Journal of Mahani Mathematical Research
Shahid Bahonar University of Kerman
Print ISSN: 2251-7952 Online ISSN: 2645-4505

# USING FRAMES IN GMRES-BASED ITERATION METHOD FOR SOLVING OPERATOR EQUATIONS

H. Jamali ✉ and R. Pourkani

Article type: Research Article

(Received: 11 August 2023, Received in revised form 21 October 2023)

(Accepted: 19 November 2023, Published Online: 19 November 2023)

Abstract. In this paper, we delve into frame theory to create an innovative iterative method for resolving the operator equation $Lu = f$. In this case, $L : H \to H$, a bounded, invertible, and self-adjoint linear operator, operates within a separable Hilbert space denoted by $H$. Our methodology, which is based on the GMRES projective method, introduces an alternate search space, which brings another dimension to the problem-solving process. Our investigation continues with the assessment of convergence, where we look at the corresponding convergence rate. This rate is intricately influenced by the frame bounds, shedding light on the effectiveness of our approach. Furthermore, we investigate the ideal scenario in which the equation finds an exact solution, providing useful insights into the practical implications of our work.

*Keywords*: Hilbert space, Operator equation, Frame, Preconditioning, GMRES iteration, Convergence rate.
*2020 MSC*: Primary 65F10, 65F08.

## 1. Introduction

Projection methods are the most recently practical iterative techniques for solving large linear systems of equations

$$Lu = f, \tag{1}$$

where $L : H \to H$ is a bounded, invertible, and self-adjoint linear operator on a separable Hilbert space $H$. By using this approach, we can extract canonically an approximation $u_n$ to the exact solution $u$ of the linear system from a subspace $\mathcal{K} \subseteq H$, called search subspace, provided that

$$f - Lu_n \perp \mathcal{L},$$

where $\mathcal{L} \subseteq H$ is another (probably the same) subspace called the subspace of constraints, of the equal dimension. $\mathcal{L}$ can be equal to $\mathcal{K}$ or it can be equal to $L\mathcal{K}$. For more details, we refer the interested reader to the book by Saad [14]. In the meantime, GMRES (Generalized Minimum Residual Method) is of great importance in projection methods that utilizes Krylov subspaces $\mathcal{K} = \mathcal{K}_m(L, r_0)$ with Arnoldi orthonormal basis.

✉ jamali@vru.ac.ir, ORCID: 0000-0003-4569-2955
https://doi.org/10.22103/jmmr.2023.21932.1480
Publisher: Shahid Bahonar University of Kerman
How to cite: H. Jamali, R. Pourkani, *Using frames in GMRES-based iteration method for solving operator equations*, J. Mahani Math. Res. 2024; 13(2): 107 - 119.

The goal of this paper is to study the application of frames in the GMRES method for solving operator equation (1). In [4, 8–10, 12] some numerical algorithms for solving this system have been developed by using wavelets and frames.
The great advantage of converting the GMRES method into FGMRES (Frame Generalized Minimum Residual Method) version is that we can see the convergence rate in this approach is formed by the upper and lower bounds of the frame, so we can control the convergence rate by choosing an appropriate frame with desired values of bounds. Furthermore in this setting, by employing a tight frame for designing iteration, even the exact solution of (1) is obtained in the first step of the iteration. These properties turn FGMRES into an applicable tool for approximating solutions of operator equations with prescribed accuracy or even finding the exact solution.
The method is designed on the basis of preconditioning the operator equation $Lu = f$, using frames and then applying the GMRES iteration method but with an orthonormal basis other than Arnoldi type.
We will now give a brief history about the Arnoldi method and GMRES iterative method, in the following subsection and definitions plus basic properties of frames in the next section. For more information about frames and GMRES method we refer the reader to the books by Christensen [7] and Saad [14] respectively.

1.1. **History.** The Arnoldi method is a technique for constructing an orthogonal basis of a Krylov subspace, which is a subspace spanned by successive powers of a matrix applied to a vector. The Arnoldi method was introduced by W. E. Arnoldi in 1951 [1] as a generalization of the Lanczos method(see [14] section 6.6), which is restricted to symmetric matrices. The Arnoldi method can be used for various purposes, such as solving linear systems, computing eigenvalues and eigenvectors, and performing matrix factorizations.
One of the most important applications of the Arnoldi method is the generalized minimal residual method (GMRES), which is an iterative method for solving nonsymmetric linear systems. GMRES was developed by Yousef Saad and Martin H. Schultz in 1986 [15] as an extension and improvement of the minimal residual method (MINRES), which was proposed by Christopher C. Paige and Michael A. Saunders in 1975 [13]. MINRES requires that the matrix is symmetric, but has the advantage of low memory and computational cost. GMRES relaxes the symmetry assumption, but requires more storage and operations as the iteration progresses. GMRES approximates the solution of a linear system by finding the vector in the Krylov subspace that minimizes the residual norm. To construct the Krylov subspace, GMRES uses the Arnoldi method to generate an orthonormal basis and an upper Hessenberg matrix that satisfies a matrix equation involving the original matrix. The solution is then obtained by solving a smaller least squares problem involving the Hessenberg matrix and the right-hand side vector.

GMRES is one of the most widely used iterative methods for solving nonsymmetric linear systems, especially when the matrix is large and sparse. It has many variants and enhancements, such as restarted GMRES, preconditioned GMRES, flexible GMRES, and deflated GMRES, that aim to improve its convergence, robustness, and efficiency. GMRES is also related to other iterative methods based on Krylov subspaces, such as FOM, QMR, BiCG, and CGS( [14] chapters 6 and 7).
Authors in [3] proposed a new iterative method based on frames to solve the operator equation (1). In another article by these authors [2], frames are used instead of wavelet basis in the Galerkin adaptive method to solve the equation $Lu = f$.
The Galerkin methods deal with variational principles and orthogonality conditions to derive approximate solutions for boundary value problems. The generalization of this approach to partial differential equations was done by Galerkin, who introduced the concept of weak formulation and used test functions from the same space as the trial functions. The Galerkin methods can also be combined with adaptive strategies, such as mesh refinement, error estimation, and goal-oriented optimization, to improve the accuracy and efficiency of the solution. We used the idea in [3] for the GMRES iterative method to solve operator equation(1) and achieving a new method called FGMRES with the help of frames. As mentioned above, the GMRES method is an iterative method for solving nonsymmetric linear systems that approximates the solution by the vector in a Krylov subspace with minimal residual norm. Studies in references [11] and [12] are related to the applications of frames in Chebyshev and conjugate gradient methods, as well as Richardson and Chebyshev methods for solving operator equations, respectively. These methods are similar to the GMRES-based iteration method presented in this paper in that they are all iterative methods for solving linear systems of equations. However, the specific techniques and algorithms used in each method differ. In particular, the GMRES method is a Krylov subspace method that seeks to minimize the residual of the linear system over a subspace of increasing dimension. The Chebyshev and conjugate gradient methods, on the other hand, are iterative methods that seek to minimize the error of the linear system over a subspace of fixed dimension. The Richardson method is a simple iterative method that involves multiplying the residual by a scalar factor and adding it to the current approximation. Overall, while FGMRES presented in this paper and these methods share some similarities in their iterative nature and use of subspaces, they differ in their specific techniques and algorithms.

## 2. **Preliminaries**

Throughout this paper, $H$ will be a separable Hilbert space and $\Lambda$ denotes a countable index set. In the next subsection afterward, we introduce the notion of preconditioning of an operator equation and we describe how a frame is used to precondition an operator equation.

2.1. **Frames.** We begin by defining the concept of the frame.

**Definition 2.1.** Let $(\psi_\lambda)_{\lambda\in\Lambda} \subset H$. Then $(\psi_\lambda)_{\lambda\in\Lambda}$ is a frame for $H$, if there exist constants $0 < A \leq B < \infty$ such that

$$A \parallel f \parallel_H^2 \leq \sum_{\lambda\in\Lambda} |< f, \psi_\lambda >|^2 \leq B \parallel f \parallel_H^2, \quad \forall f \in H.$$

The constants $A$ and $B$ are called the lower and upper frame bounds, respectively. If $A = B$, we call $(\psi_\lambda)_{\lambda\in\Lambda}$ an $A$-*tight frame*, and if $A = B = 1$, it is a *Parseval frame*.

We associate to a frame $(\psi_\lambda)_{\lambda\in\Lambda}$, the *synthesis operator*

$$T : \ell_2(\Lambda) \to H, \; T((c_\lambda)_{\lambda\in\Lambda}) = \sum_{\lambda\in\Lambda} c_\lambda \psi_\lambda,$$

and the *analysis operator*

$$T^* : H \to \ell_2(\Lambda), \; T^*(f) = (< f, \psi_\lambda >)_{\lambda\in\Lambda}.$$

For a frame $\Psi = (\psi_\lambda)_{\lambda\in\Lambda}$, the operator

$$S = TT^* : H \to H, \; S(f) = \sum_{\lambda\in\Lambda} < f, \psi_\lambda > \psi_\lambda,$$

is called *frame operator* which is positive, self-adjoint and invertible, which satisfies

$$AI_H \leq S \leq BI_H. \tag{2}$$

Also, it has been shown that if $(\psi_\lambda)_{\lambda\in\Lambda}$ is a frame for $H$ and if $L$ is bounded onto operator on $H$, then the sequence $(L(\psi_\lambda))_{\lambda\in\Lambda}$ would be a frame for $H$ too. Moreover, if $L$ is also a self-adjoint operator and $S$ is the frame operator of $(\psi_\lambda)_{\lambda\in\Lambda}$, then $LSL$ is the frame operator of $(L(\psi_\lambda))_{\lambda\in\Lambda}$. For more details we refer the reader to [7], [6].

2.2. **Preconditioning.** Preconditioning is any form of implicit or explicit modification of an original linear system that yields easier solving or faster convergence by a given iterative method. This is an effective technique for solving differential equations, integral equations, and related problems [7], [5]. The abstract approach is to multiply both sides of (1) by an operator $M$, and then apply a suitable iterative method. We choose $M$ here to be formed by using a given frame. To gain an insight, we write $Lu = f$ as

$$u = (I - L)u + f,$$

then for given $u_0 \in H$, define for $k \geq 0$,

$$u_{k+1} = (I - L)u_k + f. \tag{3}$$

Since $Lu - f = 0$, we can write

$$\begin{aligned} u_{k+1} - u &= (I - L)u_k + f - u - (f - Lu) \\ &= (I - L)u_k - u + Lu \\ &= (I - L)(u_k - u). \end{aligned}$$

Hence

$$\| u_{k+1} - u \|_H \leq \| I - L \|_{H \to H} \| u_k - u \|_H,$$

so that the sequence (3) converges if

$$\| I - L \|_{H \to H} < 1. \tag{4}$$

Let us take $M$ an operator which approximates $L^{-1}$ i.e. $M \approx L^{-1}$ or $ML \approx I$. Then, the last term implies $\| I - ML \|_{H \to H} \ll 1$, so in view of (4) the convergence of iterative sequence (3) associated to preconditioned operator equation

$$MLu = Mf, \tag{5}$$

is much faster than the original one.

One way to obtain $M$ is using frames. For this concern, we note to the following lemma.

**Lemma 2.2.** *Let $\Psi = (\psi_\lambda)_{\lambda \in \Lambda}$ be a frame for $H$ with frame operator $S$, and $L$ be as in (1). Suppose that $A$ and $B$ are the frame bounds of the frame $L\Psi = (L(\psi_\lambda))_{\lambda \in \Lambda}$. Then*

$$\left\| I - \frac{2}{A+B} LSL \right\|_{H \to H} \leq \frac{B-A}{A+B}. \tag{6}$$

*Proof.* See [11]. ☐

Since $\frac{B-A}{A+B} < 1$, we can thus take $M := \frac{2}{A+B} LS$ for preconditioning (1), and thus in the remainder of the discussion we consider alternatively the following operator equation

$$MLu = Mf. \tag{7}$$

## 3. GMRES Method by Using Frames

First of all, for any given frame $\Psi = (\psi_\lambda)_{\lambda \in \Lambda}$ with frame bounds $A$ and $B$ and frame operator $S$, we note that since $LSL$ is a positive definite operator, we can thus define the following $LSL$-norm

$$\|f\|_{LSL} = \langle LSLf, f \rangle^{\frac{1}{2}} = \left\| (LSL)^{\frac{1}{2}} f \right\|, \ \forall f \in H,$$

with corresponding inner product

$$\langle f, g \rangle_{LSL} = \langle LSLf, g \rangle, \ \forall f, g \in H.$$

To continue, we define the recurrence sequence

$$v_{n+1} := LSLv_n - \frac{\langle LSLv_n, LSLv_n \rangle}{\langle v_n, LSLv_n \rangle} v_n$$

$$- \frac{\langle LSLv_n, LSLv_{n-1}\rangle}{\langle v_{n-1}, LSLv_{n-1}\rangle} v_{n-1}, \; n \geq 0, \tag{8}$$

with $v_{-1} = 0$, $v_0 = \frac{2}{A+B} LSLu$. For this sequence, we have some pleasant properties exhibited in the two following lemmas.

**Lemma 3.1.** *Let $u$ be the exact solution of (7) and let us define the space*

$$\mathcal{K}_n := span \left\{ \left( \frac{2}{A+B} LSL \right)^i u : \, 1 \leq i \leq n \right\} = span \left\{ (LSL)^i \, u : \, 1 \leq i \leq n \right\},$$

*then for vectors $v_i$ defined by (8), we have*

$$\{v_0, v_1, \ldots, v_{n-1}\} \subset \mathcal{K}_n. \tag{9}$$

*Proof.* We verify the claim by induction. It is obvious for $n = 1$. Assume that the theorem holds true for all $k \leq n$. For $k = n+1$, by (8) and the definition of $\mathcal{K}_n$ we get

$$v_n = LSLv_{n-1} - \frac{\langle LSLv_{n-1}, LSLv_{n-1}\rangle}{\langle v_{n-1}, LSLv_{n-1}\rangle} v_{n-1} - \frac{\langle LSLv_{n-1}, LSLv_{n-2}\rangle}{\langle v_{n-2}, LSLv_{n-2}\rangle} v_{n-2}, \tag{10}$$

where the right-hand side of (10) belongs to $LSL\mathcal{K}_n + \mathcal{K}_n \subset \mathcal{K}_{n+1}$. From here, the result follows as desired. □

The following theory further elucidates the properties of the set introduced in equation (9).

**Lemma 3.2.** *The system $\{v_0, v_1, \ldots, v_{n-1}\}$, forms an orthogonal basis for $\mathcal{K}_n$ with respect to the inner product $\langle \cdot, \cdot \rangle_{LSL}$.*

*Proof.* By virtue of (8), the theorem follows obviously for $n = 1, 2$. Now, we assume that the theorem holds for $k = n$, namely, $\langle v_n, LSLv_i\rangle = 0$ for all $i = 0, \ldots, n-1$, and that $\{v_0, v_1, \ldots, v_n\}$ is an $LSL$-orthogonal basis for $\mathcal{K}_{n+1}$. The step of $k = n+1$ can be followed immediately for $i = n-1, n$ via (8). For $i < n-1$, since $LSLv_i \in LSL\mathcal{K}_{n-1}$, induction hypothesis yields $LSLv_i = \sum_{j=0}^{n-1} c_j v_j$ for some coefficients $c_j \in \mathbb{C}$. Therefore, using the $LSL$-orthogonality of $v_i$ for $i \leq n$, we obtain for $i < n-1$,

$$\begin{aligned} \langle v_{n+1}, LSLv_i\rangle = & \left\langle LSLv_n - \tfrac{\langle LSLv_n, LSLv_n\rangle}{\langle v_n, LSLv_n\rangle} v_n \; - \tfrac{\langle LSLv_n, LSLv_{n-1}\rangle}{\langle v_{n-1}, LSLv_{n-1}\rangle} v_{n-1}, LSLv_i \right\rangle \\ = & \langle LSLv_n, LSLv_i\rangle = \left\langle LSLv_n, \textstyle\sum_{j=0}^{n-1} c_j v_j \right\rangle = 0. \end{aligned}$$

For the remainder, it turns out that $\{v_0, v_1, \ldots, v_{n-1}\}$ is indeed a basis for $\mathcal{K}_n$ since

$$n = \dim \{v_0, v_1, \ldots, v_{n-1}\} \leq \dim \mathcal{K}_n = n.$$

As we desired. □

To continue, for each $m$, we define the tridiagonal matrix $H_m = [h_{ij}]_{m+1\times m}$ such that,

$$\begin{cases} h_{ij} = 0, \text{ i}\neq j-1, j, j+1, \\ h_{j-1,j} = \frac{\langle LSLv_j, LSLv_{j-1}\rangle}{\langle v_{j-1}, LSLv_{j-1}\rangle}, \\ h_{jj} = \frac{\langle LSLv_j, LSLv_j\rangle}{\langle v_j, LSLv_j\rangle}, \\ h_{j+1,j} = 1. \end{cases}$$

We can therefore easily see that

$$v_{j+1} = (LSL)\, v_j - \sum_{i=1}^{j} h_{ij} v_i, \quad 1 \leq j \leq m.$$

From this, we can follow

$$(LSL)\, v_j = \sum_{i=1}^{j+1} h_{ij} v_i, \quad j = 1, \ldots, m,$$

and consequently

$$(LSL)\, \mathbb{V}_m = \mathbb{V}_{m+1}^T H_m, \tag{11}$$

where $\mathbb{V}_m$ denotes the $n \times m$ matrix with column vectors $v_1, \ldots, v_m$.

Here, we intend to compute an approximate solution as

$$u_m^{k+1} = u_0^k + \delta_m \in u_0^k + \mathcal{K}_m, \tag{12}$$

for some $m$, such that

$$\left(\frac{2}{A+B} LS\right) f - \left(\frac{2}{A+B} LSL\right) u_m^{k+1} \perp_{LSL} \left(\frac{2}{A+B} LSL\right) \mathcal{K}_m, \tag{13}$$

or equivalently

$$r_m^{k+1} := (LS)\, f - (LSL)\, u_m^{k+1} \perp_{LSL} (LSL)\, \mathcal{K}_m. \tag{14}$$

From (12) we see that

$$u_m^{K+1} = u_0^k + \delta_m = u_0^k + \mathbb{V}_m x_m,$$

where $x_m \in \mathbb{R}^n$. Accordingly, by (12) and by putting $r_0^k = (Ls)f - (LSL)u_0^k$, we can write

$$(LS)\, f - (LSL)\, u_m^{k+1} = r_0^k - (LSL)\, \mathbb{V}_m x_m,$$

which in view of (14) yields

$$(LSL\mathbb{V}_m)^T (LSL\mathbb{V}_m)\, x_m = (LSL\mathbb{V}_m)^T r_0^k. \tag{15}$$

On the other hand, by virtue of (11), we can derive the following relations

$$(LSL\mathbb{V}_m)^T (LSL\mathbb{V}_m) = \left(\mathbb{V}_{m+1}^T H_m\right)^T \left(\mathbb{V}_{m+1}^T H_m\right) = H_m^T H_m, \tag{16}$$

and

(17)

$$(LSL\mathbb{P}_m)^T r_0^k = \left(\mathbb{V}_{m+1}^T H_m\right)^T r_0^k = H_m^T \mathbb{V}_{m+1} r_0^k = H_m^T \mathbb{V}_{m+1} (LSL)\, (u - u_0^k),$$

which in case of $u_0^k = 0$, the relation (17) turns into

$$(LSL\mathbb{V}_m)^T r_0^k = \frac{A+B}{2} H_m^T \mathbb{V}_{m+1} v_0 = \frac{A+B}{2} H_m^T e_1. \tag{18}$$

By substituting expressions (16) and (18) into (15) we obtain

$$H_m^T H_m x_m = \frac{A+B}{2} H_m^T e_1,$$

which is actually equivalent to

$$x_m = \arg\min_{x \in \mathbb{R}^m} \left\| H_m x - \frac{A+B}{2} e_1 \right\|_{LSL}.$$

To solve the above norm minimization problem, following Saad [14], we use Givens rotations

$$\Omega_j := J\left( j+1, j, c_j := \frac{h_{jj}}{\sqrt{(h_{jj})^2 + 1}}, s_j := \frac{1}{\sqrt{(h_{jj})^2 + 1}} \right), \quad 1 \le j \le m.$$

For these rotations, we can see the following expressions

$$\Omega_m \Omega_{m-1} \cdots \Omega_1 H_m = \begin{pmatrix} R_m \\ 0 \end{pmatrix} \tag{19}$$

where $R_m = [r_{ij}]_{m \times m}$ is an upper triangular matrix with the entries

$$r_{ij} = \begin{cases} c_j h_{jj} + s_j, & i = j, \\ c_j h_{j,j+1} + s_j h_{j+1,j+1}, & i = j-1, \\ s_j h_{j+1,j+2}, & i = j-2, \\ 0, & o.w. \end{cases}$$

and also

$$\Omega_m \Omega_{m-1} \cdots \Omega_1 \left( \frac{A+B}{2} e_1 \right) = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \\ \gamma_{m+1} \end{pmatrix} = \begin{pmatrix} g_m \\ \gamma_{m+1} \end{pmatrix}, \tag{20}$$

where for $1 \le i \le m$, we have $\gamma_i = \frac{A+B}{2}(-1)^{i+1} c_i \prod_{j=0}^{i} s_j$ with $s_0 = 1$, and $\gamma_{m+1} = \prod_{i=1}^{m} s_i$.

Finally, it could be seen that $x_m = R_m^{-1} g_m$, and consequently

$$\left\| H_m x_m - \frac{A+B}{2} e_1 \right\|_{LSL} = |\gamma_{m+1}|.$$

Concerning to above discussion, if $\Psi = (\psi_\lambda)_{\lambda \in \Lambda}$ is a frame for $H$ with frame operator $S$, and $L$ be as in (1) and if $A$, $B$ are the frame bounds of the frame $L\Psi = (L(\psi_\lambda))_{\lambda \in \Lambda}$, FGMRES can be defined algorithmically as follows:

**FGMRES** $[L, S, \epsilon, A, B] \to u_\epsilon$ [1]

$i \leftarrow 0$, $u_0^i \leftarrow 0$, $v_0 \leftarrow \frac{2}{A+B} LSf$, $v_1 \leftarrow \frac{\langle LSLv_0, LSLv_0\rangle}{\langle v_0, LSLv_0\rangle} v_0$ compute $r_0^i \leftarrow (LS)f - (LSL)u_0^i$ $\left(\frac{B^4-A^4}{B^4}\right)^{i/2} \left\|r_0^i\right\|_{LSL} \geq \epsilon$ $i \leftarrow i+2$, $i \leq m$ $h_{i-1,i} \leftarrow \frac{\langle LSLv_i, LSLv_{i-1}\rangle}{\langle v_{i-1}, LSLv_{i-1}\rangle}$ $h_{ii} \leftarrow \frac{\langle LSLv_i, LSLv_i\rangle}{\langle v_i, LSLv_i\rangle}$ $h_{i+1,i} \leftarrow 1$ $i \geq 2$
$v_i := (LSL)v_{i-1} - h_{i-1,i-1}v_{i-1} - h_{i-2,i-1}v_{i-2}$ Put tridiagonal matrix $H_i \leftarrow \{h_{ji}\}_{i-1 \leq j \leq i+1, 1 \leq i \leq m}$ $x_i \leftarrow argmin \left\|H_i x - \frac{A+B}{2} e_1\right\|_{LSL}$, $u_i \leftarrow \mathbb{V}_i x_i$ $u_0^i \leftarrow u_i$
$u_\epsilon := u_0^i$ As one can clearly see, we need only compute $(h_{ij})$ for $i = j-1, j, j+1$ to obtain the matrix $H_m$. This makes it worthwhile using FGMRES rather than GMRES.

In the following example, $\mathcal{H}$ is considered to be a finite-dimensional subspace of $\underline{L}^2(-1,1)$. Since the sequence $\{x^i\}_{i=0}^{\infty}$ of functions is a linearly independent set in $\underline{L}^2(-1,1)$, it follows that $\mathcal{H} = span\{1, x, x^2, x^3, \ldots, x^{n-1}\}$ is a subspace of $\underline{L}^2(-1,1)$ of dimension $n$ in which $\{x^i\}_{i=0}^{n-1}$ is a basis. Using the Gram-Schmidt orthonormalization algorithm [14], we obtain the orthonormal basis corresponding to this basis as $\{e_1(x), e_2(x), \ldots, e_n(x)\}$. As shown in the following example, we employ the orthonormal property of the basis to create a frame. Let

$$\{f_i\}_{i=1}^{2^{n+2}-2} = \{e_1(x), e_1(x), e_2(x), e_2(x), e_2(x), e_2(x), e_n(x), ..., e_n(x)\}, \tag{21}$$

so that each $e_i(x)$ is repeated $2^i$ times. It is easy to show that $\{f_i\}_{i=1}^{2^{n+2}-2}$ is a frame for $\mathcal{H}$ with lower bound $A = 2$ and upper bound $B = 2^n$.

**Example 3.3.** *According to the previous statements, if we let $n = 7$, then $\mathcal{H} = span\{1, x, x^2, x^3, \ldots, x^6\}$ and $\{e_1(x), e_2(x), ..., e_7(x)\}$ is the corresponding orthonormal basis obtained from applying the Gram-Schmidt algorithm to the basis $\{x^i\}_{i=0}^{6}$. As mentioned earlier, $\{f_i\}_{i=1}^{2^9-2}$ is a frame for $\mathcal{H}$, and it is easy to show that the frame operator $S$ of this frame is $Sf(x) = \sum_{i=1}^{7} 2^i \langle f(x), e_i(x)\rangle e_i(x)$.*

*Now, let* $L = \begin{bmatrix} 2 & 1 & 0 & 3 & 0 & 5 & 4 \\ 1 & 6 & 3 & 0 & -4 & 7 & 1 \\ 0 & 3 & 5 & 2 & 4 & 1 & 8 \\ 3 & 0 & 2 & 8 & 0 & 0 & 3 \\ 0 & -4 & 4 & 0 & 5 & 1 & 0 \\ 5 & 7 & 1 & 0 & 1 & 4 & 5 \\ 4 & 1 & 8 & 3 & 0 & 5 & 1 \end{bmatrix}$ *and $m = 5$, then by using FGMRES algorithm and help of MATLAB coding with* $f = \begin{bmatrix} -3 & 2 & 3 & 5 & 7 & 0 & 1 \end{bmatrix}^T$ *and $\epsilon = 0.001$ we obtain,*

$$H_m = \begin{bmatrix} 6085.6 & 3011500 & 0 & 0 & 0 \\ 1 & 2168.8 & 13144 & 0 & 0 \\ 0 & 1 & 713.7 & 187140 & 0 \\ 0 & 0 & 1 & 568.01 & 7567 \\ 0 & 0 & 0 & 1 & 95.23 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ *and*

$$R_m = \begin{bmatrix} 6085.6 & 3011500 & 2.16 & 0 & 0 \\ 0 & 1668 & 13145 & 112.19 & 0 \\ 0 & 0 & 705.83 & 187140 & 10.72 \\ 0 & 0 & 0 & 302.88 & 7567.3 \\ 0 & 0 & 0 & 0 & 70.26 \end{bmatrix}.$$

*In this case, we can conclude* $u_\epsilon = \begin{bmatrix} -2.3 & 3.2 & -0.5 & 3.1 & 4 & 0.75 & -3.46 \end{bmatrix}^T$ *that is very close to the exact solution of* $Lu = f$.

In the sequel, we study the convergent of FGMRES algorithm.

3.1. **Convergence Analysis.** Here, we study the convergence of FGMRES under the already known assumption that $LSL$ is a positive definite operator, where $L$ is as in (4) and $S$ is the frame operator of a frame $(\psi_\lambda)_{\lambda\in\Lambda}$. As one may expect, the convergence rate obtained via FGMRES is directly computed by using frame bounds of $(L(\psi_\lambda))_{\lambda\in\Lambda}$.

First of all, we present here an auxiliary lemma.

**Lemma 3.4.** *[14] Let $A$ be a positive definite operator and assume that $\mathcal{L} = A\mathcal{K}$. Then a vector $u_m$ is the result of an (oblique) projection method onto $\mathcal{K}$ $LSL$-orthogonally to $\mathcal{L}$ with the starting vector $u_0^k$ if and only if it minimizes the $LSL$-norm of the residual vector $b - Au$ over $u \in u_0^k + \mathcal{K}$, i.e., if and only if*

$$\left\|b - Au_m^{k+1}\right\|_{LSL} = \min_{u\in u_0^k+\mathcal{K}} \|b - Au\|_{LSL}.$$

**Theorem 3.5.** *Let $LSL$ be as mentioned, then for each $m$ the residual vector*

$$r_m^{k+1} = (LS)f - (LSL)u_m^{k+1},$$

*of FGMRES method satisfies*

$$\left\|r_m^{k+1}\right\|_{LSL} \le \left(\frac{B^4 - A^4}{B^4}\right)^{1/2} \left\|r_0^k\right\|_{LSL}. \tag{22}$$

*Proof.* We first give the proof for the case $m = 1$. In this case, if $u_0^k = 0$, then $\mathcal{K}_1 = \langle (LSL)u\rangle = \langle (LSf)\rangle = \langle r_0^k\rangle$. We first note that by taking $u_0^k = 0$ and by considering (14) we can see

$$\begin{aligned}\left\langle r_m^{k+1}, (LSL)r_0^k\right\rangle_{LSL} &= \left\langle (LSf) - (LSL)u_m^{k+1}, (LSL)(LSf) - (LSL)(LSL)u_0^k\right\rangle_{LSL} \\ &= \left\langle (LSf) - (LSL)u_m^{k+1}, (LSL)(LSf)\right\rangle_{LSL} \\ &= \left\langle (LSf) - (LSL)u_m^{k+1}, (LSL)(LSLu)\right\rangle_{LSL} = 0\end{aligned}$$

Therefore again by (14), we have

$$\begin{aligned}\left\|r_m^{k+1}\right\|_{LSL}^2 &= \left\langle r_m^{k+1}, r_{k+1}\right\rangle_{LSL} \\ &= \left\langle r_m^{k+1}, r_0^k - \frac{\left\langle (LSL)r_0^k, r_0^k\right\rangle_{LSL}}{\left\langle (LSL)r_0^k, (LSL)r_0^k\right\rangle_{LSL}}(LSL)(LSLu)\right\rangle_{LSL}\end{aligned}$$

$$= \left\langle r_m^{k+1}, r_0^k - \frac{\langle (LSL)r_0^k, r_0^k\rangle_{LSL}}{\langle (LSL)r_0^k, (LSL)r_0^k\rangle_{LSL}}(LSL)r_0^k \right\rangle_{LSL} = \langle r_m^{k+1}, r_0^k\rangle_{LSL}$$

$$= \left\langle r_0^k - \frac{\langle (LSL)r_0^k, r_0^k\rangle_{LSL}}{\langle (LSL)r_0^k, (LSL)r_0^k\rangle_{LSL}}(LSL)r_0^k, r_0^k \right\rangle_{LSL}$$

$$= \langle r_0^k, r_0^k\rangle_{LSL} - \frac{\langle (LSL)r_0^k, r_0^k\rangle_{LSL}}{\langle (LSL)r_0^k, (LSL)r_0^k\rangle_{LSL}} \langle (LSL)r_0^k, r_0^k\rangle_{LSL}$$

$$= \|r_0^k\|_{LSL}^2 \left(1 - \frac{\langle (LSL)r_0^k, r_0^k\rangle_{LSL}}{\langle (LSL)r_0^k, (LSL)r_0^k\rangle_{LSL}} \frac{\langle (LSL)r_0^k, r_0^k\rangle_{LSL}}{\langle r_0^k, r_0^k\rangle_{LSL}}\right)$$

$$(23) \qquad = \|r_0^k\|_{LSL}^2 \left(1 - \frac{\langle (LSL)r_0^k, r_0^k\rangle_{LSL}^2}{\langle r_0^k, r_0^k\rangle_{LSL}^2} \frac{\|r_0^k\|_{LSL}^2}{\|(LSL)r_0^k\|_{LSL}^2}\right).$$

On one hand, we can deduce the following

$$(24) \qquad \frac{\|r_0^k\|_{LSL}^2}{\|(LSL)r_0^k\|_{LSL}^2} \geq \frac{\|r_0^k\|_{LSL}^2}{\|LSL\|_{LSL\to LSL}^2 \|r_0^k\|_{LSL}^2} \geq \frac{1}{B^2}.$$

On the other hand, since $(LSL)$ is a self-adjoint operator, it could be seen that

$$\frac{\langle (LSL)r_0^k, r_0^k\rangle_{LSL}}{\langle r_0^k, r_0^k\rangle_{LSL}} = \frac{\|LSLr_0^k\|^2}{\langle LSLr_0^k, r_0^k\rangle} \geq \frac{A^2 \|r_0^k\|^2}{\langle LSLr_0^k, r_0^k\rangle}$$

$$(25) \qquad = \frac{A^2}{\frac{\langle LSLr_0^k, r_0^k\rangle}{\langle r_0^k, r_0^k\rangle}} \geq \frac{A^2}{\|LSL\|_{LSL\to LSL}} \geq \frac{A^2}{B}.$$

Now by substituting (24) and (25) into (23), we arrive at the result.

For $m > 1$, we can deduce (22) because the subspace $\mathcal{K}_m$ contains the initial residual vector $r_0^k$ at each restart. Since by Lemma 25 the algorithm FGMRES minimizes the residual $LSL$-norm in the subspace $\mathcal{K}_m$, at each outer iteration, the residual $LSL$-norm will be reduced by as much as the result of one step of the case $m = 1$. Therefore, the inequality (22) is satisfied by residual vectors produced after each outer iteration and the FGMRES method converges. □

Since $\left(\frac{B^4 - A^4}{B^4}\right)^{1/2} < 1$, FGMRES converges to the exact solution of (1) for any initial guess. This convergence rate suggests also that the more closely to be to a tight frame, the faster convergence of $\{u_n\}$ to the exact solution of (1) is expected.

## 4. Conclusion

In this paper, we explored within frame theory to devise a novel iterative method for solving the operator equation $Lu = f$, where $L$ is a bounded, invertible, and self-adjoint linear operator employed in a separable Hilbert space $H$. Our methodology, which was inspired by the GMRES projective method, presented an alternate search space, broadening the boundaries of problem-solving.
Our investigation was expanded to the assessment of convergence, where we evaluated the corresponding convergence rate, which was deeply influenced by the frame bounds. This provided important insights into the efficacy of our method. We also looked at the ideal scenario, in which the equation finds an exact solution, to shed light on the practical implications of our strategy.
The amalgamation of frame theory, iterative methods, and operator equations has shown promise in enhancing our problem-solving capabilities within the realm of bounded, invertible, and self-adjoint linear operators. This study opens the door to further investigations and applications in diverse fields that rely on these foundational mathematical concepts.

## 5. Conflict of interest

The authors declare no conflict of interest.

## References


[1] Arnoldi, W. E. (1951). *The principle of minimized iterations in the solution of the matrix eigenvalue problem.* Quarterly of Applied Mathematics, 9, 17–29. https://doi.org/10.1090/QAM/2F42792

[2] Asakari Hemmat, A., & Jamali, H. (2011). *Adaptive Galerkin frame methods for solving operatoreEquation.* U.P.B. Scientific Bulletin, Serries A, 73(2), 129–138. https://api.semanticscholar.org/CorpusID:211140973

[3] Asakari Hemmat, A., & Jamali, H. (2012). *Approximated solutions to operator equations based on the frame bounds.* Journal of communication in Mathematics and Application, 3(3), 253–259. https://doi.org/10.26713/cma.v3i3.209

[4] Beylkin, G., Coifman, R.R., & Rokhlin, V. (1991). *Fast wavelet transforms and numerical algorithms I.* Communication in Pure and Applied Mathematics, 1, 141–183. https://doi.org/10.1002/CPA.3160440202

[5] Brezinski, C. (1997). *Projection Methods for System of Equations.* Elsevier, Amsterdam.

[6] Casazza, P. G. (2000). *The art of frame theory.* Taiwanese Journal of Mathematics, 4 129–201. https://doi.org/10.11650/twjm/1500407227

[7] Christensen, O. (2003). *An Introduction to Frames and Riesz Bases.* Birkhauser, Boston.

[8] Cohen, A. (2003). *Numerical Analysis of Wavelet Methods.* Elsevier.

[9] Cohen, A., & DeVore, W. (2001). *Adaptive wavelet methods for elliptic operator equations: convergence rates.* Mathematics of Computation, 70, 27–75. https://doi.org/10.1090/S0025-5718-00-01252-7

[10] Dahlke, S., Fornasier, M., & Raasch, T. (2007). *Adaptive frame methods for elliptic operator equations.* Advances in Computational Mathematics, 27, 27–63. https://doi.org/10.1007/s10444-005-7501-6

[11] Jamali, H., & Afroomand, E. (2017). *Applications of frames in Chebyshev and conjugate gradient methods.* Bulletin of the Iranian Mathematical Society, 43, 1265–1279.

[12] Jamali, H., & Ghaedi, S. (2017). *Applications of frames of subspaces in Richardson and Chebyshev methods for solving operator equations.* Mathematical Communications, 22, 13–23.
[13] Paige, c. c., & Saunders, M. A. (1975). *Solution of Sparse Indefinite Systems of Linear Equations.* SIAM Journal of Numerical Analysis, 12, 617–629. https://doi.org/10.1137/0712047
[14] Saad, Y. (2000)*Iterative Methods for Sparse Linear Systems.* PWS press, New York.
[15] Saad, Y., & Schultz M. H. (1986). *GMRES: A Generalized Minimal Residual Algorithm for Solving Nonsymmetric Linear Systems.* SIAM Journal on Scientific and Statistical Computing, 7, 856–869. https://doi.org/10.1137/0907058

Hassan Jamali
Orcid number: 0000-0003-4569-2955
Department of Mathemaics
Vali-e-Asr University of Rafsanjan
Rafsanjan, Iran
*Email address*: jamali@vru.ac.ir

Reza Pourkani
Orcid number: 0000-0002-1212-9763
Department of Mathematics
vali-e-asr University of Rafsanjan
Rafsanjan, Iran
*Email address*: r.poorkani@gmail.com