THE BEHAVIOR OF NON-IDENTICAL AND DEPENDENT STANDBY COMPONENTS IN A COHERENT SYSTEM

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Abstract. In this paper, a coherent system consisting of non-identical dependent active components and equipped with non-identical dependent standby components is considered. The main object of this study is the random quantity which account the number of surviving standby components when the system is failed. We represent the distribution function of the corresponding random variable in terms of system signature.

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1. Introduction

A commonly used technique to increase the reliability or the availability of a system is a redundant standby. A hot standby has the same failure rate as the active components of the system, while a warm standby has a failure rate larger

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than an active component. A cold standby means that the redundant component is inactive, and the likelihood of failure is assumed to be zero.

Recently, Eryilmaz [3] studied the reliability of a k-out-of-n system equipped with a single warm standby component. He carried out a reliability analysis for a system equipped with a single warm standby unit and obtain an explicit expression for the reliability function of the system for arbitrary lifetime distributions. The behavior of warm standby components with respect to a coherent system was considered by Eryilmaz [2]. In particular, he studied the number of surviving warm standby components at the time of system failure in terms of system signature and discussed its potential utilization with a certain optimization problem, when the system has identical components with identical components. We shall to consider a coherent system with non-identical components equipped with non-identical redundant standby. Because of non-identical active components and standby redundancies, we may assume that the standby components are either warm or hot. We also consider the possible dependencies between the active components or between standby redundancies using an Archimedean copula which is a very convenient subclass of copulas and has a close connection to Laplace transforms. For a thorough discussion on Archimedean copulas, one may refer to Joe [5] and Nelsen [7]. If the active components of a system or standby redundancies are independent their lifetimes have a joint distribution function that is the product of marginal distribution functions. Some copulas belonging to the class of Archimedean copulas reduce to a multivariate distribution function with product of marginal distribution functions for certain values of dependent parameter. Therefore, the results of this paper reduce to corresponding results of [3] and [2] in special cases and hence may be used for a system with dependent as well as independent components.

To be specific, we assume that the n underlying variables are jointly distributed according to an Archimedean copula. If a copula C_ψ has the form

$$C_\psi(u_1, ..., u_n) = \psi\left(\sum_{i=1}^{n} \psi^{-1}(u_i)\right), \tag{1}$$

where $\psi : \Re_+ \mapsto [0,1]$ is a completely monotone generator function, i.e. $(-1)^n \psi^{(n)} \geq 0$, $n \geq 2$, such that $\psi(0) = 1$ and $\lim_{x\to\infty} \psi(x) = 0$, then it is called an Archimedean copula (see e.g. McNeil and Nèslehovà [6]). ψ is said to be the generator function of this Archimedean copula. Let $G_\psi(u) = \exp\{-\psi^{-1}(u)\}$ and M_ψ be the distribution function with Laplace Transform ψ. Then, we can obtain an equivalent representation of (1) as

$$C_\psi(u_1, \ldots, u_n) = \int_0^\infty \prod_{j=1}^n G_\psi^\alpha(u_i) dM_\psi(\alpha). \tag{2}$$

This representation is the key to all subsequent developments.

Now, consider a random vector $\mathbf{X} = (X_1, \ldots, X_n)$ with joint distribution function

$$\psi\left(\sum_{i=1}^n \psi^{-1}(F_i(x_i))\right) = \int_0^\infty \prod_{i=1}^n G_\psi^\alpha(F_i(x_i)) dM_\psi(\alpha)\,, \tag{3}$$

where F_i is the marginal distribution functions, $i = 1, \ldots, n$. Let us further assume that the function G_ψ and F_i have the first derivative g_ψ and f_i, $i = 1, \ldots, n$, respectively. Then, the joint density function of $\mathbf{X}$ equals

$$\int_0^\infty \prod_{i=1}^n \alpha g_\psi(F_i(x_i)) f_i(x_i) G_\psi^{\alpha-1}(F_i(x_i)) dM_\psi(\alpha)\,, \tag{4}$$

which yields

$$\begin{aligned}
&\Pr\Big(X_1 < x_1, \ldots, X_k < x_k, X_{k+1} > x_{k+1}, \ldots, X_n > X_n\Big) \\
&= \int_0^\infty \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_k} \prod_{s=1}^k \alpha g_\psi(F_s(w_s)) G_\psi^{\alpha-1}({}_sF(w_s)) f_s(w_s) dw_s \\
&\quad \times \int_{x_{k+1}}^\infty \cdots \int_{x_n}^\infty \prod_{s=k+1}^n \alpha g_\psi(F_s(v_s)) G_\psi^{\alpha-1}(F_s(v_s)) f_s(v_s) dv_s dM_\psi(\alpha) \\
&= \int_0^\infty \prod_{s=1}^k G_\psi^\alpha(F_s(x_s)) \prod_{s=k+1}^n \{1 - G_\psi^\alpha(F_s(x_s))\} dM_\psi(\alpha),
\end{aligned} \tag{5}$$

for $k = 1, 2, \ldots, n$. For more details, one may refer to Joe [5] or Nelsen [7].

We use an Archimedean copula for considering the dependency of components because of its feasibility and also because the reliability of the system with standby redundancies with the mentioned setup has a closed form. This class was used by Rezapour et. al. [9] and Rezapour and Alamatsaz [10] to study a $(n-k+1)$-out-of-n system with dependent components. Rezapour et. al. [8] also considered reliability properties of a system whose components are distributed according to an Archimedean copula.

We investigate the distribution and the expected value of the number of surviving standby components at the moment that the system fails. More explicitly, let

$X_1, \ldots, X_n$ be random lifetimes of the active components with joint distribution function (3) and $Y_1, \ldots, Y_m$ be those of the standby components with joint density function

$$\int_0^\infty \prod_{i=1}^m \alpha g_\phi(H_i(x_i)) h_i(x_i) G_\phi^{\alpha-1}(H_i(x_i)) dM_\phi(\alpha)\,, \tag{6}$$

where ϕ is a completely monotone generator of an Archimedean copula and H_i is the distribution function with corresponding density function h_i, $i = 1, \ldots, n$, $G_\phi(u) = \exp\{-\phi^{-1}(u)\}$ and $g_\phi(u) = G'_\phi(u)$.

We also assume that the random vectors $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_m)$ has joint survival functions

$$\psi\left(\sum_{i=1}^n \psi^{-1}(\bar{F}_i(x_i))\right) = \int_0^\infty \prod_{i=1}^n G_\psi^\alpha(\bar{F}_i(x_i)) dM_\psi(\alpha)\,, \tag{7}$$

and

$$\phi\left(\sum_{i=1}^m \phi^{-1}(\bar{H}_i(y_i))\right) = \int_0^\infty \prod_{i=1}^m G_\phi^\alpha(\bar{H}_i(x_i)) dM_\phi(\alpha)\,, \tag{8}$$

respectively, where $\bar{H}_i$ ($\bar{F}_i$) is marginal survival function of Y_i (X_i) with derivative $-h_i$, $i = 1, \ldots, m$ ($-f_i$, $i = 1, \ldots, n$).

Now, suppose that the system has an arbitrary coherent structure φ. Thus the lifetime of the system is represented by $T = \varphi(X_1, \ldots, X_n)$ and the number of warm standby components that are alive at the time of system failure is $S_m = \#\{j \leq m : Y_j > T\}$. This random quantity is potentially useful for understanding the behavior of warm standby components with respect to a coherent system and can be used to determine the optimal number of standby components.

The following notation, will be used throughout this paper. The set $N_{m:s}^{-i}$, $s > i$, consists of all permutations $(j_1, \ldots, j_{s-i}, j_{s+1}, \ldots, j_m)$ of $\{1, \ldots, m\}$ for which $j_1 < \cdots < j_{s-i}$ and $j_{s+1} < \cdots < j_m$, $N_{m:s(l)}^{-i}$, $s > i$, contains all permutations $(j_1, \ldots, j_l)$ of $j_1, \ldots, j_{s-i}$, $N_{m\backslash s(l)}^{-i}$ contains all permutations $(j_1, \ldots, j_l)$ of $j_{s+1}, \ldots, j_m$, and $N_{s(l)}$ contains all permutations $(j_1, \ldots, j_l)$ of $1, \ldots, s$. In the case of $i = 0$, we will denote $N_{m:s}^{-i}$, $N_{m:s(l)}^{-i}$, and $N_{m\backslash s(l)}^{-i}$ by $N_{m:s}$, $N_{m:s(l)}$, and $N_{m:s\backslash(l)}$, respectively.

In Section 2, we derive the distribution function and the expected value of S_m for both dependent non-identically distributed and exchangeable components.

2. The number of surviving standby components at failure time of the system

Let T be the lifetime of a coherent system consisting of n components whose lifetimes are $X_1, \ldots, X_n$ with joint density function (4). The representation of the density function $f_{X_1,\ldots,X_n}$ given in (4) can be written as

$$f_{\mathbf{X}}(x_1, \ldots x_n) = \int_0^\infty f_{\mathbf{X}_\alpha}(x_1, \ldots x_m) dM_\psi(\alpha), \tag{9}$$

where $\mathbf{X} = (X_1, \ldots, X_n)$ and $f_{\mathbf{X}_\alpha}(x_1, \ldots x_m)$ is defined as the integrand in (4). Notice that $f_{\mathbf{X}_\alpha}$ is the joint density function of independent random variables $X_{i,\alpha}$ with distributions $G_\psi^\alpha(F_i(\cdot))$, $i = 1, \ldots, n$, $\alpha \geq 0$. Hence, the density function $f_{\mathbf{X}}$ can be seen as a M_ψ-mixture of the distributions of indepdnet random variables. Now, if if $X_{i:n}$ $(X_{i:n;\alpha})$, $i = 1, \ldots, n$ are the ith order statistics of the random vector $\mathbf{X}$ $(\mathbf{X}_\alpha)$, with similar arguments as in the proof of Theorem 1 in [4] we have

$$\begin{aligned}
P(T < t) &= \sum_{i=1}^n P(T = X_{i:n}, T < t) \\
&= \int_0^\infty \sum_{i=1}^n P(T = X_{i:n;\alpha}, T < t) dM_\psi(\alpha) \\
&= \int_0^\infty \sum_{i=1}^n P(T = X_{i:n;\alpha}) P(X_{i:n;\alpha} < t | T = X_{i:n;\alpha}) dM_\psi(\alpha) \\
&= \int_0^\infty \sum_{i=1}^n P(T = X_{i:n;\alpha}) P(X_{i:n;\alpha} < t) dM_\psi(\alpha) \\
&= \sum_{i=1}^n P(T = X_{i:n}) P(X_{i:n} < t) = \sum_{i=1}^n p_i F_{i:n}(t)\,, \quad t > 0\,, \qquad (10)
\end{aligned}$$

where $p_i = P(T = X_{i:n})$ and $F_{i:n}$ is the distribution function of $X_{i:n}$, $i = 1, \ldots, n$.. The vector $\mathbf{p} = (p_1, \ldots, p_n)$ is called the system signature. If the random variables $X_1, \ldots, X_n$ are exchangeable, i.e. $P(X_1 < x_1, \ldots, X_n < x_n) = P(X_{\pi(1)} < x_1, \ldots, X_{\pi(n)} < x_n)$ holds for any permutation $\pi = (\pi(1), \ldots, \pi(n))$, the signature of the system does not depend on the distribution of $X_1, \ldots, X_n$. Thus the system with exchangeable components has the same signature vector as the system with independent and identically distributed (i.i.d.) components. See Eryilmaz [2] for more details. If $F_i = F$ for $i = 1, \ldots, n$ in (4), the random variables $X_1, \ldots, X_n$ are exchangeable and therefore, the ith signature of a coherent system such the lifetime of its components has density function (4) is given by

$$p_i = \frac{r_{n-i+1(n)}}{\binom{n}{i}} - \frac{r_{n-i(n)}}{\binom{n}{i}}\,, \quad i = 1, \ldots, n\,, \tag{11}$$

where $r_i(n)$ is the number of path sets of the structure with exactly i working components (Boland [1]).

Example 2.1. *Consider the three component coherent system with i.i.d. components pictured in Figure 1 below. The failure times X_1, X_2 and X_3 of the three components of this system can be ordered in $3! = 6$ ways, and these six possible permutations are equally likely due to the i.i.d. assumption. It is easy to show that the signature of this system is $\mathbf{p} = (1/3, 2/3, 0)$ and that that the five distinct coherent systems of order 3 have the signatures $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1/3,2/3,0)$ and $(0,2/3,1/3)$.*

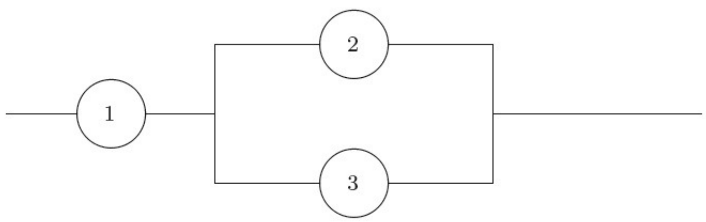

FIGURE 1. A 3-component system with structure function $\varphi(x) = x_1(x_2 + x_3 x_2 x_3)$.

In the following we obtain the distribution function of the random variable S_m in terms of signature for both dependent and non-identical and exchangeable component lifetimes.

Theorem 2.2. *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be lifetimes having joint density functions (4) and (6), respectively. If the system with lifetime $T = \varphi(X_1, \ldots, X_n)$ has signature $\mathbf{p} = (p_1, \ldots, p_n)$, then*

$$\begin{aligned} P(S_m = s) = \sum_{N_{m:s}} \sum_{l=0}^{s} \sum_{N_{m:s(l)}} (-1)^l \int_0^\infty \phi\Bigg(& \sum_{k \in N_{m:s(l)}} \phi^{-1}(H_{j_k}(t)) \\ & + \sum_{k=s+1}^{m} \phi^{-1}(H_{j_k}(t)) \Bigg) dF_T(t)\,, \end{aligned} \tag{12}$$

where $f_T(t) = \sum_{i=1}^{n} p_i f_{i:n}(t)$ *and*

$$f_{i:n}(t) = \sum_{N^{-1}_{n:i}} \frac{f_{j_i}(t)}{\psi'(\psi^{-1}(F_{j_i}(t)))} \sum_{l=0}^{n-i} (-1)^l \sum_{N^{-1}_{n\setminus i(l)}} \psi'\Big(\sum_{k=1}^{i-1} \psi^{-1}(F_{j_k}(t)) + \psi^{-1}(F_{j_i}(t)) + \sum_{k \in N^{-1}_{n\setminus i(l)}} \psi^{-1}(F_{j_k}(t)) \Big), \tag{13}$$

Proof. By the definition of S_m and the conditions of the theorem, one may write

$$P(S_m = s) = \sum_{N_{m:s}} P(Y_{j_1} > T, \ldots, Y_{j_s} > T, Y_{j_{s+1}} < T, \ldots, Y_{j_m} < T),$$

By conditioning on T and an arguments similar to those in Section 1, we have

$$\begin{aligned} P(S_m = s) &= \sum_{N_{m:s}} \int_0^\infty \int_0^\infty \prod_{k=1}^{s} \{1 - G^\alpha_\phi(H_{j_k}(t))\} \prod_{k=s+1}^{m} G^\alpha_\phi(H_{j_k}(t)) dM_\phi(\alpha) dF_T(t) \\ &= \sum_{N_{m:s}} \sum_{l=0}^{s} (-1)^l \sum_{N_{m:l(s)}} \int_0^\infty \int_0^\infty \prod_{k \in N_{l(s)}} G^\alpha_\phi(H_{j_k}(t)) \\ &\quad \times \prod_{k=s+1}^{m} G^\alpha_\phi(H_{j_k}(t)) dM_\phi(\alpha) dF_T(t), \end{aligned} \tag{14}$$

where the second equality follows by

$$\prod_{k=1}^{s} \{1 - G^\alpha_\phi(H_{j_k}(t))\} = \sum_{l=0}^{s} (-1)^l \prod_{k \in N_{l(s)}} G^\alpha_\phi(H_{j_k}(t))$$

By definition, $G_\phi(u) = \exp(-\phi^{-1}(u))$ and (14) reduces to

$$\sum_{N_{m:s}} \sum_{l=0}^{s} (-1)^l \sum_{N_{m:s(l)}} \int_0^\infty \int_0^\infty \exp\Big(-\alpha\Big(\sum_{k \in N_{l(s)}} \phi^{-1}(H_{j_k}(t)) + \sum_{k=s+1}^{n} \phi^{-1}(H_{j_k}(t)) \Big)\Big) dM_\phi(\alpha) dF_T(t). \tag{15}$$

Now, (12) follows by the identity $\phi(x) = \int_0^\infty e^{-\alpha u} dM_\phi(\alpha)$ and (15). Moreover, $f_T(t) = \sum_{i=1}^n p_i f_{i:n}(t)$ follows by (10) and the density function of $X_{i:n}$ equals

$$\begin{aligned}
&f_{i:n}(t)\\
&= \sum_{N_{n:i}^{-1}} \Pr\Big(X_{j_1} < t, \ldots, X_{j_{i-1}} < t, X_{j_{i+1}} > t, \ldots, X_{j_n} > t | X_{j_i} = t\Big) f_{j_i}(t),\\
&= \sum_{N_{n:i}^{-1}} \int_0^\infty \prod_{k=1}^{i-1} G_\psi^\alpha(F_{j_k}(t)) \alpha g_\psi(F_{j_i}(t)) f_{j_i}(t) G_\psi^{\alpha-1}(F_{j_i}(t))\\
&\qquad\qquad \times \prod_{k=i+1}^{n} \{1 - G_\psi^\alpha(F_{j_k}(t))\} dM_\psi(\alpha)\\
&= \sum_{N_{n:i}^{-1}} \int_0^\infty \prod_{k=1}^{i-1} G_\psi^\alpha(F_{j_k}(t)) \alpha g_\psi(F_{j_i}(t)) f_{j_i}(t) G_\psi^{\alpha-1}(F_{j_i}(t))\\
&\qquad \times \sum_{l=0}^{n-i} (-1)^l \sum_{N_{n\setminus i(l)}^{-1}} \prod_{k \in N_{n\setminus i(l)}^{-1}} G_\psi^\alpha(F_{j_k}(t)) dM_\psi(\alpha)\\
&= \sum_{N_{n:i}^{-1}} \frac{f_{j_i}(t)}{\psi'(\psi^{-1}(F_{j_i}(t)))} \sum_{l=0}^{n-i} (-1)^l \sum_{N_{n\setminus i(l)}^{-1}} \int_0^\infty (-\alpha) \exp\Bigg(-\alpha\Big(\sum_{k=1}^{i-1} \psi^{-1}(F_{j_k}(t))\\
&\qquad\qquad + \psi^{-1}(F_{j_i}(t)) + \sum_{k \in N_{n\setminus i(l)}} \psi^{-1}(F_{j_k}(t))\Big)\Bigg) dM_\psi(\alpha)\,,
\end{aligned} \tag{16}$$

where the last equality follows by $g_\psi(u) = -\frac{G_\psi(u)}{\psi'(\psi^{-1}(u))}$ and $G_\psi(u) = \exp(-\psi^{-1}(u))$. Therefore, (13) follows by $\psi'(u) = \int_0^\infty (-\alpha) e^{-\alpha u} dM_\psi(\alpha)$ and (16) and this completes the proof. □

Corollary 2.3. *Under assumption of Theorem 2.2, if $F_i = F$ for $i = 1, \ldots, n$ and $H_j = H$ for $j = 1, \ldots, m$, then*

$$P(S_m = s) = \sum_{l=0}^{s} (-1)^l \binom{m}{s} \binom{s}{l} \int_0^\infty \phi\Big((m - s + l)\phi^{-1}(H(t))\Big) dF_T(t)\,, \tag{17}$$

where $f_T(t) = \sum_{i=1}^n p_i f_{i:n}(t)$ and

$$f_{i:n}(t) = i\binom{n}{i} \frac{f(t)}{\psi'(\psi^{-1}(F(t)))} \sum_{l=0}^{n-i} (-1)^l \binom{n-i}{l} \psi'\Big((i + l)\psi^{-1}(F(t))\Big)\,, \tag{18}$$

Remark 2.4. *Under assumption of Corollary 2.3, if the underling Archimedean copula is generated by $\psi(x) = e^{-x}$, we have independence copula and $\psi'(t)\Big|_{t=z\psi^{-1}(x)} =$*

$-x^z$, which yields the probability mass function of S_m as obtained in Theorem 1. of Eryilmaz [2].

In the following, we obtain the mean value of the number of surviving standby components at the moment that the system fails.

Corollary 2.5. *Under assumption of Theorem 2.2, we have*

$$E(S_m) = \sum_{i=1}^{m}\sum_{j=1}^{m}\int_0^\infty \bar{H}_i(t) f_{i:n}(t)dt\,. \tag{19}$$

Proof. By definition of S_m, we have

$$E(S_m) = E\Big(\sum_{j=1}^{m} I(Y_j > T)\Big) = \sum_{j=1}^{m} P(Y_j > T) = \sum_{j=1}^{m}\int_0^\infty P(Y_j > t)dF_T(t)\,,$$

where $I(A)$ is the indicator function of A. Therefore, the results follows by (10). □

Example 2.6. *If the assumption of Corollary 2.3 holds and copulas belong to the Ali-Mikhail-Haq family with generators $\psi(u) = \frac{1-\theta_1}{e^t-\theta_1}$ and $\phi(u) = \frac{1-\theta_2}{e^t-\theta_2}$ for $\theta_1, \theta_2 \in [0,1)$, and $H(x) = F(x) = 1-e^{-\lambda}$, then the probability mass function of S_m reduces to*

$$\sum_{l=0}^{s}(-1)^l\binom{m}{s}\binom{s}{l}\int_0^\infty \frac{(1-\theta_2)(1-e^{-\lambda t})^{m-s+l}}{\Big(1-\theta_2 e^{-\lambda t}\Big)^{m-s+l} - \theta_2(1-e^{-\lambda t})^{m-s+l}} dF_T(t)\,,$$

where $f_T(t) = \sum_{i=1}^{n} p_i f_{i:n}(t)$ and

$$\begin{aligned} f_{i:n}(t) = i\binom{n}{i}&\frac{\lambda e^{-\lambda t}}{\frac{1-e^{-\lambda t}}{1-\theta_1}\Big(1-\theta_1 e^{-\lambda t}\Big)}\sum_{p=0}^{n-i}(-1)^p\binom{n-i}{p} \\ &\times \frac{(1-\theta_1)(1-e^{-\lambda t})^{p+i}}{\Big(1-\theta_1 e^{-\lambda t}\Big)^{i+p} - \theta_1(1-e^{-\lambda t})^{i+p}} \end{aligned}$$

Therefore, we have

$$\begin{aligned} P(S_m = s) = \sum_{l=0}^{s}\sum_{p=0}^{n-i}(-1)^{l+p}\int_0^1 &\frac{\binom{m}{s}\binom{s}{l}(1-\theta_2)i\binom{n}{i}\binom{n-i}{p}}{\Big(1-\theta_2 u\Big)^{m-s+l} - \theta_2(1-u)^{m-s+l}} \\ &\times \frac{(1-u)^{m-s+l+p+i-1}}{\Big(1-\theta_1 u\Big)^{i+p+1} - \theta_1(1-u)^{i+p}(1-\theta_1 u)} du\,, \end{aligned}$$

The last integral may be evaluated using a mathematical software.

In the following results, we consider the mass function of S_m when the random vectors $\mathbf{X}$ and $\mathbf{Y}$ have joint survival functions (7) and (8), respectively. The proof is similar to those above.

Corollary 2.7. *If the system with lifetime $T = \varphi(X_1, \ldots, X_n)$ has signature $\mathbf{p} = (p_1, \ldots, p_n)$, then*

$$P(S_m = s) = \sum_{N_{m:s}} \sum_{l=0}^{m-s} \sum_{N_{m\setminus s(l)}} (-1)^l \int_0^\infty \phi\Bigg(\sum_{k=0}^{s} \phi^{-1}(\bar{H}_{j_k}(t)) + \sum_{k\in N_{m\setminus s(l)}} \phi^{-1}(\bar{H}_{j_k}(t)) \Bigg) dF_T(t)\,, \tag{20}$$

where $f_T(t) = \sum_{i=1}^n p_i f_{i:n}(t)$ and

$$f_{i:n}(t) = \sum_{N^{-1}_{n:i}} \frac{f_{j_i}(t)}{\psi'(\psi^{-1}(\bar{F}_{j_i}(t)))} \sum_{l=0}^{i} (-1)^l \sum_{N^{-1}_{n:i(l)}} \psi'\Bigg(\sum_{k\in N^{-1}_{n:i}} \psi^{-1}(\bar{F}_{j_k}(t)) + \psi^{-1}(\bar{F}_{j_i}(t)) + \sum_{k=i+1}^{n} \psi^{-1}(\bar{F}_{j_k}(t)) \Bigg)\,, \tag{21}$$

Corollary 2.8. *Under assumption of Corollary 2.7, if $\bar{F}_i = \bar{F}$ for $i = 1, \ldots, n$ and $\bar{H}_j = \bar{H}$ for $j = 1, \ldots, m$, then*

$$P(S_m = s) = \sum_{l=0}^{m-s} (-1)^l \binom{m}{s} \binom{m-s}{l} \int_0^\infty \phi\Big((s+l)\phi^{-1}(\bar{H}(t)) \Big) dF_T(t)\,, \tag{22}$$

where $f_T(t) = \sum_{i=1}^n p_i f_{i:n}(t)$ and

$$f_{i:n}(t) = i\binom{n}{i} \frac{f(t)}{\psi'(\psi^{-1}(\bar{F}(t)))} \sum_{l=0}^{i} (-1)^l \binom{i}{l} \psi'\Big((n-i+l+1)\psi^{-1}(\bar{F}(t)) \Big)\,, \tag{23}$$

Example 2.9. *If the assumption of Corollary 2.8 holds and the copulas belong to the Gumbel family with generators $\psi(t) = \exp(-t^{1/\theta_1})$, $\phi(t) = \exp(-t^{1/\theta_2})$ for $\theta_1, \theta_2 \geq 1$, $f(x) = \lambda_1 e^{-\lambda_1 x}$ and $h(x) = \lambda_2 e^{-\lambda_2 x}$, $\lambda_1, \lambda_2 > 0$, then the probability mass function of S_m reduces to*

$$P(S_m = s) = \sum_{l=0}^{m-s} (-1)^l \binom{m}{s} \binom{m-s}{l} \int_0^\infty \exp\Big(-(s+l)^{1/\theta_2} \lambda_2 t \Big) dF_T(t)\,,$$

where $f_T(t) = \sum_{i=1}^{n} p_i f_{i:n}(t)$ *and*

$$
\begin{aligned}
f_{i:n}(t) = i\binom{n}{i}\frac{\lambda_1 e^{-\lambda_1 t}}{(\lambda_1 t)^{\theta_1(1/\theta_1-1)}e^{-\lambda_1 t}}\sum_{l=0}^{i}(-1)^l\binom{i}{l}\Big((n-i+l+1)(\lambda_1 t)^{\theta_1}\Big)^{1/\theta_1-1} \\
\times \exp\Big(-(n-i+l+1)^{1/\theta_1}\lambda_1 t\Big)\Big) \\
= i\binom{n}{i}\lambda_1\sum_{l=0}^{i}(-1)^l\binom{i}{l}(n-i+l+1)^{1/\theta_1-1}\exp\Big(-(n-i+l+1)^{1/\theta_1}\lambda_1 t\Big)\Big).
\end{aligned}
$$

Hence, the probability mass function of S_m *is*

$$
\begin{aligned}
&\sum_{l=0}^{m-s}(-1)^l\binom{m}{s}\binom{m-s}{l}i\binom{n}{i}\lambda_1\sum_{q=0}^{i}(-1)^q\binom{i}{q}(n-i+q+1)^{1/\theta_1-1} \\
&\quad\times\int_0^\infty \exp\Big(-\Big((s+l)^{1/\theta_2}\lambda_2+(n-i+q+1)^{1/\theta_1}\lambda_1\Big)t\Big)dt \\
&= \sum_{l=0}^{m-s}\sum_{q=0}^{i}\frac{(-1)^{l+q}\binom{m}{s}\binom{m-s}{l}i\binom{n}{i}\lambda_1\binom{i}{q}(n-i+q+1)^{1/\theta_1-1}}{(s+l)^{1/\theta_2}\lambda_2+(n-i+q+1)^{1/\theta_1}\lambda_1}.
\end{aligned}
$$

In Table 1 we calculate the value of $E(S_m)$ *for a system* $p = (1/3, 2/3, 0)$ *for different value of* λ_1 , λ_2 , θ_1 *and* θ_2 . *Since the cases* $\theta_1 = \theta_2 = 1$ *considers a systems with independent components we can infer that the dependency between the components of a system is a crucial object that should be considered. For example when* $\lambda_1 = \lambda_2 = \theta_1 = 1$, *and* $m = 5$ *by increasing* θ_2 , *to* 2 , *which reduce dependency between standby components,* $E(S_m)$ *is decreases. But, when we increase it to* 3 , $E(S_m)$ *is increase, but it is steel less than its value for independent case. When we increase the value of* θ_1 , *to* 2 *which decreases the dependency between the components of the system,* $E(S_m)$ *is increase. By changing* θ_1 *to* 3 , $E(S_m)$ *is decreases but steel it is greater than its value for independent case.*

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Table 1. Values of the $E(S_m)$ in Example 2.9 when $p=(1/3,2/3,0)$ for different value of λ_1, λ_2, θ_1 and θ_2.

λ_1	λ_2	θ_1	θ_2	m	$E(S_m)$
1	1	1	1	5	0.4167
1	1	1	2	5	0.2870
1	1	1	3	5	0.2112
1	1	2	1	5	0.4421
1	1	3	1	5	0.4358
2	2	1	1	5	0.8194
2	2	5	1	5	0.6265
2	2	1	5	5	0.4874
2	2	5	2	3	0.8683
2	2	5	3	2	0.8716
2	5	2	1	1	0.4956
2	5	2	2	3	1.9604
2	5	2	3	2	1.3339

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