ON THE SHEARLET TRANSFORM USING HYPERBOLIC FUNCTIONS

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Abstract. In this paper, we focus on the study of shearlet transform which is defined by using the hyperbolic functions. As a result we check an admissibility condition such that implies the reconstruction formula. To this end, we will use the concept of the classical shearlet, which indicates the position and direction of a singularity.

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1. Introduction

In recent years, much effort has been spent to design directional representation systems for images such as curvelets [2], ridgelets [3] and shearlets [7] and corresponding transforms (this list is not complete). Among these transforms, the shearlet transform stands out since it steams from a square-integrable group representation [4] and has the corresponding useful mathematical properties. Moreover, similarly as wavelets are related to Besov spaces via atomic decomposition, shearlets correspond to certain function spaces, the so-called shearlet coorbit spaces [5]. In

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addition, shearlets provide an optimally sparse approximation in the class of piecewise smooth functions with C^2 singularity curves, i.e.,

$$\|f - f_N\|_{L^2}^2 \leq CN^{-2}(logN)^3 \qquad as \quad N \longrightarrow \infty,$$

where f_N is the non-linear shearlet approximation of function f from this class, obtained by taking N the largest shearlet coefficients in absolute value.
Shearlets have been applied to a wide field of image processing takes, see, e.g., [7, 9, 10, 11]. In [8] the authors showed how the directional information encoded by the shearlet transform can be used in image segmentation.

The shearlets, provide an alternative approach to the curvelets, and exhibit some very distinctive features. In fact, similarly to the curvelets, the shearlets are a multiscale directional system and are also optimal in approximating 2-D smooth functions with discontinuities along C^2-Curves.
However unlike the curvelets, the shearlets form an affine system. That is, they are generated by dilating and translating one single generating function, where the dilation matrix is the product of a parabolic scaling matrix and a shear matrix. In particular, the shearlets can be regarded as coherent states arising from a unitary representation of a particular locally compact group, called the shearlet group. Recently in [1] the authors characterize irreducible as well as square-integrable sub-representations of the shearlet group representation in 2-D. This allows one to employ the theory of uncertainty principles to study the accuracy of the shearlet parameters. Another consequence of the group structure of the shearlets is that they are associated with a generalized Multiresolution Analysis, and this is particularly useful in both their theoretical and numerical applications.

This article is organized as follows: section 2 is devoted to some preliminaries and notations contains the definitions of the parabolic shearlet transform. In section 3 we introduce some special hyperbolic functions and show the special properties. The continuous shearlet transform via hyperbolic functions and its properties are investigated in section 4.

2. Preliminaries and notations

In this section we will review briefly some preliminaries about shearlet transform which is introduce in [6] by Kutyniok and et all.

Let $\mathbb{S} = \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ be the shearlet group, equipped with the group multiplication

$$(a, s, t)(a', s', t') = (aa', s' + s\sqrt{a'}, t' + S_{s'}A_{a'}t),$$

where $A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}$ is an isotropic dilation matrix and $S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ is the shear matrix, for $a \in \mathbb{R}^+$ and $s \in \mathbb{R}$. For $\psi \in L^2(\mathbb{R}^2)$ the continuous shearlet transform generated by ψ is the map

$$\mathcal{SH}_\psi : L^2(\mathbb{R}^2) \to L^2(\mathbb{S}),$$

defined by

$$f \mapsto \mathcal{SH}_\psi f(a, s, t) = \langle f, \psi_{a,s,t}\rangle, \quad f \in L^2(\mathbb{R}^2), \quad (a, s, t) \in \mathbb{S}.$$

The shearlet transform $\mathcal{SH}_\psi$ is invertible if the function ψ fulfils the admissibility condition

$$\int_{\mathbb{R}^2} \frac{|\hat{\psi}(\xi_1, \xi_2)|^2}{|\xi_1|^2} \mathrm{d}\xi_1 \mathrm{d}\xi_2 < \infty.$$

Let $\psi_1 \in L^2(\mathbb{R})$ be a function satisfying the discrete Caldern condition, i.e.,

$$\textstyle\sum_{j\in\mathbb{Z}} |\hat{\psi}_1(2^{-j}\xi)|^2 = 1, \quad \xi \in \widehat{\mathbb{R}},$$
$$\textstyle\sum_{k\in\mathbb{Z}} |\hat{\psi}_2(\xi + k)|^2 = 1, \quad \xi \in \widehat{\mathbb{R}}.$$

One typically chooses $\hat{\psi}_2$ to be a smooth bump function. Then $\psi \in L^2(\mathbb{R}^2)$ given by

$$\hat{\psi}(\xi) = \hat{\psi}_1(\xi_1)\hat{\psi}_2\left(\tfrac{\xi_2}{\xi_1}\right), \quad \xi = (\xi_1, \xi_2) \in \widehat{\mathbb{R}^2}. \tag{1}$$

It is shown that the shearlet of the form defined as in (1), fulfils the admissibility condition [6].

3. Hyperbolic functions with special properties

In this section we define some special hyperbolic functions with some special properties, which will be needed in the sequel.

We start by defining a function $F : \mathbb{R} \to \mathbb{R}$ such that

$$F(x) := \begin{cases} 0 & x < 0 \\ f(x) & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}, \tag{2}$$

where f is a continuous function on $[0, 1]$ such that $f(0) = 0$, $f(1) = 1$ and is symmetry around the point $(\frac{1}{2}, \frac{1}{2})$.

Recall that (the graph of) a function $f : \mathbb{R} \to \mathbb{R}$ is point symmetric with respect to (a, b), if and only if $f(a+x)-b = -f(a-x)+b$, for all $x \in \mathbb{R}$, which is equivalent to $f(x) + f(2a - x) = 2b$, for all $x \in \mathbb{R}$. Thus, for a symmetric function around $(\frac{1}{2}, \frac{1}{2})$, we have $f(x) + f(1 - x) = 1$. therefore f is increasing function on $[0, 1]$. We will see that it is a useful property for f to be symmetry around $(\frac{1}{2}, \frac{1}{2})$.
Now we define the function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(w) := \begin{cases} \sinh(F(|w| - 1)) & 1 \le |w| < 2 \\ \cosh(F(\frac{|w|}{2} - 1)) & 2 \le |w| \le 4 \\ 0 & otherwise \end{cases}. \tag{3}$$

Then h is a piecewise smooth function, that is a continuous functions which is symmetric, positive and real function. Also $supph = [-4, -1] \cup [1, 4]$ and furthermore $h(\pm 2) = 1$. Because of the symmetry of h, we restrict ourselves in the following analysis to the case that $w > 0$.
Let $h_j(\cdot) := h(2^{-j}\cdot)$ for $j \in \mathbb{N}_0$. Thus $supph_j = 2^j[1, 4] = [2^j, 2^{j+2}]$ and $h_j(2^{j+1}) = 1$. Obviously all these properties are satisfied for h_j^2. For $j_1 \ne j_2$ the overlap between the support of $h_{j_1}^2$ and $h_{j_2}^2$ is empty except for $|j_1 - j_2| = 1$. Thus, for h_j^2 and h_{j+1}^2, the $supp(h_j^2) \cap supp(h_{j+1}^2) = [2^{j+1}, 2^{j+2}]$. In this interval $h_j^2 = cosh^2(F(\frac{2^{-j}}{2}|w| - 1))$ and $h_{j+1}^2 = sinh^2(F(2^{-(j+1)}|w| - 1))$. Therefore in this interval we have

$$h_j^2(w) - h_{j+1}^2 = cosh^2(F(2^{-j-1}|w| - 1)) - sinh^2(F(2^{-j-1}|w| - 1)) = 1.$$

As a summarize of all these results we have

$$(h_j^2 - h_{j+1}^2)(w) = \begin{cases} h_j^2 & 2^j \leq w < 2^{j+1} \\ 1 & 2^{j+1} \leq w \leq 2^{j+2} \\ h_{j+1}^2 & 2^{j+2} \leq w \leq 2^{j+3} \\ 0 & o.w \end{cases} . \tag{4}$$

Consequently, we have the following lemma.

Lemma 3.1. *For each h_j defined as above, the relations*

$\sum_{j=-1}^{\infty}(-1)^{j+1}h_j^2(w) = \sum_{j=-1}^{\infty}(-1)^{j+1}h^2(2^{-j}w) = 1 \qquad |w| \geq 1$

and

$$\sum_{j=-1}^{\infty}(-1)^{j+1}h_j^2(w) = \begin{cases} 0 & |w| \leq \frac{1}{2} \\ sinh^2(F(2|w|-1)) & \frac{1}{2} < |w| < 1 \\ 1 & |w| \geq 1 \end{cases} , \tag{5}$$

hold true.

Proof. Since h_j and h_{j+1} are not equal to zero in each interval $[2^{j+1}, 2^{j+2}]$ for $j \geq -1$, so it is sufficient to prove that $h_j^2 - h_{j+1}^2 \equiv 1$ in this interval. Since $w \in [2^{j+1}, 2^{j+2}]$, so $2^{-j}w \in [2, 4]$ and also $2^{-j-1}w \in [1, 2]$. Hence we get that

$$\begin{aligned}(h_j^2 - h_{j+1}^2)(w) &= h^2(2^{-j}w) - h^2(2^{-j-1}w) \\ &= cosh^2(F(\frac{1}{2} \cdot 2^{-j}w - 1)) - sinh^2(F(2^{-j-1}w - 1)) \\ &= cosh^2(F(2^{-j-1}w - 1)) - sinh^2(F(2^{-j-1}w - 1)) \\ &= 1.\end{aligned}$$

The second relation follows by straightforward computation.

□

Now we define the function $\psi_1 : \mathbb{R} \to \mathbb{R}$ via its Fourier transform such that

$$\hat{\psi}_1(w) := \sqrt{h^2(2w) + h^2(w)}, \tag{6}$$

where h is defined as in (3).

The following theorem states an important property of ψ_1.

Theorem 3.1. *Let $\hat{\psi}_1$ be the function defined as in (6). Then $supp\hat{\psi}_1 = [-4, -\frac{1}{2}]\cup [\frac{1}{2}, 4]$ and fulfils the following equality*

$$\sum_{j\geq 0}(-1)^j(\hat{\psi}_1(2^{-2j}w))^2 = 1 \qquad |w| > 1.$$

Proof. By using the definition of h, it is easy to show that $supp\hat{\psi}_1 = [-4, -\frac{1}{2}]\cup[\frac{1}{2}, 4]$. Also by means of the property of h, we have

$$\begin{aligned}\sum_{j\geq 0}(-1)^j(\hat{\psi}_1(2^{-2j}w))^2 &= \sum_{j=0}^{\infty}(-1)^j(h^2(2.2^{-2j}w) + h^2(2^{-2j}w))\\ &= \sum_{j=0}^{\infty}(-1)^j(h^2(2^{-2j+1}w) + h^2(2^{-2j}w)).\end{aligned} \tag{7}$$

So by Lemma 3.1, we have

$$\begin{aligned}\sum_{j\geq 0}(-1)^j(\hat{\psi}_1(2^{-2j}w))^2 &= \sum_{j=-1}^{\infty}(-1)^{j+1}h^2(2^{-j}w)\\ &= 1.\end{aligned} \tag{8}$$

□

As a result of Theorem 3.1 and By (2) we have

$$\sum_{j\geq 0}(-1)^j(\hat{\psi}_1(2^{-2j}w))^2 = \begin{cases} 0 & |w| \leq \frac{1}{2} \\ sinh^2(F(2|w|-1)) & \frac{1}{2} < |w| < 1 \\ 1 & |w| \geq 1 \end{cases}. \tag{9}$$

Now we define the function $\psi_2 : \mathbb{R} \to \mathbb{R}$, in the Fourier domain, by

$$\hat{\psi}_2 := \begin{cases} \sqrt{F(1+w)} & w \leq 0 \\ \sqrt{F(1-w)} & w > 0 \end{cases}. \tag{10}$$

To investigate some properties of ψ_2, we need the following two lemmas.

Lemma 3.2. *The function F, which is defined in (2) is point symmetric with respect to $(\frac{1}{2}, \frac{1}{2})$, i.e., $F(x) + F(1-x) = 1$ for all $x \in \mathbb{R}$.*

Proof. The proof is clear from the definition of F.

□

Note that $\hat{\psi}_2$ is symmetric with respect the points $(0, y)$ for all $y \in \mathbb{R}$.

Lemma 3.3. *The function $\hat{\psi}_2$ fulfils the following condition*

$$\hat{\psi}_2^2(w-1) + \hat{\psi}_2^2(w) + \hat{\psi}_2^2(w+1) = 1 \qquad for |w| \leq 1.$$

Proof. By the definition of ψ_2, we have

$$\hat{\psi}_2^2(w) = \begin{cases} F(1+w) & w \leq 0 \\ F(1-w) & w > 0 \end{cases}.$$

So by the fact that for $0 \leq w \leq 1$, $F(-w) = 0$, we have

$$\begin{aligned} \hat{\psi}_2^2(w-1) + \hat{\psi}_2^2(w) + \hat{\psi}_2^2(w+1) &= F(1+w-1) + F(1-w) + F(1-w-1) \\ &= F(w) + F(1-w) + F(-w) \\ &= 1, \end{aligned}$$

and similarly we obtain for $-1 \leq w < 0$ that

$$\begin{aligned} \hat{\psi}_2^2(w-1) + \hat{\psi}_2^2(w) + \hat{\psi}_2^2(w+1) &= F(1+w-1) + F(1+w) + F(1-w-1) \\ &= F(-|w|) + F(1-|w|) + F(|w|) \\ &= 1. \end{aligned}$$

□

One can see in the proof of Lemma 3.3, that the sum in both cases has two (different) summands, in particular

$$1 = \hat{\psi}_2^2(w-1) + \hat{\psi}_2^2(w) + \hat{\psi}_2^2(w+1) = \begin{cases} \hat{\psi}_2^2(w-1) + \hat{\psi}_2^2(w) & 0 \leq w \leq 1 \\ \hat{\psi}_2^2(w) + \hat{\psi}_2^2(w+1) & -1 \leq w < 0 \end{cases}.$$

Now we can state a property of ψ_2 in the following theorem, By means of the two previous lemmas.

Theorem 3.2. *The function $\hat{\psi}_2$, which is defined in (10), fulfils the following equality*

$$\sum_{k=-2^j}^{2^j} |\hat{\psi}_2(k+2^j w)|^2 = 1 \qquad for |w| \leq 1, j \geq 0. \tag{11}$$

Proof. With $\tilde{w} := 2^j w$ the formula in (11) becomes

$$\sum_{k=-2^j}^{2^j} |\hat{\psi}_2(k+\tilde{w})|^2 = 1 \qquad for |\tilde{w}| \leq 2^j, j \geq 0.$$

For a fixed (but arbitrary) $w^* \in [-2^j, 2^j] \subset \mathbb{R}$, since $supp\hat{\psi}_2 = [-1,1]$, so $\hat{\psi}_2(w^* + k) \neq 0$ for $-1 \leq w^* + k \leq 1$. Thus, for $w^* \in \mathbb{Z}$, just the summands for $k \in \{-w^* - 1, -w^*, -w^* + 1\}$ do not vanish. But for $k = -w^* \pm 1$, we have that $w^* + k = \pm 1$ and $\hat{\psi}_2(\pm 1) = 0$. In this case the entire sum reduces to one summand $k = -w^*$ such that

$$\sum_{k=-2^j}^{2^j} |\hat{\psi}_2(k+w^*)|^2 = |\hat{\psi}_2(-w^*+w^*)|^2 = |\hat{\psi}_2(0)|^2 = 1.$$

If $w^* \notin \mathbb{Z}$ and $w^* > 0$, the only non-zero summand appears for $k \in \{[w^*], [w^*] - 1\}$. Thus, for $0 < r^+ := w^* - [w^*] < 1$,

$$\begin{aligned}\sum_{k=-2^j}^{2^j} |\hat{\psi}_2(k+w^*)|^2 &= |\hat{\psi}_2(-[w^*]+w^*)|^2 + |\hat{\psi}_2(-[w^*]-1+w^*)|^2 \\ &= |\hat{\psi}_2(r^+)|^2 + |\hat{\psi}_2(1-r^+)|^2,\end{aligned}$$

which is equal to 1 by Lemma 3.3. Analogously, we obtain for $w^* \notin \mathbb{Z}$ such that $w^* < 0$, the remaining non-zero summands are those for $k \in \{[w^*], [w^*] + 1\}$. For $-1 < r^- := [w^*] + w^* < 0$ we have

$$\begin{aligned}\sum_{k=-2^j}^{2^j} |\hat{\psi}_2(k+w^*)|^2 &= |\hat{\psi}_2([w^*]+w^*)|^2 + |\hat{\psi}_2([w^*]+1+w^*)|^2 \\ &= |\hat{\psi}_2(r^-)|^2 + |\hat{\psi}_2(1+r^-)|^2.\end{aligned}$$

So $\hat{\psi}_2(x) = \hat{\psi}_2(-x)$, Lemma 3.3 lead one obtains

$$|\hat{\psi}_2(r^-)|^2 + |\hat{\psi}_2(1+r^-)|^2 = |\hat{\psi}_2(|r^-|)|^2 + |\hat{\psi}_2(1-|r^-|)|^2 = 1.$$

□

4. The continuous shearlet transform associated with hyperbolic functions

To define a usable shearlet transform, we need functions with special properties. In this section, we define shearlet transform via hyperbolic functions, which are introduced in section 3.

Recall that the shearlet transform $\psi_{a,s,t}$ emerges by dilation, shearing and translation of a function $\psi \in L^2(\mathbb{R}^2)$ as follows

$$\begin{aligned} \psi_{a,s,t}(x) &:= a^{-\frac{3}{4}}\psi(A_a^{-1}S_s^{-1}(x-t)) \\ &= a^{-\frac{3}{4}}\psi(\begin{pmatrix} \frac{1}{a} & -\frac{s}{a} \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix}(x-t)). \end{aligned} \tag{12}$$

We assume that $\hat{\psi}$ can be written as

$$\hat{\psi}(w_1, w_2) = \hat{\psi}_1(w_1)\hat{\psi}_2(\frac{w_2}{w_1}), \tag{13}$$

where ψ_1, ψ_2 are defined in (6) and (10) respectively. Consequently, we obtain for the Fourier transform,

$$\begin{aligned} \hat{\psi}_{a,s,t}(w) &= a^{-\frac{3}{4}}\psi(\begin{pmatrix} \frac{1}{a} & -\frac{s}{a} \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix}(.-t))\ \hat{}\ (w) \\ &= a^{-\frac{3}{4}}e^{-2\pi i<w,t>}\psi(\begin{pmatrix} \frac{1}{a} & -\frac{s}{a} \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix}.)\ \hat{}\ (w) \\ &= a^{-\frac{3}{4}}e^{-2\pi i<w,t>}(a^{-\frac{3}{2}})^{-1}\hat{\psi}(\begin{pmatrix} a & 0 \\ s\sqrt{a} & \sqrt{a} \end{pmatrix}w) \\ &= a^{\frac{3}{4}}e^{-2\pi i<w,t>}\hat{\psi}(aw_1, \sqrt{a}(sw_1 + w_2)) \\ &= a^{\frac{3}{4}}e^{-2\pi i<w,t>}\hat{\psi}_1(aw_1)\hat{\psi}_2(a^{-\frac{1}{2}}(\frac{w_2}{w_1} + s)). \end{aligned}$$

The shearlet transform $\mathcal{SH}_\psi(f)$ of a function $f \in L^2(\mathbb{R})$ can now be defined as follows

$$\begin{aligned}
\mathcal{SH}_\psi(f) &:=< f, \psi_{a,s,t} > \\
&=< \hat{f}, \hat{\psi}_{a,s,t} > \\
&= \int_{\mathbb{R}^2} \hat{f}(w)\overline{\hat{\psi}_{a,s,t}(w)}\mathrm{d}w \\
&= a^{\frac{3}{4}} \int_{\mathbb{R}^2} \hat{f}(w)\hat{\psi}_1(aw_1)\hat{\psi}_2(a^{-\frac{1}{2}}(\frac{w_2}{w_1}+s))e^{-2\pi i<w,t>}\mathrm{d}w \\
&= a^{\frac{3}{4}}\mathcal{F}^{-1}(\hat{f}(w)\hat{\psi}_1(aw_1)\hat{\psi}_2(a^{-\frac{1}{2}}(\frac{w_2}{w_1}+s)))(t).
\end{aligned}$$

In the next theorem we will show that the shearlet ψ, fulfils the admissibility condition which means that we have the reconstruction formula for every function in $L^2(\mathbb{R}^2)$.

Theorem 4.1. *The shearlet transform, which is defined by using hyperbolic functions in (13), is invertible, i.e the function ψ fulfils the following admissibility property:*

$$\int_{\mathbb{R}^2} \frac{|\hat{\psi}(w_1, w_2)|^2}{|w_1|^2}\mathrm{d}w_1\mathrm{d}w_2 < \infty.$$

Proof. Easy calculations show that any shearlet function of the form $\hat{\psi}(w_1, w_2) = \hat{\psi}_1(w_1)\hat{\psi}_2(\frac{w_2}{w_1})$, in which $\hat{\psi}_1$ and $\hat{\psi}_2$ are continuous and $supp\hat{\psi}_1 \subset [-b, -a] \cup [a, b]$ and $supp\hat{\psi}_2 \subset [-c, c]$ is admissible. On the other hand $\hat{\psi}_1$, $\hat{\psi}_2$ are continuous and as it is shown in section 3, $supp\hat{\psi}_1 \subset [-4, -\frac{1}{2}] \cup [\frac{1}{2}, 4]$ and $supp\hat{\psi}_2 \subset [-1, 1]$; so the function ψ fulfils the admissibility condition. □

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