LINEAR HYPOTHESIS TESTING USING D_{LR} METRIC

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Abstract. Several practical problems of hypotheses testing can be under a general linear model analysis of variance which would be examined. In analysis of variance, when the response random variable Y, has linear relationship with several random variables X, another important model as analysis of covariance can be used. In this paper, assuming that Y is fuzzy and using D_{LR} metric, a method for testing the linear hypothesis has been proposed based on fuzzy techniques. In fact, in this method a set of confidence intervals has been used for creating fuzzy test statistic and fuzzy critical values. In addition, the proposed method has been mentioned for the reforming of the hypothesis testing when there is an uncertaity in accepting or rejecting hypotheses. Finally, by presenting two examples this method is illustrated. The result are illustrated by the means of some case studies.

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1. Introduction and Background

Several practical problems of hypotheses testing can be under a general linear model analysis of variance which would be examined. Analysis of variance is a common and popular method used in the analysis of experimental designs. Many authors have studied this topic from various aspects for fuzzy environments. For instance, in [1] one-way and two-way analysis of variance using a set of confidence intervals for the variance parameter has been carried out. In [12] analysis of variance for fuzzy data is discussed by considering the α-cuts of fuzzy data via introducing the pessimistic and optimistic degrees and solving an optimization problem. One-way analysis of variance is presented in [8] to a case where observed data are fuzzy observations.

In analysis of variance, when the response random variable Y, has linear relationship with several random variables x, another important model as analysis of covariance can be used. Sometimes, in one-way analysis of covariance, the observed value of test statistic is close to the quantiles of statistical distributions and there is uncertainty with regard to accepting the null hypothesis H_0. In this paper an approach is presented to deal with this problem.

Buckley [2] introduced a method for estimating the parameters in statistical models. His method produces a fuzzy estimator using a set of confidence intervals for the required parameter. Using this estimator, a fuzzy test statistic and, subsequently, fuzzy critical values are produced. This fuzzy test statistic is used to perform the statistical hypotheses test. This issue has been studied by several other authors in different ways. In [3] an explicit and unique membership function has been derived for fuzzy estimators. In [10] Buckley's method is extended to the case where the statistical hypotheses are fuzzy. In [4] it is shown that when the crisp test statistic distribution is not symmetric, Buckley's method results in producing a fuzzy estimation where the membership degree for the point estimation of the required parameter never equals one. A solution to overcome this weakness is provided in [4] and another solution is presented in [1]. In [5] analysis of covariance is discussed by considering solution presented in [4]. It has been shown that this solution reduces to the Buckley's method when crisp test statistic distribution is symmetric. In this paper, by using [5], the data are fuzzy. It has been shown that this solution reduces to the Buckley's method when crisp test statistic distribution is symmetric. We having no interaction between X and treatment and this paper is a general case of [5].

The rest of this paper is organized as follows. In section 2, the necessary concepts of fuzzy sets theory are discussed and some basic concepts of LR fuzzy number, D_{LR} metric are described. In section 3, a brief review of one-way analysis of covariance is presented. In section 4, fuzzy test statistics and fuzzy critical values are produced and based on them decision rules are presented. In section 5, two examples are provided to illustrate the method. Finally, a conclusion is provided in section 6.

2. Preliminaries

In this section, we study some concepts of LR fuzzy numbers, D_{LR} metric and Buckley's method.

2.1. LR Fuzzy Numbers.

A particular class of fuzzy sets very useful in practice is determined by 3 values: the center, the left spread and the right spread. This type of fuzzy data is the LR fuzzy number.

Definition 2.1. An LR fuzzy number $\tilde{A} = \left(A^m, A^l, A^r\right)$ is characterized by the following membership function

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{A^m - x}{A^l}\right) & x \le A^m \\ R\left(\frac{x - A^m}{A^r}\right) & x \ge A^m, \end{cases}$$

$where$A$^m \in \mathbb{R}$ is the center, $A^l \in \mathbb{R}^+$ and $A^r \in \mathbb{R}^+$ are, respectively, the left and the right spread and, L and R are functions verifying the properties of the class of fuzzy sets $F_c(\mathbb{R})$, such that $L(0) = R(0) = 1$ and $L(x) = R(x) = 0$, $\forall x \in \mathbb{R}\backslash[0, 1]$. If $A^l = A^r$ the fuzzy number $\tilde{A}$ is referred to as symmetrical. The most used LR fuzzy numbers are the triangular ones, whose membership function is

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \dfrac{A^m - x}{A^l} & A^m - A^l \le x \le A^m \\ \\ 1 - \dfrac{x - A^m}{A^r} & A^m \le x \le A^m + A^r, \end{cases}$$

$$\tilde{A}_\alpha = [A^m - L^{-1}(A^l), A^m + R^{-1}(A^l)].$$

If $\tilde{A} = \left(A^m, A^l, A^r\right)$ and $\tilde{B} = \left(B^m, B^l, B^r\right)$ be LR fuzzy numbers, these operations can be alternatively determined by considering the fuzzy set $\tilde{A} + \tilde{B}$, we have

$$\tilde{A} + \tilde{B} = ((A+B)^m, (A+B)^l, (A+B)^r),$$

$$\begin{cases} (A+B)^m = A^m + B^m \\ (A+B)^l = A^l + B^l \\ (A+B)^r = A^r + B^r, \end{cases}$$

and $\forall \gamma \in \mathbb{R}$, $\gamma\tilde{A} = ((\gamma(A))^m, (\gamma(A))^l, (\gamma(A))^r)$, is the fuzzy set so that

$$\begin{cases} (\gamma(A))^m = \gamma A^m \\ (\gamma(A))^l = \gamma A^l \\ (\gamma(A))^r = \gamma A^r & \gamma > 0, \end{cases}$$

$$\begin{cases} (\gamma(A))^m = \gamma A^m \\ (\gamma(A))^l = \gamma A^r \\ (\gamma(A))^r = \gamma A^l & \gamma < 0. \end{cases}$$

2.2. $D_{LR}Metric$**.** we define a distance between fuzzy numbers which will be used later.

Definition 2.2. In [12] have defined a distance D_{LR} between two LR fuzzy numbers $\tilde{A} = \left(A^m, A^l, A^r\right)$ and $\tilde{B} = \left(B^m, B^l, B^r\right) \in F_{LR}$ as follows

$$\begin{aligned} D^2_{LR}(\tilde{A}, \tilde{B}) &= (A^m - B^m)^2 + \left[\left(A^m - \lambda A^l\right) - \left(B^m - \lambda B^l\right)\right]^2 \\ &+ \left[(A^m + \rho A^r) - (B^m + \rho B^r)\right]^2 \\ &= 3(A^m - B^m)^2 + \lambda^2\left(A^l - B^l\right)^2 + \rho^2 (A^r - B^r)^2 \\ &- 2\lambda \left(A^m - B^m\right)\left(A^l - B^l\right) + 2\rho \left(A^m - B^m\right)\left(A^r - B^r\right), \end{aligned}$$

where $\lambda = \int_0^1 L^{-1}(\omega)\, d\omega$ and $\rho = \int_0^1 R^{-1}(\omega)\, d\omega$ represent the influence of the shape of the membership function on the distance. In particular, λ (or ρ) less than 0.5 represents an imprecision decreasing rapidly; λ (or ρ) equal to 0.5 represents an imprecision decreasing linearly and λ (or ρ) greater than 0.5 represents an imprecision decreasing slowly.

Proposition 2.1. Let $\tilde{A} = \left(A^m, A^l, A^r\right)$, $\tilde{B} = \left(B^m, B^l, B^r\right)$ and be LR fuzzy numbers, and $\gamma > 0$. Then

1. $D^2_{LR}(\tilde{A}, \tilde{B}) = D^2_{LR}(\tilde{B}, \tilde{A})$,
2. $D^2_{LR}(\gamma\tilde{A}, \gamma\tilde{B}) = \gamma^2 D^2_{LR}(\tilde{A}, \tilde{B})$.

Proof. By means of define of D_{LR} metric, it is easy to prove this proposition.

1. We have

$$\begin{aligned} D^2_{LR}(\tilde{A},\tilde{B}) &= (A^m - B^m)^2 + \left[\left(A^m - \lambda A^l\right) - \left(B^m - \lambda B^l\right)\right]^2 \\ &+ \left[(A^m + \rho A^r) - (B^m + \rho B^r)\right]^2 \\ &= (-(B^m - A^m))^2 + \left[-\left(\left(B^m - \lambda B^l\right) - \left(A^m - \lambda A^l\right)\right)\right]^2 \\ &+ \left[-((B^m + \rho B^r) - (A^m + \rho A^r))\right]^2 = D^2_{LR}(\tilde{B},\tilde{A}), \end{aligned}$$

2. Also, for $\gamma > 0$, we have

$$\begin{aligned} D^2_{LR}(\gamma\tilde{A},\gamma\tilde{B}) &= ((\gamma A)^m - (\gamma B)^m)^2 + \left[\left((\gamma A)^m - \lambda(\gamma A)^l\right) - \left((\gamma B)^m - \lambda(\gamma B)^l\right)\right]^2 \\ &+ \left[((\gamma A)^m + \rho(\gamma A)^r) - ((\gamma B)^m + \rho(\gamma B)^r)\right]^2 \\ &= (\gamma(A^m - B^m))^2 + \left[\gamma\left(\left(A^m - \lambda A^l\right) - \left(B^m - \lambda B^l\right)\right)\right]^2 \\ &+ \left[\gamma((A^m + \rho A^r) - (B^m + \rho B^r))\right]^2 = \gamma^2 D^2_{LR}(\tilde{A},\tilde{B}). \end{aligned}$$

□

2.3. **Buckley's Method.** Buckley's method results in producing a fuzzy number to estimate the required param- eter from a statistical distribution whose α-cuts are $(1-\alpha)100\%$ confidence intervals, α
$\in [0.01, 1]$. The following definition is given, to clarify the discussion that is presented in this paper.

Definition 2.3. A fuzzy number $\tilde{\theta}$ is an unbiased fuzzy estimator for parameter θ from a statistical distribution if:

(i) the α-cuts of $\tilde{\theta}$ are $(1-\alpha)100\%$ confidence intervals for θ, with $\alpha \in [0.01, 1]$ and $\tilde{\theta}[\alpha] = \tilde{\theta}[0.01]$ for $\alpha \in [0, 0.01)$,

(ii) if $\hat{\theta}$ is an unbiased point estimator for θ then $\tilde{\theta}(\hat{\theta}) = 1$.

Similar to conventional statistics, a fuzzy estimator is a rule for calculating a fuzzy es-timation of an unknown parameter based on observed data: thus the rule and it's re- sult (the fuzzy estimation) are distinguished. For a fuzzy estimation an explicit and uni- que membership function is given by the following theorem [3].

Theorem 2.1. Suppose that $X_1, X_2, \ldots, X_n$ is a random sample of size n from a distribution with unknown parameter θ. If, based on observation $x_1, x_2, \ldots, x_n$, we consider $[\theta_1(\alpha), \theta_2(\alpha)]$, as a $(1-\alpha)100\%$ confidence interval for θ, then the fuzzy estimation of θ is a fuzzy set with the following unique membership function

$$\tilde{\theta}(u) = min\left\{\theta_1^{-1}(u), [-\theta_2]^{-1}(-u), 1\right\}.$$

3. One-way Analysis of Covariance

In this section one-way analysis of covariance is briefly reviewed, for more details see [7, 9]. For the linear model $\tilde{y}_{ij} = \mu_i + \beta(x_{ij} - \bar{x}_{..}) + \epsilon_{ij}$, where ϵ_{ij}'s have a normal distribution with a random variables which have a linear relationship with $\tilde{y}_{ij}$'s, $\bar{x}_{..} = \sum_{i=1}^{a}\sum_{j=1}^{n_i} x_{ij}/\sum_{i=1}^{a} n_i$, β and μ_i's are unknown parameters, for $i = 1, 2, \ldots, a$ and $j = 1, 2, \ldots, n_i$.
Taking into account the above linear model we are interested to test the following statistical hypotheses: $H_0 : \beta = 0$ and $H_0 : \mu_1 = \mu_2 = \ldots = \mu_a$
To simplify the discussion we use the following notations.

$$S_{yy} = \sum_{i=1}^{a}\sum_{j=1}^{n_i} D^2_{LR}(\tilde{y}_{ij}, \bar{\tilde{y}}_{..}), \qquad S_{xx} = \sum_{i=1}^{a}\sum_{j=1}^{n_i} D^2_{LR}(x_{ij}, \bar{x}_{..}),$$

$$E_{yy} = \sum_{i=1}^{a}\sum_{j=1}^{n_i} D^2_{LR}(\tilde{y}_{ij}, \bar{\tilde{y}}_{i.}), \qquad E_{xx} = \sum_{i=1}^{a}\sum_{j=1}^{n_i} D^2_{LR}(x_{ij}, \bar{x}_{i.}),$$

and for $\alpha \in (0, 1)$

$$S_{xy} = \sum_{i=1}^{a}\sum_{j=1}^{n_i}(\tilde{y}_{ij} - \bar{\tilde{y}}_{..})_{\alpha=1}(x_{ij} - \bar{x}_{..})_{\alpha=1}, \qquad SSE = E_{yy} - (E^2_{xy}/E_{xx}),$$

$$E_{xy} = \sum_{i=1}^{a}\sum_{j=1}^{n_i}(\tilde{y}_{ij} - \bar{\tilde{y}}_{i.})_{\alpha=1}(x_{ij} - \bar{x}_{i.})_{\alpha=1}, \qquad SSE^{'} = S_{yy} - (S^2_{xy}/S_{xx}).$$

Now the critical region based on generalized likelihood ratio (GLR) method [9] for testing hypotheses in (1) is $F1 \geq k$, where k is a real number and

$$F1 = \frac{E^2_{xy}/E_{xx}}{SSE/(N - a - 1)},$$

where

$$N = \sum_{i=1}^{a} n_i.$$

The pivotal quantity SSE/σ^2 has the distribution χ^2 with $N - a - 1$ degree of freedom and $E^2_{xy}/(\sigma^2 E_{xx})$, under the null hypothesis H_0 in (1), has the distribution χ^2 with 1 degree of freedom. So both of these pivotal quantities can be used to produce confidence intervals for σ^2. It can be shown that, under the null hypothesis H_0 in (1), $F1$ has the distribution F with 1 and $N - a - 1$ degrees of freedom. The null hypothesis $H_0 : \beta = 0$ is rejected if the observed value of $F1$ statistic is equal or greater than $F_{1-\gamma,1,N-a-1}$, where $F_{1-\gamma,1,N-a-1}$ is $(1 - \gamma)$'th quantile of the distribution F with with 1 and $N - a - 1$ degrees of freedom and $\gamma \in (0, 1)$ is the significance level of testing.

Also, the critical region based on GLR method for testing hypotheses in (2) is $F2 \geq k$, where k is a real number and

$$F2 = \frac{(SSE^{'} - SSE)/(a-1)}{SSE/(N-a-1)},$$

The mathematical term $(SSE^{'} - SSE)/\sigma^2$, under the null hypothesis H_0 in (2), has the distribution χ^2 with $a-1$ degree of freedom; and this pivotal quantity can be used to produce the confidence intervals for the parameter σ^2. It can be shown that, under the hypothesis H_0 in (2), $F2$ has the distribution F with $a-1$ and $N-a-1$ degrees of freedom and the null hypothesis $H_0 : \mu_1 = \mu_2 = \ldots = \mu_a$ is rejected if the observed value of $F2$ statistic is equal or greater than $F_{1-\gamma,a-1,N-a-1}$, $(1-\gamma)$'th quantile of the distribution F with $a-1$ and $N-a-1$ degrees of freedom.

Remark 3.1. Note that E^2_{xy}/E_{xx}, under the hypothesis H_0 in (1), $(SSE^{'} - SSE)/(a-1)$, under the hypothesis H_0 in (2) and $SSE/(N-a-1)$ are unbiased point estimators of the parameter σ^2.

4. One-way Analysis of Covariance based on Fuzzy Test Statistic

In this section we first consider the issue of testing the statistical hypotheses in (1).

Theorem 4.1. In one-way analysis of covariance model, if we consider $SSE/(N-a-1)$ as an unbiased point estimator for parameter σ^2, then an unbiased fuzzy estimator for σ^2 is $\widetilde{\sigma^2}$ with α-cuts $\widetilde{\sigma^2}[\alpha]$, where

$$\widetilde{\sigma^2}[\alpha] = \begin{cases} \left[SSE/\chi^2_{1-\alpha+\alpha p',N-a-1}, SSE/\chi^2_{\alpha p',N-a-1}\right] & 0.01 \leq \alpha \leq 1 \\ \\ \widetilde{\sigma^2}[0.01] & 0 \leq \alpha < 0.01, \end{cases}$$

and p' is obtained from the relation $\chi^2_{p',N-a-1} = N-a-1$.

Proof. Based on the pivotal quantity SSE/σ^2, a $(1-\alpha)100\%$ confidence interval for σ^2 is $\left[SSE/\chi^2_{1-\alpha+\alpha p,N-a-1}, SSE/\chi^2_{\alpha p,N-a-1}\right]$ for any $\alpha \in (0,1)$ and $p \in (0,1)$. When $\alpha = 1$ and $p = p'$, satisfying $\chi^2_{p',N-a-1} = N-a-1$, this interval becomes the point $SSE/(N-a-1)$ the unbiased point estimator for σ^2. Now fixing $p = p'$ and varying α from 0.01 to 1 we obtain nested intervals which are the α-cuts of a fuzzy number, say $\widetilde{\sigma^2}$. Finally, $\widetilde{\sigma^2}[\alpha] = \widetilde{\sigma^2}[0.01]$ for $\alpha \in [0, 0.01)$, we have the unbiased fuzzy estimator $\widetilde{\sigma^2}$ for σ^2. □

Lemma 4.1. The membership function of fuzzy estimator $\widetilde{\sigma^2}$ in Theorem 4.1 is as follows:

$$\widetilde{\sigma^2}(x) = \begin{cases} \frac{1-G(SSE/x)}{1-p'} & \frac{SSE}{\chi^2_{0.99+0.01p',N-a-1}} \leq x \leq \frac{SSE}{N-a-1} \\ \\ \frac{G(SSE/x)}{p'} & \frac{SSE}{N-a-1} \leq x \leq \frac{SSE}{\chi^2_{0.01p',N-a-1}} \\ \\ 0 & \text{otherwise,} \end{cases}$$

where G is the cumulative distribution function of a χ^2 variable with $N-a-1$ degree of freedom.

Proof. By Theorem 4.1, we have $\theta_1(\alpha) = SSE/\chi^2_{1-\alpha+\alpha p',N-a-1}$ for $\alpha \in [0.01, 1]$. Hence, $\theta_1^{-1}(x) = [1 - G(\frac{SSE}{x})]/(1-p')$. Also $\theta_2(\alpha) = SSE/\chi^2_{\alpha p',N-a-1}$, therefore $[-\theta_2]^{-1}(-x) = G(\frac{SSE}{x})/p'$. Based on Theorem 2.1, we have

$$\widetilde{\sigma^2}(x) = min\{\theta_1^{-1}(x), [-\theta_2]^{-1}(-x), 1\}.$$

So,

$$\widetilde{\sigma^2}(x) = \begin{cases} \frac{1-G(SSE/x)}{1-p'} & \frac{SSE}{\chi^2_{0.99+0.01p',N-a-1}} \leq x \leq \frac{SSE}{N-a-1} \\ \\ \frac{G(SSE/x)}{p'} & \frac{SSE}{N-a-1} \leq x \leq \frac{SSE}{\chi^2_{0.01p',N-a-1}} \\ \\ 0 & \text{otherwise.} \end{cases}$$

□

Theorem 4.2. Under the null hypothesis $H_0 : \beta = 0$, if we consider E_{xy}^2/E_{xx} as an unbiased point estimator for parameter σ^2, then an unbiased fuzzy estimator for σ^2 is $\widetilde{\sigma^2_{H_0 1}}$ with α-cuts $\widetilde{\sigma^2_{H_0 1}}[\alpha]$, where

$$\widetilde{\sigma^2_{H_0 1}}[\alpha] = \begin{cases} \left[E_{xy}^2/(E_{xx}\chi^2_{1-\alpha+\alpha p'',1}), E_{xy}^2/(E_{xx}\chi^2_{\alpha p'',1})\right] & 0.01 \leq \alpha \leq 1 \\ \\ \widetilde{\sigma^2_{H_0 1}}[0.01] & 0 \leq \alpha < 0.01, \end{cases}$$

and p'' is obtained from the relation $\chi^2_{p'',1} = 1$.

Proof. We use the pivotal quantity $E_{xy}^2/(E_{xx}\sigma^2)$. The proof is now similar to that of Theorem 4.1. □

Theorem 4.3. The fuzzy test statistic for testing $H_0 : \beta = 0$ against $H_1 : \beta \neq 0$ is $\widetilde{F1}$ with α-cuts

$$\widetilde{F1}[\alpha] = \begin{cases} \big[(f1)_1(\alpha)F1,\ (f1)_2(\alpha)F1\big] & 0.01 \leq \alpha \leq 1 \\ \\ \widetilde{F1}[0.01] & 0 \leq \alpha < 0.01, \end{cases}$$

$$(f1)_1(\alpha) = \chi^2_{\alpha p', N-a-1} \big/ \left[(N-a-1)\chi^2_{1-\alpha+\alpha p'', 1}\right],$$

$$(f1)_2(\alpha) = \chi^2_{1-\alpha+\alpha p', N-a-1} \big/ \left[(N-a-1)\chi^2_{\alpha p'', 1}\right].$$

Proof. Using the equality $\widetilde{F1}[\alpha] = \widetilde{\sigma^2_{H_0 1}}[\alpha] / \widetilde{\sigma^2}[\alpha]$ and interval arithmetic, fuzzy test statistic follows from Buckley's method. □

Note 4.1. Since the test statistic is a fuzzy number, then critical value is also a fuzzy number with α-cuts

$$\widetilde{CV1}[\alpha] = \begin{cases} \big[(cv1)_1(\alpha),\ (cv1)_2(\alpha)\big] & 0.01 \leq \alpha \leq 1 \\ \\ \widetilde{CV_1}[0.01] & 0 \leq \alpha < 0.01, \end{cases}$$

where

$$(cv1)_1(\alpha) = (f1)_1(\alpha) F_{1-\gamma,1,N-a-1},$$

is obtained from the relation $P\big[(f1)_1(\alpha)F1 > (cv1)_1(\alpha)\big] = \gamma$, where $\gamma \in (0,1)$ is the significance level of the test. Similarly, we obtain $(cv1)_2(\alpha) = (f1)_2(\alpha) F_{1-\gamma,1,N-a-1}$.

Decision rule 4.1. The decision rule is considered as follows. After observing the data,

(i) if $F_{1-\gamma,1,N-a-1} \leq F1$, then the hypothesis $H_0 : \beta = 0$ is rejected.

(ii) if $F_{1-\gamma,1,N-a-1} > F1$, then the area A_1 and also A_T the total area under the triangle $\widetilde{F1}$ are calculated. If $A_1/A_T \leq \phi$, then the null hypothesis $H_0 : \beta = 0$ is accepted. Otherwise it is rejected, where $\phi \in [0,1]$, which depends on the decision maker desire. In this paper we set $\phi = 0.3$.

In the sequel we consider testing the statistical hypotheses in (2) based on a fuzzy test statistic.

Theorem 4.4. Under the null hypothesis $H_0 : \mu_1 = \mu_2 = \ldots = \mu_a$, If we consider $(SSE^{'} - SSE)/(a-1)$ as an unbiased point estimator for parameter σ^2, then an unbiased fuzzy estimator for σ^2 is $\widetilde{\sigma^2_{H_0 2}}$ with α-cuts $\widetilde{\sigma^2_{H_0 2}}[\alpha]$, where

$$\widetilde{\sigma^2_{H_0 2}}[\alpha] = \begin{cases} \left[\frac{SSE^{'}-SSE}{\chi^2_{1-\alpha+\alpha p''',a-1}}, \frac{SSE^{'}-SSE}{\chi^2_{\alpha p''',a-1}}\right] & 0.01 \le \alpha \le 1 \\ \\ \widetilde{\sigma^2_{H_0 2}}[0.01] & 0 \le \alpha < 0.01, \end{cases}$$

and p''' is obtained from the relation $\chi^2_{p''',a-1} = a-1$.

Proof. We use pivotal quantity $(SSE^{'} - SSE)/\sigma^2$. The proof is now similar to that of Theorem 4.1. □

Theorem 4.5. The fuzzy test statistic for testing $H_0 : \mu_1 = \mu_2 = \ldots = \mu_a$ against H_1 : not all μ_i's are equal, is $\widetilde{F2}$ with α-cuts

$$\widetilde{F2}[\alpha] = \begin{cases} \left[(f2)_1(\alpha)F2,\ (f2)_2(\alpha)F2\right] & 0.01 \le \alpha \le 1 \\ \\ \widetilde{F2}[0.01] & 0 \le \alpha < 0.01, \end{cases}$$

$$(f2)_1(\alpha) = \left[(a-1)\chi^2_{\alpha p',N-a-1}\right] / \left[(N-a-1)\chi^2_{1-\alpha+\alpha p''',a-1}\right],$$

$$(f2)_2(\alpha) = \left[(a-1)\chi^2_{1-\alpha+\alpha p',N-a-1}\right] / \left[(N-a-1)\chi^2_{\alpha p''',a-1}\right].$$

Proof. Using the equality $\widetilde{F2}[\alpha] = \widetilde{\sigma^2_{H_0 2}}[\alpha]/\widetilde{\sigma^2}[\alpha]$ and interval arithmetic, fuzzy test statistic follows from Buckley's method. □

Note 4.2. Similar to Note 4.1, the critical value is a fuzzy number with α-cuts

$$\widetilde{CV}2[\alpha] = \begin{cases} \left[(f2)_1(\alpha)F_{1-\gamma,a-1,N-a-1},\ (f2)_2(\alpha)F_{1-\gamma,a-1,N-a-1}\right] & 0.01 \le \alpha \le 1 \\ \\ \widetilde{CV}2[0.01] & 0 \le \alpha < 0.01. \end{cases}$$

Decision rule 4.2. The final decision about accepting or rejecting $H_0 : \mu_1 = \mu_2 = \ldots = \mu_a$ is considered as follows. After observing data,

(i) if $F_{1-\gamma,a-1,N-a-1} \le F2$, then the hypothesis H_0 is rejected.

(ii) if $F_{1-\gamma,a-1,N-a-1} > F2$, then the area A_1 and also the total area A_T under the graph $\widetilde{F2}$ are calculated. Now, if $A_1/A_T \le \phi$ then the null hypothesis H_0 is accepted. Otherwise it is rejected, where $\phi = 0.3$.

5. Numerical Example

In this section, we analyze two data set and use mentioned method about accepting or rejecting hypothesis H_0 in hypotheses testing (1) and (2).

Example 5.1. An example is quoted from [7] that includes an experiment performed to determine if there is a difference in the breaking strength of a monofilament fiber produced by three different machines for a textile company. Clearly the strength of the fiber is also affected by its thickness. However the strength of a fiber is related to its diameter, with thicker fibers being generally stronger than thinner ones. A random sample of five fiber specimens is selected from each machine. The fiber strength (y) and the corresponding diameter (x) for each specimen are shown in Table 1, unlike the previous research ([7]), in this paper, we concider y values as the fuzzy data ($\tilde{y}$). The one-way analysis of covariance model is as follows:

$$\tilde{y}_{ij} = \mu_i + \beta(x_{ij} - \bar{x}_{..}) + \varepsilon_{ij}, \text{ for } i = 1, 2, 3 \text{ and } j = 1, 2, \ldots, 5.$$

Table 1. Breaking Strength Data ($\tilde{y}$ =strength in pounds and x = diameter in 10^{-3} inches)

Machine 1				Machine 2				Machine 3			
(y^m	y^l	y^r)	x	(y^m	y^l	y^r)	x	(y^m	y^l	y^r)	x
(36	34	37)	20	(40	39	43)	22	(35	34	36)	21
(41	40	43)	25	(48	47	49)	28	(37	35	38)	23
(39	39	42)	24	(39	36	41)	22	(42	39	45)	26
(42	41	44)	25	(45	45	45)	30	(34	31	35)	21
(49	47	50)	32	(44	42	46)	28	(32	31	36)	15

Here, we have $F1 = 60.378$ and $F2 = 2.314$. Since, $F_{0.9,1,11} = 3.23$ for $\gamma = 0.1$. So by decision rule 4.1, the null hypothesis $H_0 : \beta = 0$ is rejected.

Since $F_{0.9,2,11} = 2.86$, the values of $F_{1-\gamma,a-1,N-a-1}$ and $F2$, for $\gamma = 0.1$, are close to each other and in conventional statistics we are uncertain to accept the hypothesis $H_0 : \mu_1 = \mu_2 = \mu_3$. Therefore, we use the method presented in this paper for testing the statistical hypotheses in (2) based on a fuzzy test statistic.

We have $SSE^{'} = 143.277$ and $SSE = 100.842$. Therefore, based on Theorem 4.1 an unbiased fuzzy estimation for σ^2 is a fuzzy number with α-cuts

$$\widetilde{\sigma^2}[\alpha] = \begin{cases} \left[100.842/\chi^2_{1-\alpha+\alpha 0.557,11}, 100.842/\chi^2_{\alpha 0.557,11}\right] & 0.01 \leq \alpha \leq 1 \\ \\ \widetilde{\sigma^2}[0.01] & 0 \leq \alpha < 0.01, \end{cases}$$

and $p' = 0.557$ is obtained from the relation $\chi^2_{p',11} = 11$.

So, by Lemma 4.1, the membership function of the unbiased fuzzy estimator is given as follows:

$$\widetilde{\sigma^2}(x) = \begin{cases} \frac{1-G(100.842/x)}{1-0.557} & \frac{100.842}{\chi^2_{0.99+0.01(0.557),11}} \leq x \leq \frac{100.842}{11} \\ \\ \frac{G(100.842/x)}{0.557} & \frac{100.842}{11} \leq x \leq \frac{100.842}{\chi^2_{0.01(0.557),11}} \\ \\ 0 & \text{otherwise}, \end{cases}$$

where G is the cumulative distribution function of the distribution χ^2 with 11 degree of freedom. Also, an unbiased fuzzy estimator for σ^2 based on Theorem 4.4, under the null hypothesis $H_0 : \mu_1 = \mu_2 = \mu_3$, is a fuzzy number with α-cuts as follows:

$$\widetilde{\sigma^2_{H_0 2}}[\alpha] = \begin{cases} \left[42.435/\chi^2_{1-\alpha+\alpha 0.632,2}, 42.435/\chi^2_{\alpha 0.632,2}\right] & 0.01 \leq \alpha \leq 1 \\ \\ \widetilde{\sigma^2_{H_0 2}}[0.01] & 0 \leq \alpha < 0.01, \end{cases}$$

and $p''' = 0.632$ is obtained from the relation $\chi^2_{p''',2} = 2$.
By Theorem 4.5 and Note 4.2, the fuzzy test statistic $\widetilde{F}2$ and the fuzzy critical value $\widetilde{CV}2$, with $\gamma = 0.1$ and $F2 = 2.314$, are fuzzy numbers with the following α-cuts:

$$\widetilde{F}2[\alpha] = \begin{cases} \left[\frac{\chi^2_{\alpha 0.557,11}}{\chi^2_{1-\alpha+\alpha 0.632,2}} 0.420,\ \frac{\chi^2_{1-\alpha+\alpha 0.557,11}}{\chi^2_{\alpha 0.632,2}} 0.420\right] & 0.01 \leq \alpha \leq 1 \\ \\ \widetilde{F}2[0.01] & 0 \leq \alpha < 0.01, \end{cases}$$

and

$$\widetilde{CV}2[\alpha] = \begin{cases} \left[\frac{\chi^2_{\alpha 0.557,11}}{\chi^2_{1-\alpha+\alpha 0.632,2}} 0.52,\ \frac{\chi^2_{1-\alpha+\alpha 0.557,11}}{\chi^2_{\alpha 0.632,2}} 0.52\right] & 0.01 \leq \alpha \leq 1 \\ \\ \widetilde{CV}2[0.01] & 0 \leq \alpha < 0.01. \end{cases}$$

The intersection between the vertical line $F_{0.9,2,11} = 2.86$ and the right-hand side $\widetilde{F}2$ is obtained as a point $\alpha^* = (2.86, 0.904)$. The area $A_1 \simeq 25.852$ for

$\alpha \in [0.01, 0.904]$ and $A_T \simeq 27.596$ for $\alpha \in [0.01, 1]$(for more details see [5]). Hence, $A_1/A_T \simeq 0.936$. Since $A_1/A_T > \phi = 0.3$, the hypothesis H_0 is certainly rejected.

Example 5.2 [6] Table 5.2 presents the raw data illustrating sleep quqlity perception scores (y), according to levels of exercise frequency (x) (with three levels: frequent, infrequent and none). The table also shows data for the age of each participant this will be the covariate.In this paper, the fuzzy data ($\tilde{y}$) have been concidered as y values. The one-way analysis of covariance model is as follows:

$$\tilde{y}_{ij} = \mu_i + \beta(x_{ij} - \bar{x}..) + \varepsilon_{ij}, \text{ for } i = 1, 2, 3 \text{ and } j = 1, 2, \ldots, 10.$$

we have $F1 = 10.116$. Since, $F_{0.9,1,26} = 2.91$ for $\gamma = 0.1$. So by decision rule

Table 2. Sleep quqlity scores (SQ), in respect of exercise frequency (adjusted for age)

Frequent				Infrequent				None			
Age			SQ	Age			SQ	Age			SQ
$(y^m$	y^l	$y^r)$	x	$(y^m$	y^l	$y^r)$	x	$(y^m$	y^l	$y^r)$	x
(67	66	68)	40	(44	43	45)	18	(52	50	54)	25
(80	78	81)	47	(44	42	46)	24	(46	45	47)	14
(74	74	75)	27	(62	61	62)	31	(62	60	64)	22
(37	36	38)	27	(39	38	40)	21	(46	45	47)	22
(80	79	82)	44	(49	47	51)	28	(52	51	53)	39
(62	61	63)	30	(39	39	40)	21	(52	52	52)	34
(37	35	39)	30	(33	32	34)	24	(46	45	48)	34
(83	83	83)	49	(56	55	57)	24	(36	35	38)	25
(55	54	56)	34	(56	55	58)	37	(20	20	21)	22
(86	85	88)	37	(39	37	40)	24	(26	25	28)	25

4.1, the null hypothesis $H_0 : \beta = 0$ is rejected. Also, we have $F2 = 1.683$. Since, $F_{0.9,2,26} = 2.52$, that $F_{1-\gamma,a-1,N-a-1} > F2$, for $\gamma = 0.1$. Therefore, we use the method presented in this paper for testing the statistical hypotheses in (2) based on a fuzzy test statistic. We have $SSE^{'} = 13730.13$ and $SSE = 12156.11$. Therefore, based on Theorem 4.1 an unbiased fuzzy estimation for σ^2 is a fuzzy number with

α-cuts

$$\widetilde{\sigma^2}[\alpha] = \begin{cases} \left[12156.11/\chi^2_{1-\alpha+\alpha 0.537,26}, 12156.11/\chi^2_{\alpha 0.537,26}\right] & 0.01 \leq \alpha \leq 1 \\ \\ \widetilde{\sigma^2}[0.01] & 0 \leq \alpha < 0.01, \end{cases}$$

and $p' = 0.537$ is obtained from the relation $\chi^2_{p',26} = 26$.

So, by Lemma 4.1, the membership function of the unbiased fuzzy estimator is given as follows:

$$\widetilde{\sigma^2}(x) = \begin{cases} \frac{1-G(12156.11/x)}{1-0.537} & \frac{12156.11}{\chi^2_{0.99+0.01(0.537),26}} \leq x \leq \frac{12156.11}{26} \\ \\ \frac{G(12156.11/x)}{0.537} & \frac{12156.11}{26} \leq x \leq \frac{12156.11}{\chi^2_{0.01(0.537),26}} \\ \\ 0 & \text{otherwise}, \end{cases}$$

where G is the cumulative distribution function of the distribution χ^2 with 11 degree of freedom. Also, an unbiased fuzzy estimator for σ^2 based on Theorem 4.4, under the null hypothesis $H_0 : \mu_1 = \mu_2 = \mu_3$, is a fuzzy number with α-cuts as follows:

$$\widetilde{\sigma^2_{H_0 2}}[\alpha] = \begin{cases} \left[1574.025/\chi^2_{1-\alpha+\alpha 0.632,2}, 1574.025/\chi^2_{\alpha 0.632,2}\right] & 0.01 \leq \alpha \leq 1 \\ \\ \widetilde{\sigma^2_{H_0 2}}[0.01] & 0 \leq \alpha < 0.01, \end{cases}$$

and $p''' = 0.632$ is obtained from the relation $\chi^2_{p''',2} = 2$. By Theorem 4.5 and Note 4.2, the fuzzy test statistic $\widetilde{F2}$ and the fuzzy critical value $\widetilde{CV2}$, with $\gamma = 0.1$ and $F2 = 1.6834$, are fuzzy numbers with the following α-cuts:

$$\widetilde{F2}[\alpha] = \begin{cases} \left[\frac{\chi^2_{\alpha 0.537,26}}{\chi^2_{1-\alpha+\alpha 0.632,2}} 0.129, \ \frac{\chi^2_{1-\alpha+\alpha 0.537,26}}{\chi^2_{\alpha 0.632,2}} 0.129\right] & 0.01 \leq \alpha \leq 1 \\ \\ \widetilde{F2}[0.01] & 0 \leq \alpha < 0.01, \end{cases}$$

and

$$\widetilde{CV2}[\alpha] = \begin{cases} \left[\frac{\chi^2_{\alpha 0.537,26}}{\chi^2_{1-\alpha+\alpha 0.632,2}} 0.193, \ \frac{\chi^2_{1-\alpha+\alpha 0.537,26}}{\chi^2_{\alpha 0.632,2}} 0.193\right] & 0.01 \leq \alpha \leq 1 \\ \\ \widetilde{CV2}[0.01] & 0 \leq \alpha < 0.01. \end{cases}$$

The intersection between the vertical line $F_{0.9,2,26} = 2.52$ and the right-hand side $\widetilde{F2}$ is obtained as a point $\alpha^* = (2.56, 0.806)$. The area $A_1 \simeq 14.741$ for

$\alpha \in [0.01, 0.806]$ and $A_T \simeq 16.301$ for $\alpha \in [0.01, 1]$. Hence, $A_1/A_T \simeq 0.9043$. Since $A_1/A_T > \phi = 0.3$, the hypothesis H_0 is certainly rejected.

6. Conclusions

In this paper, by using D_{LR} metric, Buckley's method is applied to a one-way analysis of covariance and used for testing the statistical hypotheses when there is an uncertainty in accepting or rejecting the hypotheses. This method can be used for other linear models, and an interesting topic for research is the study of this method on one-way analysis of covariance when the hypotheses are fuzzy. An interesting topic for further research to examine the performance of the method presented in this study to test the hypothesis test using fuzzy linear compared to conventional methods.

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