ON AN INDEPENDENT RESULT USING ORDER STATISTICS AND THEIR CONCOMITANT

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Abstract. Let $X_1, X_2, ..., X_n$ have a jointly multivariate exchangeable normal distribution. In this work we investigate another proof of the independence of $\bar{X}$ and S^2 using order statistics. We also assume that $(X_i\ , Y_i),\ i = 1, 2, ..., n$, jointly distributed in bivariate normal and establish the independence of the mean and the variance of concomitants of order statistics.

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1. Introduction

The skew normal distribution was introduced by Azzalini (1985) for modeling normal data in the presence of a skewness parameter. The random variable Y has a skew normal distribution if

$$f_Y(y) = 2\varphi\left(y;\mu,\sigma^2\right)\Phi\left(\lambda\frac{y-\mu}{\sigma}\right), \qquad y \in \mathbb{R} \tag{1}$$

where $\varphi\left(.;\mu,\sigma^2\right)$ is the normal density with mean and variance μ, σ^2 respectively and $\Phi(.)$ denotes the standard normal distribution function.

Following Arellano-Valle and Azzalini (2006), we say that a d-dimensional random vector $\mathbf{Y}$ has a unified multivariate skew-normal distribution, denoted by $\mathbf{Y} \sim SUN_{d,m}\left(\boldsymbol{\xi},\ \boldsymbol{\delta},\ \boldsymbol{\Omega},\ \boldsymbol{\Gamma},\ \boldsymbol{\Lambda}\right)$, if it has a density function of the form

$$f_{\mathbf{Y}}\left(\mathbf{y}\right) = \phi_d\left(\mathbf{y}-\boldsymbol{\xi};\,\boldsymbol{\Omega}\right)\frac{\Phi_m\left(\boldsymbol{\delta}+\boldsymbol{\Lambda}^T\boldsymbol{\Omega}^{-1}\left(\mathbf{y}-\boldsymbol{\xi}\right);\boldsymbol{\Gamma}-\boldsymbol{\Lambda}^T\boldsymbol{\Omega}^{-1}\boldsymbol{\Lambda}\right)}{\Phi_m\left(\boldsymbol{\delta};\,\boldsymbol{\Gamma}\right)}, \qquad \mathbf{y} \in \mathbb{R}^d \tag{2}$$

where $\varphi_d(.,\ \boldsymbol{\xi},\ \boldsymbol{\Omega})$ is the density function of a multivariate normal and $\Phi_m(.;\ \boldsymbol{\Sigma})$ is the multivariate normal cumulative function with the covariance matrix $\boldsymbol{\Sigma}$. For more information about univariate and multivariate skew normals, one may refer to Azzalini and Dalla Valle (1996), Azzalini (2005), Genton (2004), etc.

Another case of the unified skew normal distribution is the singular unified skew normal distribution which defines as follow:

We say that the random vector $\mathbf{Y}$ has a singular unified skew normal distribution and write $\mathbf{Y} \sim SSUN_{d,m}(\boldsymbol{\xi},\ \boldsymbol{\delta}, \boldsymbol{\Omega},\ \boldsymbol{\Gamma},\ \boldsymbol{\Lambda})$, if both the matrices $\boldsymbol{\Gamma}$ and $\boldsymbol{\Omega}$ are of full ranks ($rank(\boldsymbol{\Gamma})$ $=m$ and $rank(\boldsymbol{\Omega})$ $=d$) but $\Sigma^* = \begin{pmatrix} \boldsymbol{\Gamma} & \boldsymbol{\Lambda}^T \\ \boldsymbol{\Lambda} & \boldsymbol{\Omega} \end{pmatrix}$ is a singular matrix with $rank(\Sigma^*)$ $<m+d$. Arellano-Valle and Azzalini (2006) introduced three types of a singular skew normal distribution. In our definition the distribution of $\mathbf{Y}$ is, for all practical purposes, again of type (2) of Arellano-Valle and Azzalini (2006). For more details, see Sheikhi and Tata (2013).

Now, let $\mathbf{X}$ be a random vector in $\mathbb{R}^n$ such that

$$\mathbf{X} \sim N_n\left(\mu\mathbf{1}_n,\ \boldsymbol{\Sigma} = \sigma^2\left\{(1-\rho)\,\mathbf{I}_n + \rho\mathbf{1}_n\mathbf{1}_n^T\right\}\right), \tag{3}$$

where ρ is the correlation coefficient between any two components of $\mathbf{X}$, $\mathbf{1}_n = (1,...,1)^T$ has n components and $\mathbf{I}_n$ is the identity matrix of dimension n. In other words, the random vector $\mathbf{X}$ has an exchangeable multivariate normal distribution.

The aim of this article is to give a new proof of the independence between the mean and variance of random variables $X_1, X_2, ..., X_n$ when the corresponding joint distribution is a multivariate normal exchangeable. The independence of the sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_{i:}$ and the sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$when $(X_1, X_2, ..., X_n)$ is a random sample coming from a normal distribution is well known; e.g., Basu (1955). Arnold (1973) has discussed the independence of squared order statistics and in connection with the skew normal distribution, Azzalini and Capitanio (1999) have used the properties of skew normal distributions to prove the Cochran theorem and Gupta and Huang (2002) have proved the independence of linear and quadratic forms of skew normal variables. Bathachria (1974) has established that given the value of order statistics, the concomitant variables are independent, while Suresh (1993) has shown that the central concomitants and extreme concomitants are asymptotically independent. In 2002, Loperfido showed that ordered statistics and skew normal distributions are intimately connected. In this paper we use this connection to present a new proof of the independence of $\bar{X}$ and S^2.

The paper is structured as follows. In Section 2 we prove the independence of the mean and the variance of exchangeable normal variables and then generalize our proof to include a normal matrix in Section 3. Section 4 is devoted to the proof of the independence of mean and variance in concomitant variables.

2. Main results

The following lemma is similar to Proposition 6 of Azzalini and Capitanio (1999).

Lemma 1: *If* $\mathbf{Y} \sim SSUN_{d,m}(\boldsymbol{\xi}, \boldsymbol{\delta}, \boldsymbol{\Gamma}, \boldsymbol{\Lambda}, \boldsymbol{\Omega})$ *where* $\boldsymbol{\Omega} = (\omega_{ij}), i, j = 1, ..., d$ *and* $\boldsymbol{\Lambda} = (\lambda_1^T, ..., \lambda_d^T)^T$, *Then the k-th component of* $\mathbf{Y}$ *is independent of the other components if the following two conditions hold:*

1) $\omega_{kj} = 0 \quad \forall j \neq k$,

2) $\lambda_k = \mathbf{0}$.

Proof: Let

$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_{-k} \\ y_k \end{pmatrix}, \ \boldsymbol{\xi} = \begin{pmatrix} \xi \mathbf{1}_{k-1} \\ \xi \end{pmatrix}, \ \boldsymbol{\Omega} = \begin{pmatrix} \boldsymbol{\Omega}_{-k-k} & \mathbf{0}^T \\ \mathbf{0} & \omega_{kk} \end{pmatrix}, \boldsymbol{\Lambda} = (\boldsymbol{\Lambda}_{-k}, \mathbf{0})^T,$$

where $\mathbf{y}_{-k}$ is the vector $\mathbf{y}$ which its kth component is removed and $\boldsymbol{\Omega}_{-k-k}$ is the covariance matrix of $\mathbf{y}_{-k}$. If the conditions (1) and (2) hold, we have

$$\varphi_d\left(\mathbf{y} - \boldsymbol{\xi}; \boldsymbol{\Omega}\right) = \varphi\left(y_k - \xi; \sigma^2\right)\varphi_{d-1}\left(\mathbf{y}_{-k} - \boldsymbol{\xi}_{-k}; \boldsymbol{\Omega}_{-k}\right)$$

and

$$\begin{aligned}
&\Phi_m\left(\boldsymbol{\delta}+\boldsymbol{\Lambda}^T\boldsymbol{\Omega}^{-1}\left(\mathbf{y}-\boldsymbol{\xi}\right);\boldsymbol{\Gamma}-\boldsymbol{\Lambda}^T\boldsymbol{\Omega}^{-1}\boldsymbol{\Lambda}\right)\\
&\qquad = \Phi_m\left(\boldsymbol{\delta}+\boldsymbol{\Lambda}_{-k}^T\boldsymbol{\Omega}_{-k}^{-1}\left(\mathbf{y}-\boldsymbol{\xi}\right);\boldsymbol{\Gamma}-\boldsymbol{\Lambda}_{-k}^T\boldsymbol{\Omega}_{-k}^{-1}\boldsymbol{\Lambda}_{-k}\right)\\
&\qquad = \Phi_m\left(\boldsymbol{\delta}+\boldsymbol{\Lambda}_{-k}^T\boldsymbol{\Omega}_{-k}^{-1}\left(\mathbf{y}_{-k}-\boldsymbol{\xi}_{-k}\right);\boldsymbol{\Gamma}-\boldsymbol{\Lambda}_{-k}^T\boldsymbol{\Omega}_{-k}^{-1}\boldsymbol{\Lambda}_{-k}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
f_{\mathbf{Y}}\left(\mathbf{y}\right) &= \varphi\left(y_k-\xi;\,\sigma^2\right)\varphi_{d-1}\left(\mathbf{y}_{-k}-\boldsymbol{\xi}_{-k};\,\boldsymbol{\Omega}_{-k}\right)\\
&\quad\times\frac{\Phi_m\left(\boldsymbol{\delta}+\boldsymbol{\Lambda}_{-k}^T\boldsymbol{\Omega}_{-k}^{-1}\left(\mathbf{y}_{-k}-\boldsymbol{\xi}_{-k}\right);\boldsymbol{\Gamma}-\boldsymbol{\Lambda}_{-k}^T\boldsymbol{\Omega}_{-k}^{-1}\boldsymbol{\Lambda}_{-k}\right)}{\Phi_m\left(\boldsymbol{\delta};\,\boldsymbol{\Gamma}\right)}.
\end{aligned}$$

which implies the assertion. ■

Suppose $S(\mathbf{X})=\{\mathbf{X}^{(i)}=\mathbf{P}_i\mathbf{X};i=1,2,...,n!\}$ where $\mathbf{P}_i$ is an $n\times n$ permutation matrix. Let $\boldsymbol{\Delta}$ be the difference matrix of dimension $(2n-3)\times n$ such that its first $n-1$ rows are of the form $\mathbf{e}_{n,i}^T-\mathbf{e}_{n,1}^T$, $i=2,...,n$, and its last $n-2$ rows are of the form $\mathbf{e}_{n,i}^T-\mathbf{e}_{n,2}^T$, $i=3,...,n$ where $\mathbf{e}_{n,1},\mathbf{e}_{n,2},...,\mathbf{e}_{n,n}$ are the n-dimensional standard unit vectors. Then $\boldsymbol{\Delta}\mathbf{X}=(X_2-X_1,\,X_3-X_1,...,X_n-X_1,\,X_3-X_2,X_4-X_2...,X_n-X_2)^T$.

We have the following lemma.

Lemma 2: The sample variance of a random sample $X_1,...,X_n$ can be written as $S^2=\frac{1}{2n(n-1)}\sum_{i=1}^n\sum_{j=1}^n(X_{i:n}-X_{j:n})^2$.

Proof: It is well known that $S^2=\frac{1}{n}\sum_{i=1}^n(X_i-\bar{X})^2=\frac{1}{2n(n-1)}\sum_{i=1}^n\sum_{j=1}^n(X_i-X_j)^2$. Since this relation considers all differnces of X_is, we can write
$S^2=\frac{1}{2n(n-1)}\sum_{i=1}^n\sum_{j=1}^n(X_{i:n}-X_{j:n})^2$. ■

Theorem 1: *Let the random vector $\mathbf{X}=(X_1,...,X_n)^T$ have a multivariate exchangeable normal distribution, then $\bar{X}$ and S^2 are independent.*

Proof*:* Since $\bar{X}=\frac{1}{n}\sum_{i=1}^n X_{i:n}$ and using Lemma 2, it is sufficient to show that $\sum_{i=1}^n X_{i:n}$ and $X_{i:n}-X_{j:n}$ are independent for all i and j and $i\neq j$. Without loss of generality we assume $i=1$ and $j=2$ and the proof reduces to showing that $\sum_{i=1}^n X_{i:n}$ is independent of $X_{1:n}-X_{2:n}$. The proof for each i and j follows exactly in the same manner.

Let $\mathbf{a}=(1,1,...,1)^T$ and $\mathbf{b}=(1,-1,0,0,...,0)^T$ be two n-dimensional vectors and $\mathbf{X}_{(n)}=(X_{1:n},\;X_{2:n},...,X_{n:n})^T$ denotes the vector of order statistics of X_i's, then $\mathbf{a}^T\mathbf{X}_{(n)} = \sum_{i=1}^{n} X_{i:n}$ and $\mathbf{b}^T\mathbf{X}_{(n)} = X_{1:n} - X_{2:n}$. The joint distribution of $\mathbf{a}^T\mathbf{X}_{(n)}$ and $\mathbf{b}^T\mathbf{X}_{(n)}$ is

$$F_{\mathbf{a}^T\mathbf{X}_{(n)},\mathbf{b}^T\mathbf{X}_{(n)}}(y_1, y_2) = P\left(\mathbf{a}^T\mathbf{X}_{(n)} \le y_1, \mathbf{b}^T\mathbf{X}_{(n)} \le y_2\right)$$

$$= \sum_{i=1}^{n!} P\left(\mathbf{a}^T\mathbf{X}^{(i)} \le y_1, \mathbf{b}^T\mathbf{X}^{(i)} \le y_2 | \mathbf{\Delta X}^{(i)} \ge \mathbf{0}\right) P\left(\mathbf{\Delta X}^{(i)} \ge \mathbf{0}\right),$$

where $\mathbf{X}^{(i)}$ is the i-th permutation of $\mathbf{X}$. Since $P\left(\mathbf{\Delta X}^{(i)} \ge \mathbf{0}\right) = (n!)^{-1}$, $i = 1, 2, ..., n!$, by exchangeability

$$F_{\mathbf{a}^T\mathbf{X}_{(n)},\mathbf{b}^T\mathbf{X}_{(n)}}(y_1, y_2) = P\left(\mathbf{a}^T\mathbf{X} \le y_1, \mathbf{b}^T\mathbf{X} \le y_2 | \mathbf{\Delta X} \ge \mathbf{0}\right).$$

Moreover

$$\begin{pmatrix} \mathbf{\Delta X} \\ \mathbf{a}^T\mathbf{X} \\ \mathbf{b}^T\mathbf{X} \end{pmatrix} \sim N_{2n-1}\left(\begin{pmatrix} \mathbf{0} \\ n\mu \\ 0 \end{pmatrix}, \begin{pmatrix} \mathbf{\Delta \Sigma \Delta}^T & \mathbf{\Delta \Sigma a} & \mathbf{\Delta \Sigma b} \\ & \mathbf{a}^T\mathbf{\Sigma a} & \mathbf{a}^T\mathbf{\Sigma b} \\ & & \mathbf{b}^T\mathbf{\Sigma b} \end{pmatrix}\right).$$

Since both $\mathbf{\Delta X}$ and $(\mathbf{a}^T\mathbf{X},\ \mathbf{b}^T\mathbf{X})^T$ are of full rank but $\begin{pmatrix} \mathbf{\Delta X} \\ \mathbf{a}^T\mathbf{X} \\ \mathbf{b}^T\mathbf{X} \end{pmatrix}$ is not, according to case 3 of Arellano-Valle and Azzalini (2006) we conclude that $(\mathbf{a}^T\mathbf{X},\ \mathbf{b}^T\mathbf{X})^T | \mathbf{\Delta X} > \mathbf{0} \sim SSUN_{2,2n-3}(\boldsymbol{\xi}\ , \mathbf{0},\ \mathbf{\Omega},\ \mathbf{\Delta \Sigma \Delta^T},\ \mathbf{\Lambda})$ where

$$\boldsymbol{\xi} = \begin{pmatrix} n\mu \\ 0 \end{pmatrix},\ \mathbf{\Omega} = \begin{pmatrix} \mathbf{a}^T\mathbf{\Sigma a} & \mathbf{a}^T\mathbf{\Sigma b} \\ & \mathbf{b}^T\mathbf{\Sigma b} \end{pmatrix},\ \mathbf{\Lambda} = \begin{pmatrix} \mathbf{\Delta \Sigma a} \\ \mathbf{\Delta \Sigma b} \end{pmatrix}.$$

We easily obtain $\mathbf{a}^T\mathbf{\Sigma b}$ =0 and $\mathbf{\Delta \Sigma a} = \mathbf{0}_{2n-3}$. Hence Lemma 1 finishes the proof.

The following two corollaries are now a direct consequence of Theorem 1.

Corollary 1. *Let the random vector* $\mathbf{X} = (X_1, X_2, ..., X_n)^T$ *have a multivariate exchangeable normal distribution, then* $\bar{X}$ *and range* $W = X_{(n)} - X_{(1)}$ *are independent.*

Corollary 2. *If* $X_1, X_2, ..., X_n$ *is a random sample arising form* $N\left(\mu, \sigma^2\right)$, *then* $\bar{X}$ *and* S^2 *are independent.*

3. The general case

We can generalize the results of the previous section to the multivariate case. Let $\mathbf{X} = (\mathbf{X}_1^T, \mathbf{X}_2^T, ..., \mathbf{X}_n^T)^T$ be an $n \times p$ matrix with a matrix normal distribution.

We write $\mathbf{X} \sim \ N_{n\times p}\,(\mathbf{M},\ \mathbf{\Sigma},\mathbf{\Xi})$ where $\mathbf{M}$ is an $n\times p$ matrix of means, $\mathbf{\Xi}$ is $p\times p$ matrix of column covariances and $\mathbf{\Sigma}$ is $n\times n$ matrix of row covariances. We assume that the random matrix $\mathbf{X}$ has an exchangeable matrix normal distribution with $\mathbf{M} = \mu[\mathbf{1}_n\vdots\mathbf{1}_n\vdots...\vdots\mathbf{1}_n]$ and $\mathbf{\Sigma} = \sigma^2\{(1-\rho)\mathbf{I}_n + \rho\mathbf{1}_n\mathbf{1}_n^T\}$. We write $vec(\mathbf{X}) \sim N_{np}\,(vec(\mathbf{M}),\ \mathbf{\Sigma}\otimes\mathbf{\Xi}\,)$ where $\otimes$ denotes the Kronecker product.

The sample mean and the sample covariance matrix of the vectors $\mathbf{X}_1,\mathbf{X}_2,...,\mathbf{X}_n$ are, respectively, $\bar{\mathbf{X}} = \frac{1}{n}\sum_{i=1}^n \mathbf{X}_i$ and $\mathbf{S} = \frac{1}{n-1}\sum_{i=1}^n(\mathbf{X}_i-\bar{\mathbf{X}})(\mathbf{X}_i-\bar{\mathbf{X}})^T$. We now order the random vectors $\mathbf{X}_1,\mathbf{X}_2,...,\mathbf{X}_n$ using multivariate ordering (See e.g. Balakrishnan, 1993) and denote $\mathbb{X}_{(n)} = (\mathbf{X}_{1:n},\mathbf{X}_{2:n},...,\mathbf{X}_{n:n})^T$. Then similar to Lemma 2, we have $\bar{\mathbf{X}} = \frac{1}{n}\sum_{i=1}^n \mathbf{X}_{i:n}$ and $\mathbf{S} = \frac{1}{2n(n-1)}\sum_{i=1}^n\sum_{j=1}^n(\mathbf{X}_{i:n}-\mathbf{X}_{j:n})(\mathbf{X}_{i:n}-\mathbf{X}_{j:n})^T$.

Theorem 2 : *If the random matrix* $\mathbf{X}^T = (\mathbf{X}_1,\mathbf{X}_2,...,\mathbf{X}_n)$ *has a matrix exchangeable normal distribution, then* $\bar{\mathbf{X}}$ *and* $\mathbf{S}^2$ *are independent.*

Proof: Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{\Delta}$ be as in Theorem 1. Then

$$F_{\mathbf{a}^T\mathbf{X}_{(n)},\mathbf{b}^T\mathbf{X}_{(n)}}\,(\mathbf{y}_1,\,\mathbf{y}_2) \;=\; P\left(\mathbf{a}^T\mathbf{X}\le\mathbf{y}_1,\mathbf{b}^T\mathbf{X}\le\mathbf{y}_2|\mathbf{\Delta X}\ge\mathbf{0}\right)$$

$$P\left(\mathbf{a}^T\mathbf{X}\le\mathbf{y}_1,\mathbf{b}^T\mathbf{X}\le\mathbf{y}_2|\mathbf{\Delta X}_1\ge\mathbf{0},\,\mathbf{\Delta X}_2\ge\mathbf{0},\ ...,\mathbf{\Delta X}_p\ge\mathbf{0}\right),$$

where $\mathbf{y}_1$ and $\mathbf{y}_2$ are p-dimensional vectors. Also,

$$\begin{pmatrix}\mathbf{\Delta X}\\ \mathbf{a}^T\mathbf{X}\\ \mathbf{b}^T\mathbf{X}\end{pmatrix}\sim N\left(\begin{pmatrix}\mathbf{0}\\ \mathbf{a}^T\boldsymbol{\mu}\\ \mathbf{b}^T\boldsymbol{\mu}\end{pmatrix},\begin{pmatrix}\mathbf{\Delta\Sigma}\otimes\mathbf{\Xi\Delta}^T & \mathbf{\Delta\Sigma}\otimes\mathbf{\Xi a} & \mathbf{\Delta\Sigma}\otimes\mathbf{\Xi b}\\ & \mathbf{a}^T\mathbf{\Sigma}\otimes\mathbf{\Xi a} & \mathbf{a}^T\mathbf{\Sigma}\otimes\mathbf{\Xi b}\\ & & \mathbf{b}^T\mathbf{\Sigma}\otimes\mathbf{\Xi b}\end{pmatrix}\right).$$

Now, $(\mathbf{a}^T\mathbf{X},\ \mathbf{b}^T\mathbf{X})^T|\mathbf{\Delta X}>\mathbf{0}\sim SSUN_{2p,(2n-3)p}(\boldsymbol{\xi}\,,\mathbf{0},\ \mathbf{\Omega},\ \mathbf{\Delta\Sigma}\otimes\mathbf{\Xi\Delta},\ \mathbf{\Lambda})$ where

$$\boldsymbol{\xi} = \begin{pmatrix} n\mu\mathbf{1}_p\\ \mathbf{0}_p\end{pmatrix},\ \mathbf{\Omega} = \begin{pmatrix}\mathbf{a}^T\mathbf{\Sigma}\otimes\mathbf{\Xi a} & \mathbf{a}^T\mathbf{\Sigma}\otimes\mathbf{\Xi b}\\ & \mathbf{b}^T\mathbf{\Sigma}\otimes\mathbf{\Xi b}\end{pmatrix}\text{ and }\mathbf{\Lambda} = \begin{pmatrix}\mathbf{\Delta\Sigma}\otimes\mathbf{\Xi a}\\ \mathbf{\Delta\Sigma}\otimes\mathbf{\Xi b}\end{pmatrix},$$

and since $\mathbf{a}^T\mathbf{\Sigma}\otimes\mathbf{\Xi b} = \mathbf{0}$ and $\mathbf{\Delta\Sigma}\otimes\mathbf{\Xi a} = \mathbf{0}$, the proof is completed. ■

Corollary 3: If $\mathbf{X}_1,\mathbf{X}_2,...,\mathbf{X}_n$ is a random sample from $N_p\,(\boldsymbol{\mu},\mathbf{\Sigma})$, then $\bar{\mathbf{X}}$ and $\mathbf{S}$ are independent.

Proof: The proof follows from Theorem 2 with
$vec(\mathbf{X}^T) \sim N_{np}\left(vec(\mathbf{M}^T), \mathbf{I}_n \otimes \mathbf{\Xi}\right)$ where $\mathbf{I}_n$ is an identity matrix.

4. Independence in concomitants

Suppose that $(X_i\ ,Y_i),\ i = 1, ..., n,$ follow the bivariate normal distribution and the joint distribution of two random vectors $\mathbf{X}$ and $\mathbf{Y}$ follows a $2n$ dimensional exchangeable multivariate normal, that is

$$\left(\begin{array}{c}\mathbf{X}\\ \mathbf{Y}\end{array}\right) \sim N_{2n}\left(\boldsymbol{\mu} = \left(\begin{array}{c}\boldsymbol{\mu}_{\mathbf{x}}\\ \boldsymbol{\mu}_{\mathbf{y}}\end{array}\right),\ \mathbf{\Sigma} = \left(\begin{array}{cc}\mathbf{\Sigma}_{xx} & \mathbf{\Sigma}_{yx}\\ \mathbf{\Sigma}_{xy} & \mathbf{\Sigma}_{yy}\end{array}\right)\right), \tag{4}$$

where

$\boldsymbol{\mu}_{\mathbf{x}} = \mu_x \mathbf{1}_n,\ \boldsymbol{\mu}_{\mathbf{y}} = \mu_y \mathbf{1}_n,$
$\mathbf{\Sigma}_{xx} = \sigma_x^2\left[\rho_x \mathbf{1}_n \mathbf{1}_n^T + (1-\rho_x)\, I_n\right],$
$\mathbf{\Sigma}_{yy} = \sigma_y^2\left[\rho_y \mathbf{1}_n \mathbf{1}_n^T + (1-\rho_y)\, I_n\right]$ and $\mathbf{\Sigma}_{xy} = \rho_{xy}\sigma_x\sigma_y\mathbf{J}$ with $\mathbf{1}_n = (1, ..., 1)^T$ and $\mathbf{J} = [1]_{n\times n}$.

Let $\mathbf{X}_{(n)} = (X_{1:n},\ X_{2:n}, ..., X_{n:n})^T$ be the vector of ordered statistics obtained from $\mathbf{X}_{n\times 1}$ and $\mathbf{Y}_{[n]} = (Y_{[1:n]},\ Y_{[2:n]}, ..., Y_{[n:n]})^T$ be the corresponding vector of concomitants. For example, $\mathbf{X}$ and $\mathbf{Y}$ may be observations on each of the two eyes, ears, lungs, ... of a person. We now denote the mean of concomitant as $\bar{Y}_{[n]} = \frac{1}{n}\sum_{i=1}^{n} Y_{[i:n]}$ and their variance as $S^2_{Y_{[n]}} = \frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}(Y_{[i:n]} - Y_{[j:n]})^2$.

Theorem 3: *Under Assumption (4), the random variables $\bar{Y}_{[n]}$ and $S^2_{Y_{[n]}}$ are independent.*

Proof: Again let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{\Delta}$ be as in Theorem 1 and it suffices to show that $\sum_{i=1}^{n} Y_{[i:n]}$ and $Y_{[1:n]} - Y_{[2:n]}$ are independent.
The joint distribution of $\mathbf{a}^T\mathbf{Y}_{[n]}$ and $\mathbf{b}^T\mathbf{Y}_{[n]}$ is

$$F_{\mathbf{a}^T\mathbf{Y}_{[n]},\mathbf{b}^T\mathbf{Y}_{[n]}}(y_1,\, y_2) = P\left(\mathbf{a}^T\mathbf{Y}_{[n]} \le y_1, \mathbf{b}^T\mathbf{Y}_{[n]} \le\ y_2\right)$$
$$= \sum_{i=1}^{n!} P\left(\mathbf{a}^T\mathbf{Y}^{(i)} \le y_1, \mathbf{b}^T\mathbf{Y}^{(i)} \le y_2 | \mathbf{\Delta}\mathbf{Y}^{(i)} \ge \mathbf{0}\right) P\left(\mathbf{\Delta}\mathbf{Y}^{(i)} \ge \mathbf{0}\right),$$

where $\mathbf{Y}^{(i)}$ stands for the i-th permutation of $\mathbf{Y}$. The last equality comes from the theorem of total probability (see e.g., Sheikhi and Tata, 2013). By exchangeability,

$F_{\mathbf{a}^T\mathbf{Y}_{[n]},\mathbf{b}^T\mathbf{Y}_{[n]}}(y_1,\, y_2) = P\left(\mathbf{a}^T\mathbf{Y} \le y_1, \mathbf{b}^T\mathbf{Y} \le y_2 | \mathbf{\Delta}\mathbf{Y} \ge \mathbf{0}\right).$

Moreover

$$\begin{pmatrix} \mathbf{\Delta X} \\ \mathbf{a}^T\mathbf{Y} \\ \mathbf{b}^T\mathbf{Y} \end{pmatrix} \sim N_{2n-1}\left(\begin{pmatrix} \mathbf{0} \\ n\mu_y \\ 0 \end{pmatrix}, \begin{pmatrix} \mathbf{\Delta\Sigma}_{xx}\mathbf{\Delta}^T & \mathbf{\Delta\Sigma}_{xy}\mathbf{a} & \mathbf{\Delta\Sigma}_{xy}\mathbf{b} \\ & \mathbf{a}^T\mathbf{\Sigma}_{yy}\mathbf{a} & \mathbf{a}^T\mathbf{\Sigma}_{yy}\mathbf{b} \\ & & \mathbf{b}^T\mathbf{\Sigma}_{yy}\mathbf{b} \end{pmatrix} \right).$$

Since $(\mathbf{\Delta X},\ \mathbf{a}^T\mathbf{Y},\ \mathbf{b}^T\mathbf{Y})^T$ is of full matrix, we conclude that $(\mathbf{a}^T\mathbf{Y},\ \mathbf{b}^T\mathbf{Y})^T|\mathbf{\Delta X} > \mathbf{0}$ $\sim SUN_{2,2n-3}(\boldsymbol{\xi}\ ,\mathbf{0},\ \mathbf{\Omega},\ \mathbf{\Delta\Sigma}_{xx}\mathbf{\Delta^T},\ \mathbf{\Lambda})$ where

$$\boldsymbol{\xi} = \begin{pmatrix} n\mu_y \\ 0 \end{pmatrix}\ \mathbf{\Omega} = \begin{pmatrix} \mathbf{a}^T\mathbf{\Sigma}_{yy}\mathbf{a} & \mathbf{a}^T\mathbf{\Sigma}_{yy}\mathbf{b} \\ & \mathbf{b}^T\mathbf{\Sigma}_{yy}\mathbf{b} \end{pmatrix}, \mathbf{\Lambda} = \begin{pmatrix} \mathbf{\Delta\Sigma}_{xy}\mathbf{a} \\ \mathbf{\Delta\Sigma}_{xy}\mathbf{b} \end{pmatrix}.$$

We readily obtain $\mathbf{a}^T\mathbf{\Sigma}_{yy}\mathbf{b}$ =0 and $\mathbf{\Delta\Sigma_{xy}a} = \mathbf{0}_{2n-3}$ and the proof is completed using Proposition 6 of Azzalini and Capitanio (2001).■

Corollary 4: If $(X_i\ ,Y_i),\ i = 1,...,n$ is a random sample form a bivariate normal distribution then $\bar{Y}_{[n]}$ and $S^2_{Y_{[n]}}$ are independent.

To illustrate our results using a numerical analysis, we perform a simulation study with 10000 iterations. We draw a paired sample of size 100 from the bivariate normal $N_2(\mu_x = 32,\ \mu_y = 32,\ \sigma^2_x = 3,\ \sigma^2_y = 3,\ \rho_{xy} = 0.5)$ and repeat this sampling 10000 times. In each paired sample, after ordering pairs based on their x values, we compute the mean value and the variance value of concomitant observations and then round their values. The result of the chi-square test, between these two statistics in 10000 iteration, showed that the mean and the variance of concomitants are independent with p-value=1.000. Figure 1 depict the scatter plot between $\bar{Y}_{[n]}$ and $S^2_{Y_{[n]}}$ before rounding their values.

5. Conclusion

In this work we use the relation between order statistics and the skew normal distribution to prove a familiar independence between the sample mean and the sample variance in an exchangeable normal family. We also extend our results to prove the independence of these two statistics in concomitants of order statistics. These results may be useful in statistical inference especially in constructing a hypothesis test to address some hypothesis about the mean of concomitants of order statistics. Also this results can be used to perform a statistical inference in a Judgment Post-Stratication analysis.

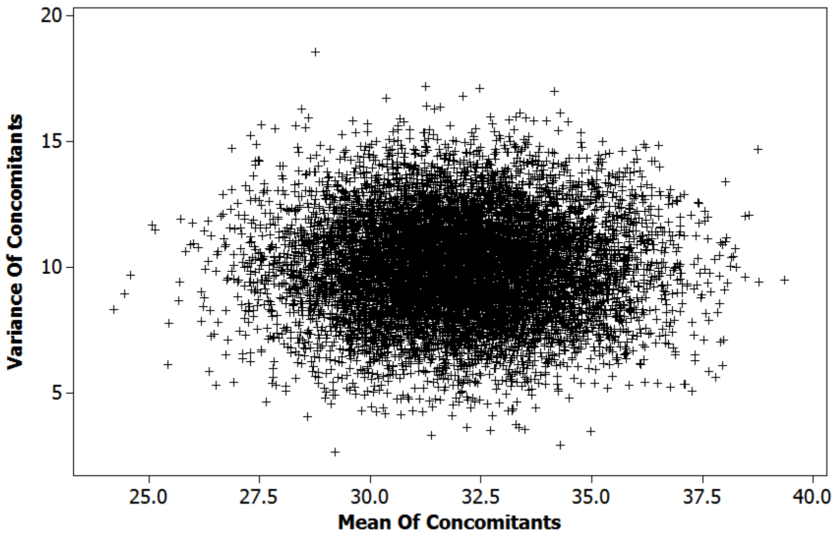

Figure 1. Scatter plot of the mean and the variance of concomitants

References

[1] Arellano-Valle R. B., Azzalini A. (2006). On the Unification of Families of skew normal Distributions. *Scand. J. Statist.* **33**, 561–574.

[2] Arnold, B. C. (1973). Independence of squared order statistics. *Comm. Statist.*, **2**, 357-362.

[3] Azzalini, A., Capitanio, A. (2001). Statistical applications of the multivariate skew normal distribution. *J. Roy. Statist. Soc. Ser. B* **61**, 579-602.

[4] Azzalini, A. (1985). A class of distributions which includes the normal ones. *Scand. J. Statist.* **12**, 171-178.

[5] Azzalini, A., Dalla Valle, A. (1996). The multivariate skew-normal distribution. *Biometrika.* **83**, 715–726.

[6] Balakrishnan, N. (1993). Multivariate normal distribution and multivariate order statistics induced by ordering linear combinations. *Stat. Prob. Lett.* **17**, 343-350.

[7] Basu, D. (1955). On statistics independent of a complete sufficient statistic. Sankhya, **15**, 377–80.

[8] Genton, M.G. (2004). *Skew-Elliptical Distributions and Their Applications: A Journey Beyond Normality*, Edited Volume, Chapman & Hall/CRC, Boca Raton, FL.

[9] Gupta, A. K., Huang, W. J. (2002). Quadratic forms in skew normal variates. *J. Math. Anal. Appl.* **273**, 558–564.

[10] Loperfido, N. (2002). Statistical implications of selectively reported inferential results. *Stat. Prob. Lett.* **56**, 13–22.

[11] Sheikhi, A., Tata, M. N. (2013). The exact joint distribution of concomitants of order statistics and their order statistics under normality. *Revstat*, **11(2)**, 121–134.

[12] Suresh, R. P. (1993). Some asymptotic results for the induced percentile selection differential. *Sankhya*-A **55**, 120-129.