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A FIXED POINT METHOD FOR THE STABILITY OF FUNCTIONAL EQUATIONS IN PROBABILISTIC NORMED QUASI-LINEAR SPACES

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Abstract. In this article, we define probabilistic normed quasi-linear spaces and provide some introductions and examples to clarify the structure of these spaces. We then investigate the generalized Hyers-Ulam stability of the (additive) Cauchy functional equation in probabilistic normed quasi-linear spaces by using a version of the fixed point theorem.

Keywords: Cauchy functional equation, Generalized Hyers-Ulam stability, Probabilistic normed quasi-linear space.
2020 MSC: 39B82, 47H10.

1. Introduction

The problem of stability for functional equations is expressed in the way that when the functional equation is replaced by an inequality, what is the difference between the answers of the inequality and the answers of the given functional equation?

In 1940, S.M. Ulam [28] stated the following question:

Let S_1 be a group and S_2 be a metric group with metric d. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $L : S_1 \to S_2$ satisfies the following condition

$$d(L(st), L(s)L(t)) < \delta,$$

for all $s, t \in S_1$, then there is a homomorphism $T : S_1 \to S_2$ such that $d(L(s), T(s)) < \epsilon$ for all $s \in S_1$?

Hyers [14] answered the Ulam's problem about Banach spaces:

Let X_1 and X_2 be Banach spaces and $\kappa : X_1 \to X_2$ be a function that for all $\rho_1, \rho_2 \in X_1$ and some $\epsilon > 0$,

$$\|\kappa(\rho_1 + \rho_2) - \kappa(\rho_1) - \kappa(\rho_2)\| \leq \epsilon.$$

Then there exists a unique additive mapping $K : X_1 \to X_2$ such that

$$\|\kappa(\rho_1) - K(\rho_1)\| \leq \epsilon,$$

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for all $\rho_1 \in X_1$.

Rassias [23] and Gavruta [12] generalized this theorem, and other researchers have presented approaches for the generalized Hyers-Ulam stability problem. For example, see [22] for more information on this topic and its generalization. The stability of the functional equations has been investigated by several researchers directly or using the fixed point alternative, as seen in [11], [15]– [24].

We know that fixed point theory is one of the most useful parts of pure mathematics in other aspects of mathematics and science, especially in economics and solving differential equations. One of the applications of fixed point theorems is to investigate the problem of various functional equation stabilities. For example, in [4], Baker used the Banach fixed point theorem to give Hyers-Ulam stability results for a nonlinear functional equation. In many physical problems, we come across nonlinear integral equations. The fixed point theory plays a significant role to obtain the solutions of such equations. In [13], the existence of a solution for nonlinear functional integral equations is investigated with the help of generalized Darbo's fixed point theorem, and an example for the application of the obtained results in the theory of integral equations is presented. In [26] has been proved the existence of the solution of non-linear functional integral equations in Banach algebra $C([0,a] \times [0,a]), a > 0$ that is used the measure of noncompactness on $C([0,a] \times [0,a])$ and a fixed point theorem, which is a generalization of Darbo's fixed point theorem for the product of operators and illustrates result with the help of an interesting example. In [27], the Petryshyn's fixed point theorem is used to prove the existence theorem for functional-integral equations and several illustrative examples.

A suitable survey of stability results for the Cauchy equation can be found in [6]. In 1987, Alsina [2] investigated the Ulam-type stability of functional equations in probabilistic normed spaces. In 2008, Mihet and Radu [19] proved the stability results for the Cauchy and Jensen functional equations in random normed spaces using the fixed point method. The stability of various functional equations was also investigated in random normed spaces, as seen in [5,8,19,20].

In this paper, we define probabilistic normed quasi-linear spaces and investigate the generalized Hyers-Ulam stability of the Cauchy functional equation in probabilistic normed quasi-linear spaces using the fixed point method.

2. **Preliminaries**

The functional equation $\kappa(\rho_1 + \rho_2) = \kappa(\rho_1) + \kappa(\rho_2)$ is called the Cauchy functional equation and any function that satisfies it, is called an additive mapping.

In 2003, Radu [22] used a version of the fixed point theorem to prove the stability of the Cauchy functional equation. Here, we present a modified version of the main result in [10], which is a fixed point theorem.

Theorem 2.1. *[1] Let (F, d) be a complete generalized metric space and let $I : F \to F$ be a contraction map with a Lipschitz constant $0 < \mu < 1$. Then* *for each given element $a \in F$, either $d(I^n a, I^{n+1}a) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*

(1) *$d(I^n a, I^{n+1}a) < \infty$ for all $n > n_0$;*
(2) *the sequence $\{I^n a\}$ converges to a fixed point a^* of I;*
(3) *a^* is the unique fixed point of I in the set $F^* := \{b \in F | d(I^{n_0}a, b) < \infty\}$;*
(4) *$d(b, a^*) \leq \frac{1}{1-\mu} d(b, Ib)$ for all $b \in F^*$.*

Park [21] and Rassias [24] used this theorem for solving the Cauchy functional equations. The problem of the stability of the additive Cauchy functional equation in random normed spaces has been investigated, see [19, 20]. In this paper by using Theorem 2.1, we prove the generalized Hyers-Ulam stability of the Cauchy functional equation in probabilistic normed quasi-linear spaces.

Throughout this article, $\mathcal{D}$ is the space of all distribution functions; that is, the space of all mappings $\mathtt{F} : \mathbb{R} \to [0, 1]$, such that $\mathtt{F}$ is left-continuous, nondecreasing, with $\inf \mathtt{F} = 0$ and $\sup \mathtt{F} = 1$. For example, ϵ_0 is the specific distribution function defined by

$$\epsilon_0(u) = \begin{cases} 0 & u \leq 0, \\ 1 & u > 0 \end{cases}$$

In 1942, Menger proposed a generalization of metric spaces, which is called a probabilistic metric space.

Definition 2.2. [7,25] A probabilistic metric space (briefly, a PM-space) is an ordered pair $(P, \mathtt{F})$, where P is an abstract set and $\mathtt{F}$ is a mapping from $P \times P$ into $\mathcal{D}$. We denote the distribution function $\mathtt{F}(p_1, p_2)$ by F_{p_1,p_2}. The functions F_{p_1,p_2} are assumed to satisfy the following conditions:

(1) $F_{p_1,p_2} = \epsilon_0$ if and only if $p_1 = p_2$;
(2) $F_{p_1,p_2}(0) = 0$;
(3) $F_{p_1,p_2} = F_{p_2,p_1}$;
(4) if $F_{p_1,p_2}(\upsilon) = 1$ and $F_{p_1,p_2}(\nu) = 1$, then $F_{p_1,p_2}(\upsilon + \nu) = 1$.

A function $\tau : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (briefly, a t-norm), if τ satisfies the following conditions:

(1) $\tau(\iota_1, \iota_2) = \tau(\iota_2, \iota_1)$;
(2) $\tau(\tau(\iota_1, \iota_2), \iota_3) = \tau(\iota_1, \tau(\iota_2, \iota_3))$;
(3) $\tau(\iota_1, 1) = \iota_1$;
(4) $\tau(\iota_1, \iota_2) \leq \tau(\iota_3, \iota_4)$ whenever $\iota_1 \leq \iota_3$ and $\iota_2 \leq \iota_4$;

for all $\iota_1, \iota_2, \iota_3, \iota_4 \in [0, 1]$.

For example, $\tau_m(\iota_1, \iota_2) = \min\{\iota_1, \iota_2\}$ is a continuous t-norm.

Definition 2.3. A probabilistic normed space (briefly, a PN-space) is an ordered triple $(P, \mathtt{F}, \tau)$, where P is a vector space, τ is a continuous t-norm and F is a mapping from P into $\mathcal{D}$ (we shall denote $\mathtt{F}(p_1)$ by F_{p_1}) satisfying the following conditions:

(1) $F_{p_1}(\upsilon) = \epsilon_0(\upsilon)$ for all $\upsilon > 0$ if and only if $p_1 = 0$;
(2) $F_{\lambda p_1}(\upsilon) = F_{p_1}(\frac{\upsilon}{|\lambda|})$ for all $p_1 \in P, \lambda \neq 0$;
(3) if $F_{p_1}(\upsilon) = 1$ and $F_{p_2}(\nu) = 1$, then $F_{p_1+p_2}(\upsilon + \nu) = 1$ for all $p_1, p_2 \in P$ and $\upsilon, \nu \geq 0$.

If for all $p_1, p_2 \in P$ and $\upsilon, \nu \geq 0$, $F_{p_1+p_2}(\upsilon + \nu) \geq \tau(F_{p_1}(\upsilon), F_{p_2}(\nu))$, then P is called a Menger PN-space.

Various authors have generalized metric spaces, including Aseev [3], who introduced quasi-linear spaces, and Dehghanizade and Modarres [9], who defined quotient spaces. In the following, we will provide a reminder of the definition of quasi-linear spaces.

Definition 2.4. [3, 30] A set Q is called a quasi-linear space (briefly, QLS) if a partial order relation $\preceq$, an algebraic addition operation and an operation of multiplication by real numbers are defined in it, such that for any elements $\xi, \zeta, \alpha, \gamma \in Q$ and any $\lambda, \theta \in \mathbb{R}$ the following conditions hold:

(1) $\xi \preceq \xi$;
(2) if $\xi \preceq \zeta$ and $\zeta \preceq \alpha$, then $\xi \preceq \alpha$;
(3) if $\xi \preceq \zeta$ and $\zeta \preceq \xi$. then $\xi = \zeta$;
(4) $\xi + \zeta = \zeta + \xi$;
(5) $\xi + (\zeta + \gamma) = (\xi + \zeta) + \gamma$;
(6) for each ξ there exists an zero element $\eta \in Q$ such that $\xi + \eta = \xi$;
(7) $\lambda.(\theta.\xi) = (\lambda.\theta).\xi$;
(8) $\lambda.(\xi + \zeta) = \lambda.\xi + \lambda.\zeta$;
(9) $1.\xi = \xi$;
(10) $0.\xi = \eta$;
(11) $(\lambda + \theta).\xi \preceq \lambda.\xi + \theta.\xi$;
(12) if $\xi \preceq \zeta$ and $\alpha \preceq \gamma$, then $\xi + \alpha \preceq \zeta + \gamma$;
(13) if $\xi \preceq \zeta$, then $\lambda.\xi \preceq \lambda.\zeta$.

A QLS Q with the partial order relation $\preceq$ is denoted by $(Q, \preceq)$ where η is the zero element of Q.

An example of quasi-linear space, that is not a linear space is $\Omega(\mathbb{R})$, is the set of all nonempty closed bounded subsets of real numbers. The algebraic addition operation on $\Omega(\mathbb{R})$ is defined by the expression

$$A + B = \overline{\{\iota + \varsigma : \iota \in A, \varsigma \in B\}},$$

and multiplication by a real number $\lambda \in \mathbb{R}$ is defined by $\lambda.A = \{\lambda.\iota : \iota \in A\}$. The partial order relation on $\Omega(\mathbb{R})$ is given by inclusion.

An element $p^{'} \in Q$ is called an additive inverse of $p \in Q$ if $p + p^{'} = \eta$. The additive inverse of an element is unique if it exists.

If any element in the quasi-linear space Q has an additive inverse element in Q, then the partial order on Q is determined by equality and consequently, Q is a linear space (see Lemma 2 in [3]).

Definition 2.5. [3] Let $(Q, \preceq)$ be a QLS. A real function $\| \, . \, \|_Q : Q \to \mathbb{R}$ is called a QN-norm, if the following conditions hold:

(1) $\|\xi\|_Q > 0$ if $\xi \neq \eta$;
(2) $\|\xi + \zeta\|_Q \leq \|\xi\|_Q + \|\zeta\|_Q$;
(3) $\|\lambda.\xi\|_Q = |\lambda|.\|\xi\|_Q$;
(4) if $\xi \preceq \zeta$, then $\|\xi\|_Q \leq \|\zeta\|_Q$;
(5) if for any $\epsilon > 0$ there exists an element $\xi_\epsilon \in Q$ such that $\xi \preceq \zeta + \xi_\epsilon$ and $\|\xi_\epsilon\|_Q \leq \epsilon$, then $\xi \preceq \zeta$.

A QLS Q with a norm defined on it is called normed quasi-linear space.

For example, $\Omega(\mathbb{R})$ is a normed quasi-linear space. The norm on $\Omega(\mathbb{R})$ is defined by

$$\|A\|_\Omega := \sup_{\iota \in A} |\iota|.$$

If Q is a normed quasi-linear space, the Hausdorff metric h_Q on Q is defined by

$$h_Q(\xi, \zeta) = inf\{y \geq 0 : \exists a_i^y \in Q,\ \xi \preceq \zeta + a_1^y,\ \zeta \preceq \xi + a_2^y,\ \|a_i^y\| \leq y, i = 1, 2\}.$$

3. Probabilistic normed quasi-linear spaces

Definition 3.1. A probabilistic normed quasi-linear space (briefly, a PNQ-space) is an ordered triple $(P, \preceq, \mathtt{F})$, where P is a quasi-linear space and $\mathtt{F}$ is a mapping from P into $\mathcal{D}$ (we denote the distribution function $\mathtt{F}(p_1)$ by F_{p_1}) satisfying the following conditions:

(1) $F_{p_1}(\upsilon) = \epsilon_0(\upsilon)$ for all $\upsilon > 0$ if and only if $p_1 = \eta$;
(2) $F_{\lambda p_1}(\upsilon) = F_{p_1}(\frac{\upsilon}{|\lambda|})$ for all $p_1 \in P, \lambda \neq 0$;
(3) if $F_{p_1}(\upsilon) = 1$ and $F_{p_2}(\nu) = 1$, then $F_{p_1+p_2}(\upsilon + \nu) = 1$ for all $p_1, p_2 \in P$ and $\upsilon, \nu \geq 0$;
(4) if $p_1 \preceq p_2$, then $F_{p_1}(\upsilon) \geq F_{p_2}(\nu)$;
(5) if for any $\epsilon > 0$ there exists an element $p_\epsilon \in P$ such that $p_1 \preceq p_2 + p_\epsilon$ and $F_{p_\epsilon}(\epsilon) = 1$, then $p_1 \preceq p_2$.

A PNQ-space P is called a Menger PNQ-space, if for all $p_1, p_2 \in P$ and $\upsilon, \nu \geq 0$ the following holds:

$$F_{p_1+p_2}(\upsilon + \nu) \geq \tau(F_{p_1}(\upsilon), F_{p_2}(\nu)),$$

where τ is a t-norm.

It is clear that if any element p in a PNQ-space P has an additive inverse $p' \in P$, then P is a PN-space.

An example of a PNQ-space is $(\Omega(\mathbb{R}), \subseteq, F)$ where F is defined by

$$F_B(\upsilon) := \frac{\upsilon}{\upsilon + \|B\|_\Omega},$$

for all $B \in \Omega(\mathbb{R})$ and $\upsilon > 0$.

Theorem 3.2. *Suppose* $(P, \preceq, \mathtt{F})$ *is a PNQ-space and* $\mathtt{F}$ *takes its values in* $\mathcal{D}_0 = \{\mathtt{F} \in \mathcal{D} : \mathtt{F}^{-1}\{1\} \neq \emptyset\}$. *If we define the function* $\|.\|_q$ *by*

$$\|p\|_q := \inf\{\upsilon \geq 0 : F_p(\upsilon) = 1\},$$

for all $p \in P$, *then* $\|.\|_q$ *is a QN-norm on* P *and* $(P, \mathtt{F}, \|.\|_q)$ *is a QN-space.*

Proof. $\|.\|_q$ satisfies the conditions (1), (2) and (3) of Definition 2.5, for details of the proof see the proof of Theorem 2.8.1 of [7]. Now, we check conditions (4) and (5):

Suppose that $p_1 \preceq p_2$. Then $F_{p_1}(\upsilon) \geq F_{p_2}(\upsilon)$. Consequently,

$$\{\upsilon \geq 0 | F_{p_2}(\upsilon) = 1\} \subseteq \{\upsilon \geq 0 | F_{p_1}(\upsilon) = 1\},$$

and so $\|p_1\|_q \leq \|p_2\|_q$.

Let $\epsilon > 0$ be given and assume that there exists an element $p_\epsilon \in P$ such that $p_1 \preceq p_2 + p_\epsilon$ and $\|p_\epsilon\|_q \leq \epsilon$. Then $F_{p_\epsilon}(\epsilon) = 1$ and so $p_1 \preceq p_2$. □

Now, let P be a PNQ-space. We define a metric on P by

$$G_{p_1,p_2}(\upsilon) := \sup\{r \in [0,1] : \exists a_i^r \in P, p_1 \preceq p_2 + a_1^r, p_2 \preceq p_1 + a_2^r, F_{a_i^r}(\upsilon) \geq r, i = 1, 2\}.$$

Since $p_1 \preceq p_2 + (p_1 - p_2)$ and $p_2 \preceq p_1 + (p_2 - p_1)$, the quantity $G_{p_1,p_2}(\upsilon)$ is defined, for any elements $p_1, p_2 \in P$, and $G_{p_1,p_2}(\upsilon) \geq F_{p_1-p_2}(\upsilon)$ for all $\upsilon \geq 0$. It is straightforward to see that the function G_{p_1,p_2} is a distribution function and satisfies all the axioms of a probabilistic metric.

To prove the main theorem, we need the following lemma.

Lemma 3.3. *For all nonzero* $\beta \in \mathbb{R}$ *and* $p_1, p_2 \in P$, $G_{\beta p_1,\beta p_2}(\upsilon) = G_{p_1,p_2}(\frac{\upsilon}{|\beta|})$.

Proof. Suppose $\epsilon > 0$ is given. There exist $a_1^r, a_2^r \in P$ such that

$$\beta p_1 \preceq \beta p_2 + a_1^r,\ \beta p_2 \preceq \beta p_1 + a_2^r,\ F_{a_i^r}(\upsilon) \geq G_{\beta p_1,\beta p_2}(\upsilon) - \epsilon.$$

Therefore,

$$p_1 \preceq p_2 + \frac{a_1^r}{\beta},\ p_2 \preceq p_1 + \frac{a_2^r}{\beta},\ F_{\frac{a_i^r}{\beta}}(\upsilon) = F_{a_i^r}(|\beta|\upsilon) \geq G_{\beta p_1,\beta p_2}(|\beta|\upsilon) - \epsilon.$$

Thus, $G_{p_1,p_2}(\upsilon) \geq G_{\beta p_1,\beta p_2}(|\beta|\upsilon) - \epsilon$ and by replacing u by $\frac{\upsilon}{|\beta|}$, we have

$$G_{p_1,p_2}(\frac{\upsilon}{|\beta|}) \geq G_{\beta p_1,\beta p_2}(\upsilon) - \epsilon. \tag{1}$$

Also, for each $\epsilon > 0$, there exist $b_1^r, b_2^r \in P$ such that

$$p_1 \preceq p_2 + b_1^r,\ p_2 \preceq p_1 + b_2^r,\ F_{b_i^r}(\frac{\upsilon}{|\beta|}) \geq G_{p_1,p_2}(\frac{\upsilon}{|\beta|}) - \epsilon.$$

So

$$\beta p_1 \preceq \beta p_2 + \beta b_1^r,\ \beta p_2 \preceq \beta p_1 + \beta b_2^r,\ F_{\beta b_i^r}(\upsilon) = F_{b_i^r}(\frac{\upsilon}{|\beta|}) \geq G_{p_1,p_2}(\frac{\upsilon}{|\beta|}) - \epsilon.$$

Consequently,

$$G_{\beta p_1,\beta p_2}(\upsilon) \geq G_{p_1,p_2}(\frac{\upsilon}{|\beta|}) - \epsilon. \tag{2}$$

Letting $\epsilon \to 0$ in (1) and (2) we complete the proof. □

Definition 3.4. A sequence $\{p_n\}$ in a PNQ-space P is said to converge to a $p \in P$ (we write $p_n \to p$) if for every $\epsilon > 0$ and $\lambda > 0$, there exists an integer $M_{\epsilon,\lambda}$ such that $G_{p_n,p}(\epsilon) > 1 - \lambda$ whenever $n > M_{\epsilon,\lambda}$.

Lemma 3.5. *Let P be a PNQ-space.*

a) The operation of multiplication by real numbers is continuous according to the probabilistic metric.

b) Suppose that $p_n \to p_0$ and $r_n \to p_0$. If $p_n \preceq q_n \preceq r_n$, for all n, then $q_n \to p_0$.

Proof. Suppose that $p_n \to p$ and $\beta \in \mathbb{R}$ is given. Then, for every $\epsilon > 0$ and $\lambda > 0$, there exists an integer $M_{\epsilon,\lambda}$ such that $G_{p_n,p}(\frac{\epsilon}{|\beta|}) > 1 - \lambda$ whenever $n > M_{\epsilon,\lambda}$. By Lemma 3.3, $G_{\beta p_n,\beta p}(\epsilon) > 1 - \lambda$. Hence, the operation of multiplication by real numbers is continuous. The proof b) is analogous. □

Lemma 3.6. *Let P be a PNQ-space such that for all $p_1, p_2 \in P$ and $\lambda, \upsilon, \nu > 0$, if $F_{p_1}(\upsilon) > 1 - \lambda$ and $F_{p_2}(\nu) > 1 - \lambda$, then $F_{p_1+p_2}(\upsilon + \nu) > 1 - \lambda$.*

a) The operation of algebraic sum is continuous.

b) If $p_n + q_n \to p$ and $q_n \to 0$, then $p_n \to p$.

Proof. Suppose that $p_n \to p$ and $q_n \to q$. Then for any $\epsilon > 0$ and any $\lambda > 0$ there exists an integer $M_{\epsilon,\lambda}$ such that the following conditions hold for $n > M_{\epsilon,\lambda}$:

$$p_n \preceq p + a_1, \ p \preceq p_n + a_2, \ F_{a_i}(\frac{\epsilon}{2}) \geq 1 - \lambda,$$

$$q_n \preceq q + b_1, \ q \preceq q_n + b_2, \ F_{b_i}(\frac{\epsilon}{2}) \geq 1 - \lambda.$$

Consequently, $F_{a_i+b_i}(\epsilon) \geq 1 - \lambda$ and therefore the operation of algebraic sum is continuous. The proof b) is analogous. □

4. Stability of the Cauchy functional equation

Throughout this section, Y is a linear space and P is a Menger complete probabilistic normed quasi-linear space with t-norm τ_M.

Theorem 4.1. *Assume that a mapping $\kappa : Y \to P$ with $\kappa(0) = \eta$ and a symetric mapping $\psi : Y \times Y \to \mathcal{D}$ satisfy the following conditions: For all $\rho_1, \rho_2 \in Y$ and $\upsilon > 0$,*

$$G_{\kappa(2\rho_1),2\kappa(\rho_1)}(\upsilon) \geq G_{\kappa(2\rho_1),\kappa(\rho_1)+\kappa(\rho_1)}(\upsilon), \tag{3}$$

$$\exists \beta \in (0, 2) : \psi(2\rho_1, 2\rho_2)(\beta\upsilon) \geq \psi(\rho_1, \rho_2)(\upsilon), \tag{4}$$

$$G_{\kappa(\rho_1+\rho_2),\kappa(\rho_1)+\kappa(\rho_2)} \geq \psi(\rho_1,\rho_2). \tag{5}$$

Then there exists a unique additive mapping $K : Y \to P$ such that for all $\rho_1 \in Y$ and all $\upsilon > 0$,

$$G_{\kappa(\rho_1),K(\rho_1)}(\upsilon) \geq \psi(\rho_1,\rho_1)((2-\beta)\upsilon).$$

Moreover,

$$\begin{aligned} K(\rho_1) &= \lim_{n\to\infty} \frac{\kappa(2^n\rho_1)}{2^n}, \\ K(2\rho_1) &= 2K(\rho_1). \end{aligned}$$

Proof. Consider the set $F = \{K | K : Y \to P, K(0) = \eta\}$ and define the generalized metric on F as follows:

$d_G(g_1, g_2) := \inf\{\rho \geq 0 | G_{g_1(\rho),g_2(\rho)}(\rho\upsilon) \geq \psi(\rho,\rho)(2\upsilon), \mathit{for\ all}\ \rho \in Y\ \mathit{and}\ \upsilon > 0\}$.

It is easy to show that d_G is a complete generalized metric on F (see [19]).

We consider the linear mapping $I : F \to F$ such that for all $\rho \in Y$,

$$I(K(\rho)) := \frac{1}{2}K(2\rho).$$

Let g_1, g_2 in F be given such that $d_G(g_1, g_2) < \epsilon$. Then for all $\rho \in Y, \upsilon > 0$

$$G_{g_1(\rho),g_2(\rho)}(\epsilon\upsilon) \geq \psi(\rho,\rho)(2\upsilon).$$

Now, if we replace ρ with 2ρ and υ with $\mu\upsilon$, then we have

$$G_{Ig_1(\rho),Ig_2(\rho)}(\frac{\beta}{2}\epsilon\upsilon) \geq \psi(2\rho,2\rho)(2\beta\upsilon).$$

Hence by (4), $G_{Ig_1(\rho),Ig_2(\rho)}(\frac{\mu}{2}\epsilon\upsilon) \geq \psi(\rho,\rho)(2\upsilon)$. Consequently, I is a strictly contractive mapping with the Lipschitz constant $\mu = \frac{\beta}{2}$.

By putting $\rho_2 = \rho_1$ in (5) and using condition (3), we obtain

$$G_{\kappa(2\rho_1),2\kappa(\rho_1)}(\upsilon) \geq \psi(\rho_1,\rho_1)(\upsilon),$$

for all $\rho_1 \in Y, \upsilon > 0$. Therefore by Lemma 3.3,

$$G_{\frac{\kappa(2\rho_1)}{2},\kappa(\rho_1)}(\frac{\upsilon}{2}) \geq \psi(\rho_1,\rho_1)(\upsilon).$$

Replacing υ by 2υ, we obtain $G_{\kappa(\rho_1),I\kappa(\rho_1)}(\upsilon) \geq \psi(\rho_1,\rho_1)(2\upsilon)$. It follows that $d_G(\kappa, I\kappa) \leq 1 < \infty$.

By Theorem 2.1, there exists a mapping $K : Y \to P$ which is a fixed point of I, i.e., $K(2\rho_1) = 2K(\rho_1)$, such that $I^n\kappa \to K$.

Moreover by Theorem 2.1, $d_G(\kappa, K) \leq \frac{1}{1-\frac{\beta}{2}}d(\kappa, I\kappa)$ and because $d_G(\kappa, I\kappa) \leq 1$, we obtain

$$G_{\kappa(\rho_1),K(\rho_1)}(\frac{2}{2-\beta}\upsilon) \geq \psi(\rho_1,\rho_1)(2\upsilon).$$

Hence for all $\rho_1 \in$ and $\upsilon > 0$,

$$G_{\kappa(\rho_1),K(\rho_1)}(\upsilon) \geq \psi(\rho_1,\rho_1)((2-\beta)\upsilon).$$

Since $d_G(I^n\kappa, K) \to 0$, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d_G(I^n\kappa, K) < \epsilon$. Consequently, $G_{I^n\kappa(\rho_1),K(\rho_1)}(\upsilon) \geq \psi(\rho_1,\rho_1)(\frac{2\upsilon}{\epsilon})$. By letting $\epsilon \to 0$, we get $G_{I^n\kappa(\rho_1),K(\rho_1)}(\upsilon) = 1$ for all $n \geq N$ and $\upsilon > 0$. Therefore,

$$K(\rho_1) = \lim_{n\to\infty} \frac{k(2^n\rho_1)}{2^n}.$$

Since τ_M is continuous, the mapping $z \to F_z$ is continuous (see [25], Chapter 12). Therefore, it follows from conditions (4), (5) that

$$\begin{aligned}
G_{K(\rho_1+\rho_2),K(\rho_1)+K(\rho_2)}(\upsilon) &= \lim_{n\to\infty} G_{\frac{\kappa(2^n(\rho_1+\rho_2))}{2^n},\frac{\kappa(2^n(\rho_1))}{2^n}+\frac{\kappa(2^n(\rho_2))}{2^n}}(\upsilon) \\
&= \lim_{n\to\infty} G_{\kappa(2^n(\rho_1+)\rho_2),k(2^n\rho_1)+k(2^n\rho_2)}(2^n\upsilon) \\
&\geq \lim_{n\to\infty} \psi(2^n\rho_1, 2^n\rho_2)(2^n\upsilon) \\
&\geq \lim_{n\to\infty} \psi(\rho_1,\rho_2)((\frac{2}{\beta})^n\upsilon) = 1.
\end{aligned}$$

This implies K is an additive mapping.

Finally, we prove that K is unique. Since the integer n_0 in Theorem 2.1 is 0, K is the unique fixed point of I in the set $F^* := \{K \in F | d(\kappa, K) < \infty\}$. Assume $K_1 : Y \to P$ is an additive mapping such that

$$G_{\kappa(\rho_1),K_1(\rho_1)}(\upsilon) \geq \psi(\rho_1,\rho_1)((2-\beta)\upsilon),$$
$$K_1(2\rho_1) = 2K_1(\rho_1),$$

for all $\rho_1 \in Y, \upsilon > 0$. Then $d(\kappa, K_1) < \infty$. On the other hands,

$$J(K_1(\rho_1)) = \frac{1}{2}K_1(2\rho_1) = K_1(\rho_1).$$

This shows that K_1 is a fixed point of I in $F^* := \{K \in F | d(\rho_1, K) < \infty\}$ and so $K_1 = K$. □

Corollary 4.2. *Let $p < 1$ and θ be nonnegative real numbers. Let Y be a normed vector space with norm $\|.\|$ and let $\kappa : Y \to P$ with $\kappa(0) = \eta$ be a mapping satisfying*

$$G_{\kappa(2\rho_1),2\kappa(\rho_1)}(\upsilon) \geq G_{\kappa(2\rho_1),\kappa(\rho_1)+\kappa(\rho_1)}(\upsilon),$$
$$G_{\kappa(\rho_1+\rho_2),\kappa(\rho_1)+\kappa(\rho_2)} \geq \frac{\upsilon}{\upsilon + \theta(\|\rho_1\|^p + \|\rho_2\|^p)},$$

for all $\rho_1, \rho_2 \in Y$ and all $\upsilon > 0$. Then there exists a unique additive mapping $K : Y \to P$ such that,

$$G_{\kappa(\rho_1),K(\rho_1)}(\upsilon) \geq \frac{(2-2^p)\upsilon}{(2-2^p)\upsilon + 2\theta\|\rho_1\|^p},$$

for all $\rho_1 \in Y$ and $\upsilon > 0$.

Proof. According to Theorem 4.1, for all $\rho_1, \rho_2 \in X$ and $\upsilon > 0$, by putting $\beta = 2^p$ and $\psi(\rho_1, \rho_2)(\upsilon) = \frac{\upsilon}{\upsilon + \theta(\|\rho_1\|^p + \|\rho_2\|^p)}$, the result is obtained. □

Theorem 4.3. *Assume that a mapping $\kappa : Y \to P$ with $\kappa(0) = \eta$ and symetric mapping $\psi : Y \times Y \to \mathcal{D}$ satisfy the following conditions: For all $\rho_1, \rho_2 \in Y$ and $\upsilon > 0$,*

$$G_{\kappa(2\rho_1), 2\kappa(\rho_1)}(\upsilon) \geq G_{\kappa(2\rho_1), \kappa(\rho_1)+\kappa(\rho_1)}(\upsilon), \tag{6}$$

$$\exists \beta > 2 : \psi(\rho_1, \rho_2)(\upsilon) \geq \psi(2\rho_1, 2\rho_2)(\beta\upsilon), \tag{7}$$

$$G_{\kappa(\rho_1+\rho_2), \kappa(\rho_1)+\kappa(\rho_2)} \geq \psi(\rho_1, \rho_2). \tag{8}$$

Then there is a unique additive mapping $K : Y \to P$ such that for all $\rho_1 \in Y, \upsilon > 0$,

$$G_{\kappa(\rho_1), K(\rho_1)}(\upsilon) \geq \psi(\rho_1, \rho_1)((\beta - 2)\upsilon).$$

Moreover,

$$\begin{aligned} K(\rho_1) &= \lim_{n\to\infty} 2^n \kappa(\frac{\rho_1}{2^n}), \\ K(2\rho_1) &= 2K(\rho_1). \end{aligned}$$

Proof. Consider the set $F = \{K | K : Y \to P, K(0) = \eta\}$ and introduce the complete generalized metric d_G on F by

$$d_G(g_1, g_2) := \inf\{\rho_1 \geq 0 | G_{g_1(a), g_2(\rho_1)}(\rho_1\upsilon) \geq \psi(\rho_1, \rho_1)(2\upsilon), for\ all\ \rho_1 \in Y\ and\ \upsilon > 0\}.$$

We consider the linear mapping $I : F \to F$ defined by

$$I(K(\rho_1)) := 2K(\frac{\rho_1}{2}),$$

for all $\rho_1 \in Y$.

Let g_1, g_2 in F be given such that $d_G(g_1, g_2) < \epsilon$. Then for all $\rho_1 \in Y, \upsilon > 0$

$$G_{g_1(\rho_1), g_2(\rho_1)}(\epsilon\upsilon) \geq \psi(\rho_1, \rho_1)(2\upsilon).$$

Now, if we replace ρ_1 with $\frac{\rho_1}{2}$ and υ with $\frac{\upsilon}{\beta}$, then by Lemma 3.3, we have

$$G_{Ig_1(\rho_1), Ig_2(\rho_1)}(\frac{2\epsilon}{\beta}\upsilon) \geq \psi(\frac{\rho_1}{2}, \frac{\rho_1}{2})(\frac{2\upsilon}{\beta}).$$

Hence by (7), $G_{Ig_1(a), Ig_2(a)}(\frac{2\epsilon}{\beta}\upsilon) \geq \psi(\rho_1, \rho_1)(2\upsilon)$ and I is a strictly contractive mapping with the Lipschitz constant $\mu = \frac{2}{\beta}$.

Putting $\rho_2 = \rho_1$ in (8) and by replacing υ with $\frac{2\upsilon}{\beta}$ in (6), we have

$$G_{\kappa(2\rho_1), 2\kappa(\rho_1)}(\frac{2\upsilon}{\beta}) \geq \psi(\rho_1, \rho_1)(\frac{2\upsilon}{\beta}).$$

Now, by putting $\rho_2 = \rho_1$ and by replacing υ with $\frac{2\upsilon}{\beta}$ in (7), we get

$$G_{\kappa(2\rho_1),2\kappa(\rho_1)}(\frac{2\upsilon}{\beta}) \geq \psi(2\rho_1, 2\rho_1)(2\upsilon).$$

Finally, by replacing ρ_1 by $\frac{\rho_1}{2}$, we obtain

$$G_{\kappa(\rho_1),I(\kappa(\rho_1))}(\frac{2\upsilon}{\beta}) \geq \psi(\rho_1, \rho_1)(2\upsilon).$$

It follows that $d_G(\kappa, I\kappa) \leq \frac{2}{\beta}$.

According to Theorem 2.1, there exists a mapping $K : Y \to P$, which is the unique fixed point of I in the set $F^* := \{K \in F | d(\kappa, K) < \infty\}$, such that $I^n\kappa \to K$ and

$$d_G(\kappa, K) \leq \frac{1}{1 - \frac{2}{\beta}} d(\kappa, I\kappa) \leq \frac{2}{\beta - 2}.$$

Hence by replacing υ with $\frac{\beta - 2}{2}\upsilon$, we get for all $\rho_1 \in Y, \upsilon > 0$,

$$G_{\kappa(\rho_1),K(\rho_1)}(\upsilon) \geq \psi(\rho_1, \rho_1)((\beta - 2)\upsilon).$$

The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 4.4. *Let $p > 1$ and θ be nonnegative real numbers. Let Y be a normed vector space with norm $\|.\|$ and let $\kappa : Y \to P$ with $\kappa(0) = \eta$ be a mapping satisfying*

$$G_{\kappa(2\rho_1),2\kappa(\rho_1)}(\upsilon) \geq G_{\kappa(2\rho_1),\kappa(\rho_1)+\kappa(\rho_1)}(\upsilon),$$

$$G_{\kappa(\rho_1+\rho_2),\kappa(\rho_1)+\kappa(\rho_2)} \geq \frac{\upsilon}{\upsilon + \theta(\|\rho_1\|^p + \|\rho_2\|^p)},$$

for all $\rho_1, \rho_2 \in Y$ and all $\upsilon > 0$. Then there exists a unique additive mapping $K : Y \to P$ such that $K(\rho_1) = \lim_{n\to\infty} 2^n k(\frac{\rho_1}{2^n})$ and

$$G_{\kappa(\rho_1),K(\rho_1)}(\upsilon) \geq \frac{(2^p - 2)\upsilon}{(2^p - 2)\upsilon + 2\theta\|\rho_1\|^p},$$

for all $\rho_1 \in Y$ and all $\upsilon > 0$.

Proof. According to Theorem 4.3, for all $\rho_1, \rho_2 \in Y$ and all $\upsilon > 0$, by putting $\beta = 2^p$ and $\psi(\rho_1, \rho_2)(\upsilon) = \frac{\upsilon}{\upsilon + \theta(\|\rho_1\|^p + \|\rho_2\|^p)}$, the result is obtained. □

The reader can show the corollaries corresponding to Corollaries 4.2 and 4.4 in [5].

5. Conclusion

We have defined probabilistic normed quasi-linear spaces and we have proved the generalized Hyers-Ulam stability of the Cauchy functional equation in probabilistic normed quasi-linear spaces by using a version of the fixed point theorem.

References

[1] Agarwal, R. P., Meehan, M., & O'Regan, D. (2001). Fixed Point Theory and Applications, Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9780511543005

[2] Alsina, C. (1987). On the stability of a functional equation arising in probabilistic normed spaces, General Inequalities 5 (Birkhäuser, Basel), 263-271. https://doi.org/10.1007/978-3-0348-7192-1_20

[3] Aseev, S. M. (1985). Quasilinear operators and their application in the theory of multivalued mappings, Trudy Mat. Inst. Steklov, 276, 25-52.

[4] Baker, J. A. (1991). The stability of certain functional equations, Proc. Amer. Math. Soc., 112, 729-732. https://doi.org/10.2307/2048695

[5] Benzarouala, Ch., Brzdcek, J., & Oubbi, L. (2023). A fixed point theorem and Ulam stability of a general linear functional equation in random normed spaces, J. Fixed Point Theory Appl., **25**, Paper No. 33. https://doi.org/10.1007/s11784-022-01034-8

[6] Brzdek, J., Fechner, W., Moslehian, M.S., & Sikorska, J. (2015). Recent developments of the conditional stability of the homomorphism equation, Banach J. Math. Anal., 9, 278-326. https://doi.org/10.15352/bjma/09-3-20

[7] Chang, S. H., Cho, Y., & Kang, S. (2001). Nonlinear Operator Theory in Probabilistic Metric Spaces, Nova Science Publishers, New York.

[8] Cho, Y. J., Rassias, T. M., & Saadati, R. (2013). Stability of Functional Equations in Random Normed Spaces, Springer Optimization and Its Applications, 86, Springer, Berlin. https://doi.org/10.1007/978-1-4614-8477-6

[9] Dehghanizade, R., & Modarres, S.M.S. (2021). Quotient spaces on quasilinear spaces, Int. J. Nonlinear Anal. Appl., 12, 781-792. http://dx.doi.org/10.22075/ijnaa.2021.20279.2142

[10] Diaz, J. B., & Margolis, B. (1968). A fixed point theorem of the alternative for contractions on a generalized complete metric space, Bull. Amer. Math. Soc., 74, 305-309. https://doi.org/10.1090/S0002-9904-1968-11933-0

[11] Gajda, Z. (1991). On stability of additive mappings, Int. J. Math. Math, Sci., 14, 431-434. https://doi.org/10.1155/S016117129100056X

[12] Gavruta, P. (1994). A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl., 184, 431-436. https://doi.org/10.1006/jmaa.1994.1211

[13] Hazarika, B., Srivastava, H. M., Arab, R., & Rabbani, M. (2019). Application of simulation function and measure of noncompactness for solvability of nonlinear functional integral equations and introduction to an iteration algorithm to find solution, Appl. Math. Comput., 360, 131-146. https://doi.org/10.1016/j.amc.2019.04.058

[14] Hyers, D. H. (1941). On the stability of the linear functional equation, Proc. Nat. Acad. Sci., 27, 222-224. https://doi.org/10.1073/pnas.27.4.222

[15] Hyers, D. H., Isac, G., & Rassias, Th. M. (1998). Stability of Functional Equations in Several Variables, Birkhäuser, Boston. https://doi.org/10.1007/978-1-4612-1790-9

[16] Jung, S.-M. (2011). Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis, Springer, New York. https://doi.org/10.1007/978-1-4419-9637-4

[17] Jung, S.-M., Popa, D., & Rassias, M. Th. (2014). On the stability of the linear functional equation in a single variable on complete metric groups, J. Glob. Optim., 59, 165-171. https://doi.org/10.48550/arXiv.1512.04709
[18] Kannappan, Pl. (2009). Functional Equations and Inequalities with Applications, Springer, New York. https://doi.org/10.1007/978-0-387-89492-8
[19] Mihet, D., & Radu, V. (2008). On the stability of the additive Cauchy functional equation in random normed spaces, J. Math. Anal. Appl., 343, 567-572. https://doi.org/10.1016/j.jmaa.2008.01.100
[20] Mihet, D., & Saadati, R. (2011). On the stability of the additive Cauchy functional equation in random normed spaces, Appl. Math. Lett., 24, 2005-2009. https://doi.org/10.1016/j.aml.2011.05.033
[21] Park, C., & Rassias, Th. M. (2009). Fixed points and stability of the Cauchy functional equation, Aust. J. Math. Anal. Appl., 6, 1-9. https://doi.org/10.1155/2009/809232
[22] Radu, V. (2003). The fixed point alternative and the stability of functional equations, Fixed Point Theory, 4, 91-96. https://api.semanticscholar.org/CorpusID:56040530
[23] Rassias, Th. M. (1978). On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc., 72, 297-300. https://doi.org/10.2307/2042795
[24] Rassias, Th. M. (2000). On the stability of functional equations in Banach spaces, J. Math. Anal. Appl., 251, 264-284. https://doi.org/10.1006/jmaa.2000.7046
[25] Schweizer, B., & Sklar, A. (1983). Probabilistic Metric Spaces, Elsevier North-Holland, New York.
[26] Srivastava, H. M., Das, A., Hazarika, B., & Mohiuddine, S. A. (2019). Existence of solution for non-linear functional integral equations of two variables in Banach algebra, Symmetry, 11(5)., 674, 2715-2737. https://doi.org/10.3390/sym11050674
[27] Srivastava, H. M., Deep, A., Abbas, S., & Hazarika, B. (2021). Sovability for a class of generalized functional-integral equations by means of Petrishyn's fixed point theorem, J. Nonlinear Convex Anal., 22, 2715-2737.
[28] Ulam, S. M. (1960). A Collection of the Mathematical Problems, Intercience Publ, New York. https://doi.org/10.1126/science.132.3428.665
[29] Wang, J. (2001). Some further generalization of the Ulam-Hyers-Rassias stability of functional equations, J. Math. Anal. Appl., 263, 406-423. https://doi.org/10.1006/jmaa.2001.7587
[30] Yilmaz, Y., Cakan, S., & Aytekin, S. (2012). Topological quasilinear spaces, Abstr. Appl. Anal., **2012**, Art. ID 951374. https://doi.org/10.1155/2012/951374

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