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# GENERALIZED CESÁRO TENSOR AND IT'S PROPERTIES

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Abstract. Recently, infinite and finite dimensional generalized Hilbert tensors have been introduced. In this paper, the authors further introduce infinite and finite dimensional generalized Cesáro tensors as a generalization of Cesáro matrices and discuss the properties of these structured tensors. Next, some upper bounds of $Z_1$-spectral radius of generalized Cesáro tensors and generalized Hilbert tensors are given, which improves the existing ones. Finally, we obtain conditions under which a generalized Cesáro tensor is column sufficient tensor.

*Keywords*: Generalized Cesáro tensor, $Z_1$-eigenvalue, Column sufficient tensor.

*2020 MSC*: 15A69, 15A18, 15A42, 65F15, 47J10.

## 1. Introduction

In linear algebra, an $n$-dimensional generalized Cesáro matrix $C_\alpha^n = (C_{i,j})$ is a square matrix with entries being the unit fractions, i.e.,

$$C_{i,j} = \begin{cases} \dfrac{1}{i+\alpha-1} & i \geq j \\ 0 & i < j, \end{cases} \qquad i,j = 1,2,\ldots,n, \tag{1}$$

where $\alpha \geq 1$, is a real number. When $\alpha = 1$, an $n$-dimensional Cesáro matrix is bounded linear operator on $\ell^p$ for $1 < p < \infty$ (here, $\ell^p(0 < p < \infty)$ is the space consisting of all real number sequences $x = (x_k)_{k=0}^{\infty}$ such that $\sum_{k=0}^{\infty} |x_k|^p < \infty$).

The well-known inequality

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{|x_k|}{n+1}\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{k=0}^{\infty}|x_k|^p,$$

which is also known as Hardy's inequality, and its result is boundedness of the Cesáro operator. The infinite Cesáro operator $C_\alpha^\infty$ has the form as in (1) or

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the matrix presentation

$$C_{i,j} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

This operator has the $\ell^p$-norm $||C||_p = p^*$, where $p^*$ is the conjugate of $p$, i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$. Some properties and applications of finite and infinite Cesáro matrices have been investigated in [1, 8, 18].

In recent years, problems related to tensors have drawn much people's attention. As a generalization of matrix theory, fruitful research achievements have been made in topics such as structured tensors [14,19]. Structured tensors mean tensors with special structure. In recent years, several kinds of structured tensors have been studied such as Hilbert tensors [12, 13, 17], Hankel tensors [15], Cauchy tensors [3], and so on [14]. Furthermore, researchers established some results on spectral theory, positive semi-definiteness and other properties of structured tensors.

Denote $[n] := \{1, 2, \ldots, n\}$. A real $m$-order , $n$-dimensional tensor(hypermatrix) $\mathcal{A} = (a_{i_1 \ldots i_m})$ is a multi-array of real entries $a_{i_1 \ldots i_m}$, where $i_j \in [n]$ for $j \in [m]$. As a natural extension of a generalized Cesáro matrix, the entries of an $m$-order infinite dimensional generalized Cesáro tensor (hypermatrix) $\mathcal{C}_\alpha^\infty = (\mathcal{C}_{i_1,\ldots,i_m})$ are defined by

$$(2) \qquad \mathcal{C}_{i_1,i_2,\ldots,i_m} = \begin{cases} \dfrac{1}{i_1 + i_3 + i_4 + \ldots + i_m - m + 2\alpha} & i_1 \geq i_2 \\ 0 & i_1 < i_2, \end{cases}$$

where $\alpha \geq 1$, is a real number and $i_1, i_2, \ldots, i_m = 1, 2, \ldots, n, \ldots$. An $m$-order, $n$-dimensional generalized Cesáro tensor is showed by $\mathcal{C}_\alpha^n = (\mathcal{C}_{i_1,i_2,\ldots,i_m})$, where $i_j \in [n]$ for $j \in [m]$. When $\alpha = 1$, generalized Cesáro tensor is called Cesáro tensor $\mathcal{C}$.

For a real vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, $\mathcal{C}_\alpha^n x^{m-1}$ is a vector whose $i^{th}$ component is

$$(\mathcal{C}_\alpha^n x^{m-1})_i = \sum_{i_2=1}^{i} \sum_{i_3,\ldots,i_m=1}^{n} \frac{x_{i_2} x_{i_3} \ldots x_{i_m}}{i + i_3 + i_4 + \ldots + i_m - m + 2\alpha}, \qquad \alpha \geq 1 \ \ \textit{and} \ \ i \in [n].$$

Then $x^T \left(\mathcal{C}_\alpha^n x^{m-1}\right)$ is a homogeneous polynomial, denoted $\mathcal{C}_\alpha^n x^m$, i.e., $\mathcal{C}\alpha^n x^m$ is a homogeneous polynomial which is given by

$$\mathcal{C}_\alpha^n x^m = x^T \left(\mathcal{C}_\alpha^n x^{m-1}\right) = \sum_{i_1=1}^{n} \sum_{i_2=1}^{i_1} \sum_{i_3,\ldots,i_m=1}^{n} \frac{x_{i_1} x_{i_2} x_{i_3} \ldots x_{i_m}}{i_1 + i_3 + i_4 + \ldots + i_m - m + 2\alpha}, \qquad \alpha \geq 1,$$

where $x^T$ is the transposition of $x$.

For a real vector $x = (x_1, x_2, \ldots, x_n, \ldots) \in \ell^1$, $\mathcal{C}_\alpha^\infty x^{m-1}$ is an infinite dimensional vector whose $i^{th}$ component is

$$(\mathcal{C}_\alpha^\infty x^{m-1})_i = \sum_{i_2=1}^{i} \sum_{i_3,\ldots,i_m=1}^{\infty} \frac{x_{i_2} x_{i_3} \ldots x_{i_m}}{i + i_3 + i_4 + \ldots + i_m - m + 2\alpha},$$

where $\alpha \geq 1$ and $i = 1, 2, ....$ Accordingly, $\mathcal{C}_\alpha^\infty x^m$ is a homogeneous polynomial which is given by

$$\mathcal{C}_\alpha^\infty x^m = \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \sum_{i_3,\ldots,i_m=1}^{\infty} \frac{x_{i_1} x_{i_2} x_{i_3} \ldots x_{i_m}}{i_1 + i_3 + i_4 + \ldots + i_m - m + 2\alpha}, \qquad \alpha \geq 1.$$

In Section 2, the authors prove that $\mathcal{C}_\alpha^\infty x^m$ and $\mathcal{C}_\alpha^\infty x^{m-1}$ are well-defined.

In 2005, Qi [16] and Lim [11] proposed the concepts of eigenvalue and $Z_2$-eigenvalue of tensors, independently. Since then, the spectral theory of tensors has attracted much attention. In a series of recent works, researchers pointed out that $Z_1$-eigenvalues have significant applications in many fields. Li et al. showed that the $Z_1$-eigenvalue and its eigenvector are useful for computing the limiting probability distribution in high order Markov chain [10]. Some bounds of $Z_1$-spectral radius of tensors can be found in [9, 13].

In the following, the notion of $Z_1$-eigenvalue was introduced by Chang and Zhang [2].

**Definition 1.1.** [2] Let $\mathcal{A}$ be an $m$-order, $n$-dimensional tensor. A pair $(\lambda, x) \in \mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$ is called an $Z_1$-eigenvalue and $Z_1$-eigenvector (or simply $Z_1$-eigenpair) of $\mathcal{A}$ if they satisfy the equation:

$$\mathcal{A}x^{m-1} = \lambda x, \qquad \|x\|_1 = \sum_{i=1}^{n} |x_i| = 1. \tag{3}$$

Let $\mathcal{A}$ be an $m$-order, $n$-dimensional tensor. By $\sigma_{Z_1}(\mathcal{A})$, we denote the $Z_1$-spectrum of $\mathcal{A}$, i.e., the set of all $Z_1$-eigenvalues of $\mathcal{A}$. Assume $\sigma_{Z_1}(\mathcal{A}) \neq \emptyset$, then the $Z_1$-spectral radius of $\mathcal{A}$, denoted $\rho(\mathcal{A})$ is defined as

$$\rho(\mathcal{A}) = \max\{|\lambda| \ : \ \lambda \in \sigma_{Z_1}(\mathcal{A})\}.$$

In Section 3, with the help of the Hilbert type inequality [7] , the authors show that the upper bound of $Z_1$-spectral radius of an $m$-order, $n$-dimensional generalized Cesáro tensor $\mathcal{C}_\alpha^n$ with $\alpha \geq 1$ is not larger than $n \sin\left(\frac{\pi}{n}\right)$. Furthermore, they show that $\sum_{i=1}^{n} \mathcal{C}_{i\,1\ldots1}$ is an upper bound for all $Z_1$-eigenvalues of $\mathcal{C}_\alpha^n$, that is independent of any choice of $\alpha$. Similarly, the authors obtain an optimal bound for $Z_1$-eigenvalues of finite dimensional generalized Hilbert tensors. By running examples for some $m$ and $n$, they showed that the obtained results are sharper than existing results.

The class of column sufficient tensors has recently arisen in connection with the tensor complementarity problem (TCP) [4]. Column sufficient tensors are linked to the existence and the convexity of the solutions set. Also, they contain many important special tensors, such as positive semi-definite tensors, Hilbert tensors, diagonally dominated tensors with nonnegative diagonal entries, double $B$-tensors, quasi-double $B$-tensors, $H$-tensors with nonnegative diagonal entries, $P$-tensors, strong Hankel tensors, $M$-tensors, and positive Cauchy tensors. for details, see [4]. In Section 4, we give conditions that a Cesáro tensor is column sufficient tensor.

## 2. Infinite dimensional generalized Cesáro tensors

In this section, firstly, the authors discuss the properties of infinite and finite dimensional generalized Cesáro tensor. Secondly, they prove that $\mathcal{C}_\alpha^\infty x^m$ and $\mathcal{C}_\alpha^\infty x^{m-1}$ are well-defined. Furthermore, the authors define two operators $B_\infty$, $F_\infty$ and show that these are the bounded operators.

*Remark* 2.1. Clearly, both $\mathcal{C}_\alpha^\infty$ and $\mathcal{C}_\alpha^n$ are nonnegative ($\mathcal{C}_{i_1,i_2,\ldots,i_m} \geq 0$) but are not symmetric tensor($\mathcal{C}_{i_1,\ldots,i_m}$ are not invariant for any permutation of the indices). Generalized Cesaro tensor is not positive definite, since for all nonzero vector $x \in \mathbb{R}^n$ need to show that $\mathcal{C}_\alpha^n x^m > 0$. But by setting

$$x = (-1, 4, 6, \ldots, 2(n-1), 2n)^T \in \mathbb{R}^n \quad (n \geq 2),$$

with $\alpha = 1$, we have

$$\mathcal{C}_\alpha^n x^m < 0.$$

**Proposition 2.2.** *Suppose that $\mathcal{C}_\alpha^\infty$ is an $m$-order infinite dimensional generalized Cesáro tensor. Then both $\mathcal{C}_\alpha^\infty x^m$ and $\mathcal{C}_\alpha^\infty x^{m-1}$ are well defined for all $x \in \ell^1$.*

*Proof.* For all non-negative integer $i_1, i_3, i_4, \ldots, i_m$, we have

$$\min_{i_1,i_3,i_4,\ldots,i_m} |i_1 + i_3 + i_4 + \ldots + i_m - m + 2\alpha| = 2\alpha - 1, \qquad \alpha \geq 1. \tag{4}$$

Now, let $x = (x_1, x_2, \ldots, x_n, \ldots) \in \ell^1$. Then, we have

$$\begin{aligned} |\mathcal{C}_\alpha^\infty x^m| &= \left| \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \sum_{i_3,\ldots,i_m=1}^{\infty} \frac{x_{i_1} x_{i_2} x_{i_3} \ldots x_{i_m}}{i_1 + i_3 + i_4 + \ldots + i_m - m + 2\alpha} \right| \\ &\leq \frac{1}{2\alpha - 1} \sum_{i_1,i_2\ldots,i_m=1}^{\infty} |x_{i_1} x_{i_2} \ldots x_{i_m}| \\ &= \frac{1}{2\alpha - 1} \left( \sum_{i=0}^{\infty} |x_i| \right)^m = \frac{1}{2\alpha - 1} (\|x\|_1)^m < \infty, \end{aligned}$$

which shows that $\mathcal{C}_\alpha^\infty x^m$ is well defined for all $x \in \ell^1$. Similarly for all $x \in \ell^1$, we have $\mathcal{C}_\alpha^\infty x^{m-1} < \infty$, and the proof is complete. □

For all real vector $x \in \ell^1$, we define

(5)
$$F_\infty x = (\mathcal{C}_\alpha^\infty x^{m-1})^{[\frac{1}{m-1}]} \qquad and \qquad B_\infty x = \begin{cases} \|x\|_1^{2-m}\mathcal{C}_\alpha^\infty x^{m-1} & x \neq \theta \\ \theta & x = \theta, \end{cases}$$

where $\theta = (0, 0, \ldots, 0)^T$. Recently, Mei and Song [12] introduced these concepts for the generalized Hilbert tensor. It is easy to see that both operators $B_\infty$ and $F_\infty$ are continuous and positively homogeneous. Inspired by the work of Mei and Song [12], the authors show that $B_\infty$ and $F_\infty$ are bounded operators.

**Theorem 2.3.** *Let $B_\infty$ and $F_\infty$ be defined by Eq.* (5)*. Assume that $\alpha \geq 1$. Then*

(*i*) *$B_\infty$ is a bounded operator from $\ell^1$ into $\ell^p$* $(1 < p < \infty)$;
(*ii*) *$F_\infty$ is a bounded operator from $\ell^1$ into $\ell^p$* $(m - 1 < p < \infty)$.

*Proof.* (*i*) For $x \in \ell^1$, we have

$$\begin{aligned} \left|(\mathcal{C}_\alpha^\infty x^{m-1})_i\right| &= \left|\lim_{n\to\infty} \sum_{i_2=1}^{i} \sum_{i_3,i_4,\ldots,i_m=1}^{n} \frac{x_{i_2}\ldots x_{i_m}}{i + i_3 + \ldots + i_m - m + 2\alpha}\right| \\ &\leq \lim_{n\to\infty} \sum_{i_3,i_4,\ldots,i_m=1}^{n} \sum_{i_2=1}^{n} \frac{|x_{i_2}\ldots x_{i_m}|}{|i + 1 + \ldots + 1 - m + 2\alpha|} \\ &\leq \frac{1}{i + 2\alpha - 2} \lim_{n\to\infty} \sum_{i_2,\ldots,i_m=1}^{n} |x_{i_2}||x_{i_3}|\ldots|x_{i_m}| \\ &= \frac{1}{i + 2\alpha - 2} \lim_{n\to\infty} \left(\sum_{k=1}^{n} |x_k|\right)^{m-1} \\ &= \frac{1}{i + 2\alpha - 2} \left(\sum_{k=1}^{\infty} |x_k|\right)^{m-1} \\ &= \frac{1}{i + 2\alpha - 2} \|x\|_1^{m-1}. \end{aligned}$$

Therefore

$$\begin{aligned}\|B_\infty x\|_p^p &= \sum_{i=1}^{\infty} |(B_\infty x)_i|^p \\ &= \sum_{i=1}^{\infty} \left|\left(\|x\|_1^{2-m}\mathcal{C}_\alpha^\infty x^{m-1}\right)_i\right|^p \\ &= \|x\|_1^{(2-m)p} \sum_{i=1}^{\infty} \left|(\mathcal{C}_\alpha^\infty x^{m-1})_i\right|^p \\ &\leq \|x\|_1^{(2-m)p} \sum_{i=1}^{\infty} \left(\frac{1}{i+2\alpha-2}\|x\|_1^{m-1}\right)^p \\ &= \|x\|_1^p \sum_{i=1}^{\infty} \left(\frac{1}{i+2\alpha-2}\right)^p.\end{aligned}$$

Since $\alpha \geq 1$, then for all positive integer $i > 1$, the series $\sum_{i=1}^{\infty} \dfrac{1}{(i+2\alpha-2)^p}$ converges for $p > 1$. Hence $B_\infty x \in \ell^p$ for all $x \in \ell^1$. In addition, setting

$$M := \left(\sum_{i=1}^{\infty} \frac{1}{(i+2\alpha-2)^p}\right)^{\frac{1}{p}},$$

then $\|B_\infty x\|_p \leq M\|x\|_1$, i.e., $B_\infty$ is a bounded operator from $\ell^1$ into $\ell^p (1 < p < \infty)$.
$(ii)$ For $m-1 < p < \infty$, it follows that

$$\begin{aligned}\|F_\infty x\|_p^p &= \sum_{i=1}^{\infty} |(F_\infty x)_i|^p \\ &= \sum_{i=1}^{\infty} \left|(\mathcal{C}_\alpha^\infty x^{m-1})_i^{\frac{1}{m-1}}\right|^p \\ &\leq \sum_{i=1}^{\infty} \left|\left(\frac{1}{i+2\alpha-2}\right)\|x\|_1^{m-1}\right|^{\frac{p}{m-1}} \\ &= \|x\|_1^p \sum_{i=1}^{\infty} \frac{1}{(i+2\alpha-2)^{\frac{p}{m-1}}}.\end{aligned}$$

Since $p > m-1$, then for all positive integer $i > 1$, the series $\sum_{i=1}^{\infty} \dfrac{1}{(i+2\alpha-2)^{\frac{p}{m-1}}}$ converges. Hence, $F_\infty$ is a bounded operator from $\ell^1$ into $\ell^p$ $(m-1 < p < \infty)$. □

## 3. Some bounds for $Z_1$-eigenvalues of finite dimensional generalized Cesáro tensors

In this section, the authors first establish the upper bound for $Z_1$-eigenvalues of finite dimensional generalized Cesáro tensor. Subsequently, an optimal upper bound for $Z_1$-eigenvalues of finite dimensional generalized Cesáro(Hilbert) tensors is given. Finally, some examples are presented to show the efficiency of our proposed bound.

The following Hilbert type inequality need to establish Theorem 3.2.

**Lemma 3.1.** *[6] Let $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$. Then*

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|x_i x_j|}{i+j-1} \le (n\sin\frac{\pi}{n})\sum_{k=1}^{n}x_k^2. \tag{6}$$

**Theorem 3.2.** *Let $\mathcal{C}_\alpha^n$ be an $m$-order, $n$-dimensional generalized Cesáro tensor with $\alpha \ge 1$. Then $n\sin\left(\frac{\pi}{n}\right)$ is an upper bound for all $Z_1$-eigenvalues of $\mathcal{C}_\alpha^n$.*

*Proof.* For $\alpha \ge 1$ and all non-zero vector $x \in \mathbb{R}^n$, we have

$$\begin{aligned}
|\mathcal{C}_\alpha^n x^m| &= \left|\sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1}\sum_{i_3,\dots,i_m=1}^{n}\frac{x_{i_1}x_{i_2}x_{i_3}\dots x_{i_m}}{i_1+i_3+i_4+\dots+i_m-m+2\alpha}\right| \\
&\le \sum_{i_2=1}^{n}|x_{i_2}|\sum_{i_1,i_3,i_4,\dots,i_m=1}^{n}\frac{|x_{i_1}x_{i_3}\dots x_{i_m}|}{i_1+i_3+1+\dots+1-m+2\alpha} \\
&= \sum_{i_2=1}^{n}|x_{i_2}|\sum_{i_1,i_3,\dots,i_m=1}^{n}\frac{|x_{i_1}||x_{i_3}|\dots|x_{i_m}|}{i_1+i_3+2\alpha-3} \\
&= \sum_{i_1=1}^{n}\sum_{i_3=1}^{n}\frac{|x_{i_1}||x_{i_3}|}{i_1+i_3+2\alpha-3}\sum_{i_2,i_4,i_5,\dots,i_m=1}^{n}|x_{i_2}||x_{i_4}||x_{i_5}|\dots|x_{i_m}| \\
&\le \sum_{i_1=1}^{n}\sum_{i_3=1}^{n}\frac{|x_{i_1}x_{i_3}|}{i_1+i_3-1}\left(\sum_{i=1}^{n}|x_i|\right)^{m-2}.
\end{aligned}$$

Then, using Lemma 3.1 for all $\alpha \ge 1$

$$|\mathcal{C}_\alpha^n x^m| \le \left(\|x\|_2^2\; n\;\sin\left(\frac{\pi}{n}\right)\right)\|x\|_1^{m-2}. \tag{7}$$

On the other hand, let $(\lambda, x)$ be a $Z_1$-eigenpair of $\mathcal{C}_\alpha^n$, i.e.,

$$\mathcal{C}_\alpha^n x^{m-1} = \lambda x, \qquad \|x\|_1 = \sum_{i=1}^{n}|x_i| = 1.$$

Then using (7), we have

$$|\lambda|\|x\|_2^2 = |\lambda x^T x| = |x^T(\lambda x)| = \left|x^T(\mathcal{C}_\alpha^n x^{m-1})\right| = |\mathcal{C}_\alpha^n x^m| \le \|x\|_2^2\|x\|_1^{m-2}\; n\sin\left(\frac{\pi}{n}\right).$$

Thus $|\lambda| \le n\;\sin\left(\frac{\pi}{n}\right)$, and the proof is complete. □

**Theorem 3.3.** *Let $\mathcal{C}_\alpha^n$ be an $m$-order $n$-dimensional generalized Cesáro tensor. Then $\sum_{i=1}^{n} \mathcal{C}_{i\,1\cdots 1}$ is an upper bound for all $Z_1$-eigenvalues of $\mathcal{C}_\alpha^n$.*

*Proof.* Let $(\lambda, x)$ be a $Z_1$-eigenpair of $C_\alpha^n$. Then (3) holds. Hence

$$\mathcal{C}_\alpha^n x^{m-1} = \lambda x, \qquad \|x\|_1 = \sum_{i=1}^{n} |x_i| = 1. \tag{8}$$

From (8), we can get

$$\lambda x_i = \sum_{i_2,\ldots,i_m=1}^{n} \mathcal{C}_{i\, i_2,\ldots,i_m}\, x_{i_2} \ldots x_{i_m} \qquad i = 1, \ldots, n. \tag{9}$$

Taking modulus in (9) and using the triangle inequality give

$$\begin{aligned}
|\lambda| = |\lambda| \sum_{i=1}^{n} |x_i| &\leq \sum_{i,i_2,\ldots,i_m=1}^{n} \mathcal{C}_{i\ i_2\cdots i_m}\ |x_{i_2}| \ldots |x_{i_m}| \\
&= \sum_{i_2,\ldots,i_m=1}^{n} \left( |x_{i_2}| \ldots |x_{i_m}| \sum_{i=1}^{n} \mathcal{C}_{i\ i_2\cdots i_m} \right) \\
&\leq \left( \sum_{i_2,\ldots,i_m=1}^{n} |x_{i_2}| \ldots |x_{i_m}| \right) \max_{i_2,\ldots,i_m \in [n]} \sum_{i=1}^{n} \mathcal{C}_{i\ i_2\cdots i_m} \\
&= \sum_{i=1}^{n} \mathcal{C}_{i\ 1\cdots 1},
\end{aligned}$$

where the last equality holds because

$$\sum_{i_2,\ldots,i_m=1}^{n} |x_{i_2}| \ldots |x_{i_m}| = \prod_{s=2,3,\ldots,m} \left( \sum_{i_s=1}^{n} |x_{i_s}| \right) = 1,$$

and

$$\max_{i_2,\ldots,i_m \in [n]} \sum_{i=1}^{n} \mathcal{C}_{i\ i_2\cdots i_m} = \sum_{i=1}^{n} \mathcal{C}_{i\ 1\cdots 1}.$$

Therefore the proof is complete. □

In Table (1), we show the efficiency of our results for some finite generalized Cesáro tensors.

Ming and Song [13] introduced the generalized Hilbert tensor as follows: For each $\lambda \in \mathbb{R}\backslash\mathbb{Z}^-$, the entries of an $m$-order infinite dimensional generalized Hilbert tensor $\mathcal{H}_\lambda^\infty = (\mathcal{H}_{i_1 i_2 \ldots i_m})$ are defined by

$$\mathcal{H}_{i_1 i_2 \ldots i_m} = \frac{1}{i_1 + i_2 + \ldots + i_m + \lambda}, \quad i_1, i_2, \ldots, i_m = 0, 1, \ldots, n, \ldots. \tag{10}$$

In the finite case, an $m$-order, $n$-dimensional generalized Hilbert tensor is represented by $\mathcal{H}_\lambda^n$. They obtained some upper bound of $Z_1$–spectral radius of

TABLE 1. Upper bounds of $Z_1$-spectral radius of $\mathcal{C}_\alpha^n$ for some $m$-order with $\alpha = 1$

| | Theorem 2.1 of [9] | Theorem 3.2 | Theorem 3.3 |
|---|---|---|---|
| $m = 3,\ n = 2$ | $\rho(\mathcal{C}_\alpha^2) \leq 1.66$ | $\rho(\mathcal{C}_\alpha^2) \leq 2$ | $\rho(\mathcal{C}_\alpha^2) \leq 1.5$ |
| $m = 4,\ n = 2$ | $\rho(\mathcal{C}_\alpha^2) \leq 2.83$ | $\rho(\mathcal{C}_\alpha^2) \leq 2$ | $\rho(\mathcal{C}_\alpha^2) \leq 1.5$ |
| $m = 5,\ n = 2$ | $\rho(\mathcal{C}_\alpha^2) \leq 4.9$ | $\rho(\mathcal{C}_\alpha^2) \leq 2$ | $\rho(\mathcal{C}_\alpha^2) \leq 1.5$ |
| $m = 3,\ n = 3$ | $\rho(\mathcal{C}_\alpha^3) \leq 2.35$ | $\rho(\mathcal{C}_\alpha^3) \leq 2.59$ | $\rho(\mathcal{C}_\alpha^3) \leq 1.83$ |
| $m = 4,\ n = 3$ | $\rho(\mathcal{C}_\alpha^3) \leq 5.72$ | $\rho(\mathcal{C}_\alpha^3) \leq 2.59$ | $\rho(\mathcal{C}_\alpha^3) \leq 1.83$ |
| $m = 3,\ n = 4$ | $\rho(\mathcal{C}_\alpha^4) \leq 3.03$ | $\rho(\mathcal{C}_\alpha^4) \leq 2.82$ | $\rho(\mathcal{C}_\alpha^4) \leq 2.08$ |

finite dimensional generalized Hilbert tensor( [13, Theorem 3.1]). Similar to the proof of Theorem 3.3, we get the following theorem which is an improvement of Theorem 3.1 of [13]:

**Theorem 3.4.** *Let $\mathcal{H}_\lambda^n$ be an $m$-order, $n$-dimensional generalized Hilbert tensor. Then $\sum_{i=1}^{n} |\mathcal{H}_{i\,1\cdots 1}|$ is an upper bound for all $Z_1$-eigenvalue of $\mathcal{H}_\lambda^n$.*

In Table (2), we show the efficiency of our results for some finite generalized Hilbert tensors.

## 4. Column sufficient tensors

To have a better understanding of Cesáro tensors, we show that $\mathcal{C}_\alpha^n$ is column adequate in $\mathbb{R}_+^n$ and there is no odd-order column sufficient Cesáro tensors.

**Definition 4.1.** [4] An $m$-order, $n$-dimensional tensor $\mathcal{A}$ is called a column sufficient tensor (or $\mathcal{A}$ is column sufficient in simple), if $x \in \mathbb{R}^n$ satisfies

$$x_i \left(\mathcal{A}x^{m-1}\right)_i \leq 0, \forall i \in [n] \Longrightarrow x_i \left(\mathcal{A}x^{m-1}\right)_i = 0, \forall i \in [n]. \tag{11}$$

For $X \subseteq \mathbb{R}^n$, if $x \in X$ and (11) holds, then $\mathcal{A}$ is called column sufficient in $X$.

**Definition 4.2.** [5] A tensor $\mathcal{A} \in T_{m,n}$ is said to be column adequate tensor, if $x \in \mathbb{R}^n$ satisfies

$$x_i \left(\mathcal{A}x^{m-1}\right)_i \leq 0, \forall i \in [n] \Longrightarrow \mathcal{A}x^{m-1} = 0, \forall i \in [n]. \tag{12}$$

For $X \subseteq \mathbb{R}^n$, if $x \in X$ and (12) holds, then $\mathcal{A}$ is called column adequate in $X$.

Table 2. Upper bounds of $Z_1$-spectral radius of $\mathcal{H}_\lambda^n$ for some $m$-order

| $m=3,\ n=2$ | | | | |
|---|---|---|---|---|
| Methods | $(\lambda \geq 1)$ $\lambda = 1$ | $(0<\lambda<1)$ $\lambda=\frac{1}{2}$ | $(-mn<\lambda<0)$ $\lambda=\frac{-3}{2}$ | $(\lambda<-mn)$ $\lambda=\frac{-15}{2}$ |
| Theorem 3.1 of [13] | $\rho(\mathcal{H}_\lambda^2) \leq 2$ | $\rho(\mathcal{H}_\lambda^2) \leq 4$ | $\rho(\mathcal{H}_\lambda^2) \leq 4$ | $\rho(\mathcal{H}_\lambda^2) \leq 1.33$ |
| Theorem 3.4 | $\rho(\mathcal{H}_\lambda^2) \leq 1.5$ | $\rho(\mathcal{H}_\lambda^2) \leq 2.66$ | $\rho(\mathcal{H}_\lambda^2) \leq 2.66$ | $\rho(\mathcal{H}_\lambda^2) \leq 0.287$ |
| $m=3,\ n=3$ | | | | |
| | $\lambda=1$ | $\lambda=\frac{1}{2}$ | $\lambda=\frac{-3}{2}$ | $\lambda=\frac{-21}{2}$ |
| Theorem 3.1 of [13] | $\rho(\mathcal{H}_\lambda^3) \leq 2.59$ | $\rho(\mathcal{H}_\lambda^3) \leq 6$ | $\rho(\mathcal{H}_\lambda^3) \leq 6$ | $\rho(\mathcal{H}_\lambda^3) \leq 2$ |
| Theorem 3.4 | $\rho(\mathcal{H}_\lambda^3) \leq 1.833$ | $\rho(\mathcal{H}_\lambda^3) \leq 3.06$ | $\rho(\mathcal{H}_\lambda^3) \leq 4.66$ | $\rho(\mathcal{H}_\lambda^3) \leq 0.31$ |
| $m=4,\ n=4$ | | | | |
| | $\lambda=1$ | $\lambda=\frac{1}{2}$ | $\lambda=\frac{-3}{2}$ | $\lambda=\frac{-33}{2}$ |
| Theorem 3.1 of [13] | $\rho(\mathcal{H}_\lambda^4) \leq 2.82$ | $\rho(\mathcal{H}_\lambda^4) \leq 8$ | $\rho(\mathcal{H}_\lambda^4) \leq 8$ | $\rho(\mathcal{H}_\lambda^4) \leq 8$ |
| Theorem 3.4 | $\rho(\mathcal{H}_\lambda^4) \leq 2.08$ | $\rho(\mathcal{H}_\lambda^4) \leq 3.35$ | $\rho(\mathcal{H}_\lambda^4) \leq 5.33$ | $\rho(\mathcal{H}_\lambda^4) \leq 0.26$ |
| $m=4,\ n=5$ | | | | |
| | $\lambda=1$ | $\lambda=\frac{1}{2}$ | $\lambda=\frac{-3}{2}$ | $\lambda=\frac{-45}{2}$ |
| Theorem 3.1 of [13] | $\rho(\mathcal{H}_\lambda^5) \leq 2.93$ | $\rho(\mathcal{H}_\lambda^5) \leq 10$ | $\rho(\mathcal{H}_\lambda^5) \leq 10$ | $\rho(\mathcal{H}_\lambda^5) \leq 2$ |
| Theorem 3.4 | $\rho(\mathcal{H}_\lambda^5) \leq 2.28$ | $\rho(\mathcal{H}_\lambda^5) \leq 3.57$ | $\rho(\mathcal{H}_\lambda^5) \leq 5.73$ | $\rho(\mathcal{H}_\lambda^5) \leq 0.24$ |

**Theorem 4.3.** *Suppose that $\mathcal{C}_\alpha^n$ is an $m$-order, $n$-dimensional Cesáro tensor. Then, the following results hold:*
*(i) $\mathcal{C}_\alpha^n$ is column adequate in $\mathbb{R}_+^n$.*
*(ii) When $m$ is odd, $\mathcal{C}_\alpha^n$ is not column adequate.*
*(iii) When $m$ is even, $\mathcal{C}_\alpha^n$ is column adequate.*

*Proof.* For any $x \in \mathbb{R}^n_+$ and $i \in [n]$, we have

$$\begin{aligned} x_i \left(\mathcal{C}^n_\alpha x^{m-1}\right)_i &= x_i \sum_{i_2=1}^{i} \sum_{i_3,i_4,\ldots,i_m=1}^{n} \frac{x_{i_2} x_{i_3} \cdots x_{i_m}}{i + i_3 + i_4 + \cdots + i_m + \alpha} \\ &= x_i \sum_{i_2=1}^{i} x_{i_2} \sum_{i_3,i_4,\ldots,i_m=1}^{n} \frac{x_{i_3} x_{i_4} \cdots x_{i_m}}{i + i_3 + i_4 + \cdots + i_m + \alpha} \\ &= x_i \sum_{i_2=1}^{i} x_{i_2} \sum_{i_3,i_4,\ldots,i_m=1}^{n} \int_0^1 t^{i+i_3+i_4+\cdots+i_m+\alpha-1} x_{i_3} x_{i_4} \cdots x_{i_m} \, \mathrm{d}t \\ &= x_i \sum_{i_2=1}^{i} x_{i_2} \int_0^1 \left( \sum_{j=1}^{n} t^{j+\frac{i+\alpha-1}{m-2}} x_j \right)^{m-2} \mathrm{d}t. \end{aligned}$$

That means for any $x \in \mathbb{R}^n_+$ and $i \in [n]$, we have

$$x_i \left(\mathcal{C}^n_\alpha x^{m-1}\right)_i \leq 0 \Longleftrightarrow x_i \sum_{i_2=1}^{i} x_{i_2} \int_0^1 \left( \sum_{j=1}^{n} t^{j+\frac{i+\alpha-1}{m-2}} x_j \right)^{m-2} \mathrm{d}t \leq 0.$$

If $x = 0$, then $\mathcal{C}^n_\alpha x^{m-1} = 0$. If $x_i > 0$, then $x_i \left(\mathcal{C}^n_\alpha x^{m-1}\right)_i \leq 0$ means that

$$\left(\mathcal{C}^n_\alpha x^{m-1}\right)_i = \sum_{i_2=1}^{i} x_{i_2} \int_0^1 \left( \sum_{j=1}^{n} t^{j+\frac{i+\alpha-1}{m-2}} x_j \right)^{m-2} \mathrm{d}t \leq 0.$$

Since $x \geq 0$,

$$\int_0^1 \left( \sum_{j=1}^{n} t^{j+\frac{i+\alpha-1}{m-2}} x_j \right)^{m-2} \mathrm{d}t \leq 0. \tag{13}$$

On the other hand,

$$\left( \sum_{j=1}^{n} t^{j+\frac{i+\alpha-1}{m-2}} x_j \right)^{m-2} \mathrm{d}t \geq 0, \quad \forall t \in [0,1].$$

Therefore

$$\int_0^1 \left( \sum_{j=1}^{n} t^{j+\frac{i+\alpha-1}{m-2}} x_j \right)^{m-2} \mathrm{d}t \geq 0.$$

Combining this with (13), we have

$$\left(\mathcal{C}_\alpha^n x^{m-1}\right)_i = \sum_{i_2=1}^{i} x_{i_2} \int_0^1 \left(\sum_{j=1}^{n} t^{j+\frac{i+\alpha-1}{m-2}} x_j\right)^{m-2} \mathrm{d}t = 0, \quad \forall i \in [n],$$

which implies that $\mathcal{C}_\alpha^n$ is column adequate in $\mathbb{R}_+^n$.
(ii) When $m$ is oded for all $x \in \mathbb{R}^n$ and $x < 0$, we have

$$x_i \left(\mathcal{C}_\alpha^n x^{m-1}\right)_i = x_i \sum_{i_2=1}^{i} \sum_{i_3,i_4,\ldots,i_m=1}^{n} \frac{x_{i_2} x_{i_3} \cdots x_{i_m}}{i + i_3 + i_4 + \cdots + i_m + \alpha} < 0.$$

It shows that

$$\left(\mathcal{C}_\alpha^n x^{m-1}\right)_i > 0, \quad \forall i \in [n],$$

which implies that $\mathcal{C}_\alpha^n$ is not a column adequate tensor.
(iii) For even $m$, if $x \in \mathbb{R}^n$, we have

$$x_i \left(\mathcal{C}_\alpha^n x^{m-1}\right)_i = x_i \sum_{i_2=1}^{i} x_{i_2} \int_0^1 \left(\sum_{j=1}^{n} t^{j+\frac{i+\alpha-1}{m-2}} x_j\right)^{m-2} \mathrm{d}t \leq 0, \quad \forall i \in [n].$$

It can be easily checked that

$$x_i \sum_{i_2=1}^{i} x_{i_2} \leq 0, \quad \forall i \in [n].$$

It follows that $x = 0$ and the desired results holds. □

**Theorem 4.4.** *[5, Theorem 3.1] A column adequate tensor is a column sufficient tensor.*

**Theorem 4.5.** *Suppose that $\mathcal{C}_\alpha^n$ is an m-order, n-dimensional Cesáro tensor. Then, the following results hold:*
*(i) $\mathcal{C}_\alpha^n$ is column sufficient in $\mathbb{R}_+^n$;*
*(ii) when m is odd, $\mathcal{C}_\alpha^n$ is not column sufficient;*
*(iii) when m is even, $\mathcal{C}_\alpha^n$ is column sufficient.*

*Proof.* (i) and (iii) follow from Theorem 3.4.
(ii) When $m$ is odd for $x = (-1, -1, \ldots, -1) \in \mathbb{R}^n$, we have

$$x_i \left(\mathcal{C}_\alpha^n x^{m-1}\right)_i = x_i \sum_{i_2=1}^{i} \sum_{i_3,i_4,\ldots,i_m=1}^{n} \frac{x_{i_2} x_{i_3} \cdots x_{i_m}}{i + i_3 + i_4 + \cdots + i_m + \alpha} < 0.$$

This implies that $\mathcal{C}_\alpha^n$ is not a column sufficient tensor.

□

## Conclusion

In this paper, we defined a new class of tensors, called generalized Cesáro tensors, and studied their properties. Also, we discussed $Z_1$-eigenpairs of a finite dimensional generalized Cesáro tensor. Furthermore, we presented a sharper bound for any $Z_1$-eigenvalue of finite dimensional generalized Cesáro tensors and also Hilbert tensor. This bound is always sharper than the bounds in [9, 13].

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