion (vi), by (2) and (1), we have

$$\begin{aligned} A^*A(A^D)^2A^\dagger A &= A^*AA^DA^DA^\dagger A \\ &= A^*AA^DA^\dagger A^DA \\ &= A^*A^DAA^\dagger AA^D \\ &= A^*A^D; \end{aligned}$$

completing the proof. □

In the following theorem, we present an equivalent definition for DD matrices. For this, we need the following lemma.

Lemma 2.3. *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $ind(A) = m$. Then the following assertions are true:*

(1) *[17, Corollary 2.5] $A^DA^\dagger = A^\dagger A^D$ if and only if $A^\dagger AA^DA = AA^DAA^\dagger$;*
(2) *[14, Theorem 3.5] A is core-EP if and only if A is m-EP.*

Theorem 2.4. *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $ind(A) = m$. theoremen A is a DD matrix if and only if A is m-EP.*

Proof. Using the fact that $AA^DA = A_1$, where A_1 is the core part of A, we see, by Lemma 2.3(i), that A is DD if and only if A is core-EP. Now, by using Lemma 2.3(ii), this is equivalent to A being m-EP; completing the proof. □

For $A, B \in \mathbb{M}_{m\times n}(\mathbb{C})$, it is said that A is below B under the star partial order (denoted by $A \leq^* B$) if $A^*A = A^*B$ and $AA^* = BA^*$; see [15, Def. 5.2.1.].

Lemma 2.5. *[8, p. 10670] Let $A, B \in \mathbb{M}_{m\times n}(\mathbb{C})$. Then $A \leq^* B$ if one of the following equivalent conditions is satisfied:*

(1) *$AA^* = BA^*$ and $A^*A = A^*B$;*
(2) *$AA^\dagger = BA^\dagger$ and $A^\dagger A = A^\dagger B$;*
(3) *$AA^\dagger = AB^\dagger$ and $A^\dagger A = B^\dagger A$.*

In the following theorem, we state a result about the star order for DD matrices.

Theorem 2.6. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. If $A \leq^* B$ and A is a DD matrix, then*

$$(A^D)^2BA^\dagger = A^\dagger B(A^D)^2 \text{ and } A^{c\dagger}A = A^DA^\dagger BA.$$

Proof. To prove the first equality, by (2) and Lemma 2.5(ii), we have

$$\begin{aligned} A^DA^\dagger = A^\dagger A^D &\Rightarrow A^DAA^DA^\dagger = A^\dagger A^DAA^D \\ &\Rightarrow A^DA^DAA^\dagger = A^\dagger AA^DA^D \\ &\Rightarrow (A^D)^2BA^\dagger = A^\dagger B(A^D)^2. \end{aligned}$$

So, the result holds. To prove the second equality, by (1), (2), Lemma 2.5(ii) and the fact that $A^{c\dagger} = A^\dagger A A^D A A^\dagger$, we have

$$\begin{aligned} A^{c\dagger}A &= A^\dagger A A^D A A^\dagger A \\ &= A^\dagger A A^D A \\ &= A^\dagger A^D A A \\ &= A^D A^\dagger A A \\ &= A^D A^\dagger B A. \end{aligned}$$

So, the proof is complete. □

3. On generalized inverses of reciprocal matrices $A \in \mathbb{M}_n(\mathbb{R})$ with $n < 4$

We recall that a matrix $A = (a_{ij}) \in \mathbb{M}_n(\mathbb{R})$ is a reciprocal matrix if $a_{ij} > 0$ and $a_{ij} = \dfrac{1}{a_{ji}}$ for all $i, j = 1, 2, \ldots, n$. Obviously, the only 1×1 reciprocal matrix is $A = [1]$, which is nonsingular. So, in this section, we study the generalized inverses of 2×2 and 3×3 singular reciprocal matrices. Note that all 2×2 reciprocal matrices are singular; because $\det(\begin{bmatrix} 1 & a \\ \frac{1}{a} & 1 \end{bmatrix}) = 0$. In the following theorem which is a key result in this section, we study the Moore-Penrose inverse and the Drazin inverse of these matrices.

Theorem 3.1. *Let $A \in \mathbb{M}_n(\mathbb{R})$ with $n \in \{2, 3\}$ be a singular reciprocal matrix. Then*

$$A^\dagger = \frac{1}{\|A\|_F^2} A^T \text{ and } A^D = \frac{1}{n^2} A,$$

where for $A = (a_{ij})$, $\|A\|_F^2 = \sum_{i,j} |a_{ij}|^2$.

Proof. To prove the first equality, for $n = 3$, consider a reciprocal matrix A as $A = \begin{bmatrix} 1 & a & b \\ \frac{1}{a} & 1 & c \\ \frac{1}{b} & \frac{1}{c} & 1 \end{bmatrix}$. Since $\det(A) = 0$, it follows that $ac = b$. Now, by taking $B := \dfrac{1}{\|A\|_F^2} \begin{bmatrix} 1 & \frac{1}{a} & \frac{1}{b} \\ a & 1 & \frac{1}{c} \\ b & c & 1 \end{bmatrix}$, and a simple computation and also by using the equation $ac = b$, we see that the equalities $BAB = B$ and $ABA = A$ hold. Obviously, $(AB)^T = AB$ and $(BA)^T = BA$, and so, by (1), we have $B = A^\dagger$; as required. By (1) and in the same manner as in the above proof, the result holds for $n = 2$, and so, the proof of the first equality is complete. To prove the second equality, similarly, by (2), the result holds; completing the proof. □

The result in Theorem 3.1 does not hold for singular reciprocal matrices with higher sizes; see the following example.

Example 3.2. *Let* $A = \begin{bmatrix} 1 & 2 & 10 & 5 \\ \frac{1}{2} & 1 & 5 & \frac{1}{2} \\ \frac{1}{10} & \frac{1}{5} & 1 & 2 \\ \frac{1}{5} & 2 & \frac{1}{2} & 1 \end{bmatrix}$. *Obviously, A is a singular reciprocal matrix. By simple computations, we can see that*

$$A^{\dagger} = \begin{bmatrix} -0.0020 & 0.0206 & -0.0164 & 0.0323 \\ -0.0698 & 0.1088 & -0.1147 & 0.5238 \\ 0.0421 & 0.1426 & -0.0869 & -0.1080 \\ 0.1188 & -0.2929 & 0.2761 & 0 \end{bmatrix}, \text{ and}$$

$$\frac{1}{\|A\|_F^2} A^T = \begin{bmatrix} 0.0063 & 0.0032 & 0.0006 & 0.0013 \\ 0.0126 & 0.0063 & 0.0013 & 0.0126 \\ 0.0632 & 0.0316 & 0.0063 & 0.0032 \\ 0.0316 & 0.0032 & 0.0126 & 0.0063 \end{bmatrix}.$$

So, $A^{\dagger} \neq \frac{1}{\|A\|_F^2} A^T$. *Moreover,*

$$\frac{1}{n^2} A = \begin{bmatrix} 0.0625 & 0.1250 & 0.6250 & 0.3125 \\ 0.0313 & 0.0625 & 0.3125 & 0.0313 \\ 0.0063 & 0.0125 & 0.0625 & 0.1250 \\ 0.0125 & 0.1250 & 0.0313 & 0.0625 \end{bmatrix}, \text{ and}$$

$$A^D = \begin{bmatrix} 0 & 0.3404 & -0.3190 & 0.4681 \\ -0.0170 & 0 & -0.2020 & 0.4894 \\ 0.0128 & 0.1617 & 0 & -0.1446 \\ 0.0277 & -0.1489 & 0.4681 & 0 \end{bmatrix},$$

and hence, $\frac{1}{n^2} A \neq A^D$. *Therefore, the result in Theorem 3.1 does not hold for* $n = 4$.

In the next theorem, we present some results about the well-known generalized inverses of reciprocal matrices.

Theorem 3.3. *Let* $A \in \mathbb{M}_n(\mathbb{R})$ *with* $n \in \{2, 3\}$ *be a singular reciprocal matrix. Then the following assertions are true:*

(1) *A is SD (i.e.,* $A^T A^{\dagger} = A^{\dagger} A^T$*);*
(2) $A^D A^{\dagger} A = A A^{\dagger} A^D$;
(3) $(A^D)^{\dagger} = \frac{n^4}{\|A\|_F^4} (A^{\dagger})^D$;
(4) A^D *is SD;*
(5) $A^2 A^{\dagger} A^D = \frac{1}{n^2} A^2$;
(6) $A^D A^{\dagger} = \frac{1}{n^2 \|A\|_F^2} A A^T$;
(7) $A^{D,\dagger} = \frac{1}{n^2 \|A\|_F^2} A^2 A^T$.

Proof. To prove the assertions (i), (ii) and (iii), by Theorem 3.1, we see that $A^TA^\dagger = A^\dagger A^T = \frac{1}{\|A\|_F^2}(A^T)^2$, $A^DA^\dagger A = AA^\dagger A^D = \frac{1}{n^2\|A\|_F^2}AA^TA$, $(A^D)^\dagger = \frac{n^2}{\|A\|_F^2}A^T$ and $(A^\dagger)^D = \frac{\|A\|_F^2}{n^2}A^T$. So, the results hold. To prove the assertion (iv), by part (i), A is an SD matrix, and so, by [2, Ex. 33 in p. 167], $(A^\dagger A^T)^D = (A^\dagger)^D(A^T)^D = (A^T)^D(A^\dagger)^D$. Hence, by part (iii) and the fact that $(A^D)^T = (A^T)^D$ (see [2, Ex. 27 in p. 166]), we have

$$\begin{aligned}(A^D)^T(A^D)^\dagger &= (A^T)^D\frac{n^4}{\|A\|_F^4}(A^\dagger)^D\\ &= \frac{n^4}{\|A\|_F^4}(A^\dagger A^T)^D\\ &= \frac{n^4}{\|A\|_F^4}(A^\dagger)^D(A^T)^D\\ &= (A^D)^\dagger(A^D)^T.\end{aligned}$$

This shows that A^D is an SD matrix. To prove the assertion (v), by part (ii), (2), (1) and Theorem 3.1, we can see that

$$\begin{aligned}A^2A^\dagger A^D &= AA^DA^\dagger A\\ &= A^DAA^\dagger A\\ &= A^DA\\ &= \frac{1}{n^2}A^2.\end{aligned}$$

So, the result in (v) holds. In view of Theorem 3.1, the assertion (vi) is obvious. To prove the assertion (vii), by Theorem 3.1 and the fact that $A^{D,\dagger} = A^DAA^\dagger$, the result holds. So, the proof is complete. □

We have the following result about the Frobenius norm of 2×2 or 3×3 reciprocal matrices. We denote by J_n the $n\times n$ matrix whose all entries are equal to 1.

Proposition 3.4. *Let A be a 2×2 or 3×3 reciprocal matrix. Then $\|A\|_F \geq n$. The equality holds if and only if $A = J_n$.*

Proof. We know that $(x+\frac{1}{x}) \geq 2$ for every $x > 0$. This shows, by a simple computation, that $\|A\|_F \geq n$; as required. Also, by the fact that $(x+\frac{1}{x}) = 2$ if and only if $x = 1$, we see that $\|A\|_F = n$ if and only if $A = J_n$. So, the proof is complete. □

By Theorem 3.1, Propositions 1.1((i) *and* (ii)) and 3.4, we have the following result about the numerical range of a singular reciprocal matrix.

Corollary 3.5. *Let A be a 2×2 or 3×3 singular reciprocal matrix. Then*

$$W(A^{\dagger}) \subseteq W(A^{D}) \subseteq W(A).$$

We also have the following equivalent conditions for singular reciprocal matrices.

Theorem 3.6. *Let A be a 2×2 or 3×3 singular reciprocal matrix. Then the following conditions are equivalent:*

(1) *A is a DD matrix;*
(2) *A is a normal matrix;*
(3) *$W(A^{\dagger}) = W(A^{D})$;*
(4) *$A = J_n$;*
(5) *A is a Hermitian matrix.*

Proof. $(i) \Rightarrow (ii)$; By Theorem 3.1, we have $A^D A^{\dagger} = \frac{1}{n^2\|A\|_F^2} AA^T$ and $A^{\dagger}A^D = \frac{1}{n^2\|A\|_F^2} A^T A$. Since A is DD, it follows that $AA^T = A^T A$; as required.

$(ii) \Rightarrow (iii)$; By the fact that every normal matrix is EP (see [15, Remark 2.2.39]), we have $A^{\dagger} = A^D$, and so, the result holds.

$(iii) \Rightarrow (iv)$; By Theorem 3.1 and Proposition 1.1((i) *and* (ii)), we have $\frac{1}{\|A\|_F^2} W(A) = \frac{1}{n^2} W(A)$. Thus, $\|A\|_F = n$, and hence, by Proposition 3.4, $A = J_n$.

$(iv) \Rightarrow (v)$; This is obvious.

$(v) \Rightarrow (i)$; Since A is a Hermitian matrix, it follows that A is EP. Hence, by Theorem 2.4, A is a DD matrix. Therefore, the proof is complete. □

Recall that a point $\alpha \in \mathbb{C}$ on the boundary of $W(A)$ is called a sharp point of $W(A)$ [9, Chapter 1], if there are angles θ_1 and θ_2 with $0 \leqslant \theta_1 < \theta_2 < 2\pi$ for which

$$\mathrm{Re}(e^{i\theta}\alpha) = \max\{\mathrm{Re}(z) : z \in W(e^{i\theta}A)\}, \text{ for all } \theta \in (\theta_1, \theta_2).$$

The following theorem shows that when the numerical range of a singular reciprocal matrix has no sharp point. For this, we need the following lemmas.

Lemma 3.7. *[4, Theorem 5] Let $A \in \mathbb{M}_n(\mathbb{C})$ and $\alpha \in \mathbb{C}$. Then α is a sharp point of $W(A)$ if and only if A is unitary similar to $\alpha I_m \oplus B$ $(m \leq n)$ such that $\alpha \notin W(B)$.*

Lemma 3.8. *[1, Theorem 2.5] Let $A \in \mathbb{M}_n(\mathbb{C})$ be a singular matrix such that the origin is a boundary point of $W(A)$. Then A^k is an EP matrix for every $k \in \mathbb{N}$.*

Theorem 3.9. *Let $A \in \mathbb{M}_n(\mathbb{R})$ with $n \in \{2, 3\}$ be a singular reciprocal matrix. Then $W(A)$ has no sharp point if and only if $A \neq J_n$.*

Proof. Obviously, if $A = J_n$, then $W(A) = [0, n]$, and so, it has two sharp points. Now, we assume that $A \neq J_n$ and we will show that $W(A)$ has no sharp point. For $n = 3$, the singularity of A implies that A has two eigenvalues $\lambda_1 = 0$ with algebraic multiplicity 2, and $\lambda_2 = 3$ with algebraic multiplicity 1. We know, by [9, Theorem 1.6.3], that every sharp point of $W(A)$ is an eigenvalue of A. So, we first claim that the origin is an interior point of $W(A)$. For this, if the origin is a boundary point of $W(A)$, then by Lemma 3.8, A is an EP matrix, and hence, by Theorem 3.1, we have $\frac{1}{\|A\|_F^2}A^TA = \frac{1}{\|A\|_F^2}AA^T$. Thus, A is normal, and so, by Theorem 3.6, $A = J_3$; a contradiction.

Now, we show that 3 is an interior point of $W(A)$. For this, if 3 is a sharp point of $W(A)$, then by Lemma 3.7, there exists a unitary matrix $U \in \mathbb{M}_3(\mathbb{C})$ such that $A = U^* \begin{bmatrix} B & 0 \\ 0 & 3 \end{bmatrix} U$, where $3 \notin W(B)$. Thus, $B = 0$ or B is similar to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Since $A \neq J_3$, it follows, by Proposition 3.4, that $\|A\|_F > 3$. Now, if $B = 0$, then $\|A\|_F = 3$; a contradiction. For the case that B is similar to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, we see that A is similar to $C := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. This shows that $\operatorname{rank}(A) = \operatorname{rank}(C)$; a contradiction, because clearly, $\operatorname{rank}(A) = 1$ and $\operatorname{rank}(C) = 2$. Therefore, 3 is an interior point of $W(A)$; as required.

For $n = 2$, since $A \neq J_2$, it follows, by [9, Theorem 1.3.6], that $W(A)$ is a closed elliptical disk in which the eigenvalues of A are the interior points of it. So, the proof is complete. □

At the end of this section, we consider the order $A \leq^* B$ in which A is below B under the star partial order. We know that if $A \leq^* B$, then $A^\dagger \leq^* B^\dagger$; see [15, Corollary 5.2.9]. But, in general, the star partial order $A^D \leq^* B^D$ does not hold when $A \leq^* B$; see the following example.

Example 3.10. *Let* $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 2 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. *Hence,* $A \leq^* B$. *By a simple computation, we have*

$$A^D = \begin{bmatrix} 0.3333 & 0.6667 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } B^D = \begin{bmatrix} 0.2 & -0.2 & 0 \\ 0.4 & 0.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It shows that $A^D \not\leq^* B^D$.

The following theorem shows that if $A \leq^* B$, then the star partial order $A^D \leq^* B^D$ is valid for singular reciprocal matrices.

Theorem 3.11. *Let* $A, B \in \mathbb{M}_n(\mathbb{R})$ *with* $n \in \{2, 3\}$ *be two singular reciprocal matrices. If* $A \leq^* B$, *then* $A^D \leq^* B^D$.

Proof. By Lemma 2.5(i), we have $AA^T = BA^T$ and $A^TA = A^TB$. So, by Theorem 3.1, we can see

$$\begin{aligned} A^D(A^D)^T &= \frac{1}{n^4}AA^T \\ &= \frac{1}{n^4}BA^T \\ &= B^D(A^D)^T; \end{aligned}$$

as required. Similarly, we can see that $(A^D)^TA^D = (A^D)^TB^D$, and hence the proof is complete. □

4. Conjugate EP matrices

Recall that $A \in \mathbb{M}_n(\mathbb{C})$ is said to be con-EP (conjugate EP) if $\mathcal{R}(A) = \mathcal{R}(A^T)$, where $\mathcal{R}(.)$ is the row space of a matrix. Obviously, A is con-EP if and only if $AA^\dagger = \overline{A^\dagger A}$. The matrix A is called con-EP$_r$ if A is con-EP and rank$(A) = r$. Obviously, A is con-EP$_r$ if and only if $A^\dagger$ is con-EP$_r$. Also, con-EP matrices coincides with that of EP matrices for the class of real matrices and the class of nonsingular matrices; but not for nonreal matrices, for example, the matrix $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$ is con-EP, but it is not EP. We know that every conjugate normal matrix (i.e., $AA^* = \overline{A^*A}$) is con-EP. The matrix A is called EP$_r$ if A is EP and rank$(A) = r$. For more information, see [12, 13]. Now, we state the following result about the numerical range of these matrices. For this, we need the following lemmas.

Lemma 4.1. *[12, Theorem 3] Let $A \in \mathbb{M}_n(\mathbb{C})$. Then A is con-EP if and only if $A\overline{A}$ is EP$_r$ and rank$(A\overline{A}) =$ rank(A).*

Lemma 4.2. *Let $A \in \mathbb{M}_n(\mathbb{C})$. If A is a con-EP matrix, then $(A\overline{A})^\dagger = \overline{A}^\dagger A^\dagger$.*

Proof. Since A is con-EP, it follows, by [12, Theorem 2], that $A = U\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}U^T$, where U is unitary and D is nonsingular. Hence, $A\overline{A} = U\begin{bmatrix} D\overline{D} & 0 \\ 0 & 0 \end{bmatrix}U^*$, and so,

$$(A\overline{A})^\dagger = U\begin{bmatrix} (D\overline{D})^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{6}$$

On the other hand, we have $A^\dagger = \overline{U}\begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*$. Thus,

$$\overline{A}^\dagger A^\dagger = U\begin{bmatrix} (\overline{D})^{-1} & 0 \\ 0 & 0 \end{bmatrix}\overline{U}^*\overline{U}\begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} (D\overline{D})^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{7}$$

Therefore, by relations (6) and (7) the result holds. □

Theorem 4.3. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be two con-EP matrices. If $A \leq^* B$, then the following assertions are true:*

(1) $A\overline{A} \leq^* B\overline{B}$. *In addition* $W(A\overline{A}) \subseteq W(B\overline{B})$;

(2) $AB = \overline{A^\dagger A}B^2 = B\overline{AA^\dagger}B$, *and* $BA = B^2\overline{AA^\dagger} = B\overline{A^\dagger A}B$.

Proof. To prove the assertion (i), at first, we claim that $A\overline{A} \leq^* B\overline{B}$. For this, we must show that $(A\overline{A})(A\overline{A})^\dagger = (A\overline{A})(B\overline{B})^\dagger$ and $(A\overline{A})^\dagger(A\overline{A}) = (B\overline{B})^\dagger(A\overline{A})$. By Lemma 2.5$(iii)$, Lemma 4.2 and the fact that $AA^\dagger = \overline{A^\dagger A}$, we have

$$\begin{aligned}
(A\overline{A})(B\overline{B})^\dagger &= A\overline{A}\,\overline{B}^\dagger B^\dagger \\
&= A\overline{A}\,\overline{A}^\dagger B^\dagger \\
&= AA^\dagger AB^\dagger \\
&= AA^\dagger AA^\dagger \\
&= A\overline{A}\,\overline{A}^\dagger A^\dagger \\
&= (A\overline{A})(A\overline{A})^\dagger.
\end{aligned}$$

In the same manner as above, we have $(B\overline{B})^\dagger(A\overline{A}) = (A\overline{A})^\dagger(A\overline{A})$. Thus, $A\overline{A} \leq^* B\overline{B}$. Since A is con-EP, it follows by Lemma 4.1, that $A\overline{A}$ is EP. Therefore, by [1, Theorem 2.9], the proof of the assertion (i) is complete.

To prove the assertion (ii), by Lemma 2.5(ii), we have $A = BA^\dagger A$ and $A = AA^\dagger B$. Hence, by the fact that $AA^\dagger = \overline{A^\dagger A}$, we can see

$$AB = BA^\dagger AB = B\overline{AA^\dagger}B, \text{ and } AB = AA^\dagger BB = \overline{A^\dagger A}B^2.$$

Also, we have

$$BA = BBA^\dagger A = B^2\overline{AA^\dagger}, \text{ and } BA = BAA^\dagger B = B\overline{A^\dagger A}B,$$

completing the proof. □

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Conflict of Interest
The authors have no conflicts of interest to declare that are relevant to the content of this article.

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