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INVARIANT SOLUTIONS AND CONSERVATION LAWS OF TIME-DEPENDENT NEGATIVE-ORDER (VNCBS) EQUATION

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Abstract. We apply the basic Lie symmetry method to investigate the time-dependent negative-order Calogero-Bogoyavlenskii-Schiff (vnCBS) equation. In this case, the symmetry classification problem is answered. We obtain symmetry algebra and create the optimal system of Lie subalgebras. We obtain the symmetry reductions and invariant solutions of the considered equation using these vector fields. Finally, we determine the conservation laws of the vnCBS equation via the Bluman-Anco homotopy formula.

Keywords: Lie algebras, Calogero-Bogoyavlenskii-Schiff equation, reduction equations, conservation laws, optimal system.
2020 MSC: 70G65, 34C14, 53C50.

1. Introduction

In 2020, Wazwaz extended the Calogero Bogoyavlenskii Schiff (CBS) equation to three new completely integrable equations [16]:

- Extended CBS (eCBS);

$$u_{xt} + u_{xxxy} + 4u_x u_{xy} + 2u_{xx}u_y + u_{xy} = 0.$$

- Time-dependent CBS (vCBS);

$$f(t)u_{xt} + \alpha u_{xxxy} + 2\beta u_x u_{xy} + \beta u_{xx}u_y = 0, \tag{1}$$

  where f is a real function.
- Time-dependent negative-order CBS (vnCBS),

$$g(t)u_{xy} + \alpha u_{xxxt} + 2\beta u_x u_{xt} + \beta u_{xxt} = 0, \tag{2}$$

  where g is a real function.

In the previous work [5], conservation laws and invariant solutions of Eq. (1) were investigated. But in this work, the same research is done regarding Eq.

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(2). The third-order linear partial differential equation called the integrable Calogero–Bogoyavlenskii–Schiff equation was recently established [15]:

$$u_t + u_{xxy} + 4uu_y + 2u_x\partial_x^{-1}u_y = 0, \tag{3}$$

which satisfies the Painlevé test for integrability. The nonlinear Eq. (3) is arguably a major vehicle to explore several soliton solutions. An applied generalization of this equation is the new Painlevé-integrable negative-order time-dependent Calogero–Bogoyavlenskii–Schiff equation (vnCBS). Nonlinear PDEs are widely used in evaluating nonlinear wave phenomena. Therefore, such equations have interested scientists for many years and have made it possible to carry out detailed studies. Korteweg-de Vries (KdV) equation is an equation that is deeply analyzed. The development of a fully integrated model, which describes the actual characteristics of the science and engineering domains, is underway, and a wide range of valuable findings are being obtained.

Indeed, studies in nonlinear equations are growing exponentially because such equations depict the modes and characteristics of nonlinear phenomena [7, 11]. These equations expand the view of scientists in terms of physical aspects, and in this regard, they find more usages in engineering and other sciences. On the other hand, through the Lie process, the symmetry categorization problem is extensively considered for various equations in different spaces [3, 4]. Recently, studies have been conducted regarding non-classical Lie symmetry and conservation laws of the nonlinear time-fractional equations [8, 9]. Special symmetry analysis and conservation laws were also studied in several cases [6, 10, 14]. In the present work, we used Lie's method to obtain symmetries of Eq. (2). Next, an optimal subalgebras system is presented associated with symmetries Lie algebra. Indeed, the Lie approach (symmetry group approach), as a computational, algorithmic technique to obtain constant group solutions, is widely utilized to solve differential equations. During the mentioned process, suitable solutions are obtained through known solutions. Also, its other applications are checking fixed solutions and reducing the order of ODEs [12].

The paper is presented in several sections as follows. The infinitesimal generators of the symmetry algebra of the equation along with several results are characterized in Section 2. The optimal ideal subalgebras systems were developed in Section 3. Following the third Section, the similarity solutions, Lie variables, and similarity reduction based on infinitesimal symmetries of Eq. (2) were discovered. In the last section, we obtain the conservation laws of Eq. (2) using the direct method and provide some concluding remarks.

2. **The symmetry computation of Eq.**(2)

In this section, using the basic Lie symmetry method, we determine the largest possible set of symmetries for Eq.(2) as follows [12, 13]:

Normally,

$$\Delta_\beta((x^1,...,x^m),(u^1,...,u^n)^{(p)})=0, \qquad 1\le\beta\le t, \tag{4}$$

is a set of PDE of order p. $(u^1,...,u^n)^{(i)}$ represents the i-order derivative of $U=(u^1,...,u^n)$ regarding $X=(x^1,...,x^m)$, $0\le i\le p$. On both X and U, infinitesimal transformations Lie group acts as:

$$\begin{aligned}\tilde{x}^i &= x^i+\varepsilon\xi^i((x^1,...,x^m),(u^1,...,u^n))+o(\varepsilon^2), \qquad 1\le i\le m,\\ \tilde{u}^j &= u^j+\varepsilon\phi_j((x^1,...,x^m),(u^1,...,u^n))+o(\varepsilon^2), \qquad 1\le j\le n,\end{aligned} \tag{5}$$

in which the infinitesimal transformations for $(x^1,...,x^m)$ and $(u^1,...,u^n)$ are denoted by ξ^i and ϕ_j respectively. A given infinitesimal generator equivalent to the transformations group (5) is

$$\begin{aligned}V=&\sum_{i=1}^{p}\xi^i((x^1,...,x^m),(u^1,...,u^n))\partial_{x^i}\\ &+\sum_{j=1}^{q}\phi_j((x^1,...,x^m),(u^1,...,u^n))\partial_{u^j}.\end{aligned} \tag{6}$$

We apply x, y and t instead of x^1, x^2 and x^3 respectively, and for simplicity we suppose that

$$\begin{gathered}\xi^j:=\xi^j(x,y,t,u), \qquad j=1,\dots,3,\\ \phi:=\phi(x,y,t,u).\end{gathered}$$

Here, an infinitesimal transformation's one-parameter Lie group is taken to apply the process for Eq.(2) as follows:

$$\begin{aligned}\tilde{x} &= x+\varepsilon\xi^1+\sigma(\varepsilon^2),\\ \tilde{y} &= y+\varepsilon\xi^2+\sigma(\varepsilon^2),\\ \tilde{t} &= t+\varepsilon\xi^3+\sigma(\varepsilon^2),\\ \tilde{u} &= u+\varepsilon\phi+\sigma(\varepsilon^2).\end{aligned}$$

If the invariant symmetry generator is

$$V=\xi^1\partial_x+\xi^2\partial_y+\xi^3\partial_t+\phi\partial_u, \tag{7}$$

then the invariance condition for Eq. (2) is written as follows [13]:

$$\begin{gathered}Pr^{(4)}V[g(t)u_{xy}+\alpha u_{xxxt}+2\beta u_xu_{xt}+\beta u_{xxt}]=0, \quad \text{whenever}\\ g(t)u_{xy}+\alpha u_{xxxt}+2\beta u_xu_{xt}+\beta u_{xxt}=0.\end{gathered}$$

Given that ξ^1, ξ^2, ξ^3, and ϕ are only dependent on x,y, t, and u. By equating the coefficients to zero, we will have:

$$\begin{cases} g(t)\xi^2_u\alpha=0, & -g(t)^2\xi^2_x=0,\\ -g(t)^2\xi^2_u=0, & 3\xi^2_{uu}\alpha^2=0,\\ 6\xi^2_{uu}\alpha^2=0, & -g^2(t)\xi^2_{uu}=0,\\ 3\xi^3_{uuu}\alpha^2=0, & 3\xi^2_{ut}\alpha^2=0.\end{cases}$$

There are 93 equations in total. Solving this set of PDEs, we earn the following results:

TABLE 1. Lie algebra for Eq.(2).

$[\,,\,]$	ϑ^1	ϑ^2	ϑ^3	ϑ^4	ϑ^5	ϑ^6
ϑ^1	0	$-\vartheta^2$	0	ϑ^4	0	ϑ^6
ϑ^2	ϑ^2	0	0	$\vartheta^1+\frac{1}{2}\vartheta^3$	0	ϑ^5
ϑ^3	0	0	0	0	$-\vartheta^5$	$-\vartheta^6$
ϑ^4	$-\vartheta^4$	$-\vartheta^1-\frac{1}{2}\vartheta^3$	0	0	$-\frac{1}{2}\vartheta^6$	0
ϑ^5	0	0	ϑ^5	$\frac{1}{2}\vartheta^6$	0	0
ϑ^6	$-\vartheta^6$	$-\vartheta^5$	ϑ^6	0	0	0

TABLE 2. Adjoint list of the Lie algebra.

Ad	ϑ^1	ϑ^2	ϑ^3	ϑ^4	ϑ^5	ϑ^6
ϑ^1	$\vartheta^1 e^{-s_1}$	ϑ^2	$\vartheta^3 e^{s_1}$	ϑ^4	$\vartheta^5 e^{s_1}$	ϑ^6
ϑ^2	$\vartheta^1+s_2\vartheta^4$	$s_2\vartheta^1+\vartheta^2+\frac{1}{2}s_2^2\vartheta^4$	$\vartheta^3+\frac{1}{2}s_2\vartheta^4$	ϑ^4	$\vartheta^5+s_2\vartheta^6$	ϑ^6
ϑ^3	ϑ^1	ϑ^2	ϑ^3	ϑ^4	$e^{-s_3}\vartheta^5$	$e^{-s_3}\vartheta^6$
ϑ^4	$\vartheta^1-s_4\vartheta^2$	ϑ^2	$-\frac{1}{2}s_4\vartheta^2+\vartheta^3$	$-s_4\vartheta^1+\frac{1}{2}s_4^2\vartheta^2+\vartheta^4$	ϑ^5	$-\frac{1}{2}s_4\vartheta^5+\vartheta^6$
ϑ^5	ϑ^1	ϑ^2	ϑ^3	ϑ^4	$s_5\vartheta^3+\vartheta^5$	$\frac{1}{2}s_5\vartheta^4+\vartheta^6$
ϑ^6	ϑ^1	ϑ^2	ϑ^3	ϑ^4	$s_5\vartheta^3+\vartheta^5$	$\frac{1}{2}s_5\vartheta^4+\vartheta^6$

Theorem 2.1. *There is a Lie algebra made by* (7) *in the point symmetries Lie group of Eq.* (2)*, with coefficients as follows:*

$$
\begin{aligned}
&g(t) = g(t),\\
&\quad\phi = -\frac{1}{4}(C_1 y - 2C_4 + 2C_2)u + \int \frac{1}{4}\frac{\left(4\left(\frac{d}{dy}F_1(y)\right) + C_1 x\right) g(t)}{\beta} dt\\
&\quad + \frac{1}{2}\frac{C_6 x}{\beta} + F_2(y),\\
&\xi^1 = F_1(y) + \frac{1}{4}\left(C_1 y - 2C_4 + 2C_2\right) x,\\
&\xi^2 = \frac{1}{2}C_1 y^2 + C_2 y + C_3,\\
&\xi^3 = \frac{y\left(\int \frac{1}{2} g(t) c_1 dt\right) + yC_6 + C_4\left(\int g(t)dt\right) + C_5}{g(t)}
\end{aligned}
$$

where $C_i \in \mathbb{R}$, $i = 1, ..., 6$, and F_2 is a real function.

Corollary 2.2. *One-parameter Lie group of Eq.(2) for every point symmetry contains the infinitesimal generators as:*

$$
\begin{aligned}
\vartheta^1 =& \frac{1}{2}x\partial_x + y\partial_y - \frac{1}{2}u\partial_u,\\
\vartheta^2 =& \partial_y,\\
\vartheta^3 =& -\frac{1}{2}x\partial_x + \partial_t + \frac{1}{2}u\partial_u,\\
\vartheta^4 =& \frac{1}{4}yx\partial_x + \frac{1}{2}y^2\partial_y + \frac{1}{2}y\partial_t + \frac{(-yu\beta + xe^t\partial_u)}{4\beta},\\
\vartheta^5 =& e^{-t}\partial_t,\\
\vartheta^6 =& ye^{-t}\partial_t + \frac{1}{2}\frac{x\partial_u}{\beta}.
\end{aligned}
\tag{8}
$$

Lie algebra is provided for Eq. (2) according to Table (1). The value column j^{th} and row i^{th}, $i, j = 1, ..., 6$ is determined by $[\vartheta^i, \vartheta^j] = \vartheta^i\vartheta^j - \vartheta^j\vartheta^i$ expression.

3. Classification of 1D subalgebras

The one parameter optimal system of Eq. (2) can be determined utilizing the symmetry group. Therefore, we need to search for invariant solutions that are not linked by a transformation in the full symmetry group. An optimal set of subalgebras is obtained. The subject of classification of 1D algebras is similar to the subject of classification of adjoint representation orbits. One representative of each group of similar subalgebras is considered to solve an optimal set of subalgebra problems [12]. The adjoint representation of each ϑ^t, $t = 1, ..., 6$ is defined as:

$$
\mathrm{Ad}(\exp(s.\vartheta^t).\vartheta^r) = \vartheta^r - s.[\vartheta^t, \vartheta^r] + \frac{s^2}{2}.[\vartheta^t, [\vartheta^t, \vartheta^r]] - \cdots, \tag{9}
$$

where $[\vartheta^t, \vartheta^r]$ is presented in Table (1) for $t, r = 1, \ldots, 6$ and s represents the parameter (see [12], page 199). Let $\mathfrak{g}$ is the Lie algebra generated by (8). The adjoint action for $\mathfrak{g}$ is based on Table (2).

Theorem 3.1. *The 1D subalgebras of Eq.(2) are as follows:*

$$
\begin{aligned}
&1)\vartheta^6 + c_1\vartheta^1 + c_2\vartheta^2 + c_3\vartheta^3 + c_4\vartheta^4,\\
&2)\vartheta^5 + c_1\vartheta^1 + c_2\vartheta^2 + c_3\vartheta^3 + c_4\vartheta^4,\\
&3)\vartheta^4 + c_1\vartheta^1 + c_2\vartheta^2,\\
&4)\vartheta^4 + c_1\vartheta^2 + c_2\vartheta^3,\\
&5)\vartheta^3 + c_1\vartheta^1 + c_2\vartheta^2,\\
&6)\vartheta^2,\\
&7)\vartheta^1,
\end{aligned}
$$

where $c_i \in \mathbb{R}$ represent arbitrary numbers for $i = 1, \dots, 6$.

Proof. Given Table (1), the center of Lie algebra is empty. Therefore, all we need is to obtain the sub-algebras of the form

$$\langle \vartheta^1, \vartheta^2, \vartheta^3, \vartheta^4, \vartheta^5, \vartheta^6 \rangle.$$

For $t = 1, \dots, 6$, the map:

$$\begin{cases} \mathrm{Ad}(\exp(s\vartheta^t).X) : \mathfrak{g} \to \mathfrak{g} \\ X \mapsto \mathrm{Ad}(\exp(s\vartheta^t).X), \end{cases}$$

represents a linear function. Given the basis $\{\vartheta^1, \dots, \vartheta^6\}$ and a vector field $X = \sum_{i=1}^{6} a_i \vartheta^i$, the functions $\mathrm{Ad}(\exp(s\vartheta^i).X)$, $1 \le i \le 6$ are reported as follows:

$$\mathrm{Ad}(\exp(s_i\vartheta^i).X) = [a_1\ a_2\ a_3\ a_4\ a_5\ a_6]\ M^i_{6\times 6} \begin{bmatrix} \vartheta^1 \\ \vartheta^2 \\ \vartheta^3 \\ \vartheta^4 \\ \vartheta^5 \\ \vartheta^6 \end{bmatrix},$$

where

$$M^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-s_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{s_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{s_1} \end{bmatrix}, \quad M^2 = \begin{bmatrix} 1 & 0 & 0 & s_2 & 0 & 0 \\ s_2 & 1 & 0 & \frac{s_2^2}{2} & 0 & 0 \\ 0 & 0 & 1 & \frac{s_2}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & s_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$M^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-s_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-s_3} \end{bmatrix}, M^4 = \begin{bmatrix} 1 & -s_4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{-s4}{2} & 1 & 0 & 0 & 0 \\ -s_4 & \frac{s_4^2}{2} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{-s_4}{2} & 1 \end{bmatrix},$$

$$M^5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & s_5 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2}s_5 & 0 & 1 \end{bmatrix}, \quad M^6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & s_5 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2}s_5 & 0 & 1 \end{bmatrix}.$$

From $\mathrm{Ad}(\exp(s_1\vartheta^1)) \circ \mathrm{Ad}(\exp(s_2\vartheta^2)) \circ \cdots \circ \mathrm{Ad}(\exp(s_6\vartheta^6))$, we can simplified X as follows:

For $a_6 \neq 0$, the coefficient a_5 is vanished by setting $s_2 = \frac{-a_5}{a_6}$. If needed, by scaling, we assume $a_6 = 1$. Hence, X gives rise to case (1).

For $a_6 = 0$ and $a_5 \neq 0$, if needed, by scaling, we assume $a_5 = 1$. Hence, X gives rise to case (2).

For $a_6 = a_5 = 0$ and $a_4 \neq 0$, the coefficients a_1 or a_3 are vanished by setting $s_1 = -\frac{a_1}{a_4}$ or $S_2 = \frac{-2a_3}{a_4}$. If needed by scaling, we assume $a_4 = 1$. Hence, X gives rise to case (4) or (3).

For $a_6 = a_5 = a_4 = 0$ and $a_3 \neq 0$, if needed by scaling, we assume $a_3 = 1$. Hence, X gives rise to case (5).

For $a_6 = a_5 = a_4 = a_3 = 0$ and $a_2 \neq 0$, the coefficient a_1 is vanished by setting $s_4 = \frac{a_1}{a_2}$. If required by scaling, we assume $a_2 = 1$. Hence, X gives rise to case (6).

For $a_6 = a_5 = a_4 = a_3 = a_2 = 0$, if needed by scaling, we assume $a_1 = 1$. Hence, X gives rise to case (7).

□

4. **Reduction of similarity in Eq.** (2)

Here, the symmetry reduction of Eq. (2) is classified by considering the subalgebras of Theorem (3.1). It is necessary to find a new equation with specific coordinates and reduce the equation. The independent variables p, q, r and h must be found in the infinitesimal generator to create these coordinates. Therefore, a new coordinate is used to express Eq. (2) through the chain law, which reduces the system.

Theorem 4.1. *The similarity variables and the reduced equations for 1D subalgebras in Theorem* (3.1) *are listed in Tables* (3) *and* (4)*.*

Proof. We present the proof with a case-by-case study. Namely, we explain the details for subalgebra $H_7 := \vartheta^1 + \vartheta^3$. The invariants related to subalgebra $H_7 := \vartheta^1 + \vartheta^3$ through characteristic equation integration are as follows:

$$\frac{dx}{0} = \frac{dy}{y} = \frac{dt}{1} = \frac{du}{0}.$$

Thus, the similarity variables are:

$$p = x, \qquad q = t - Ln(y), \qquad h = u,$$

where $h(p, q)$ meets a reduced PDE with three variables:

$$-h^2_{qp} + \alpha h^4_{qp^3} + 2\beta h_p h^2_{qp} + \beta h^2_{p^2} h_{q^2} = 0. \tag{10}$$

Tables (3) and (4) represent the similarity variables and the reduced equation for the subalgebra $\vartheta^1 + \vartheta^3$ by the case (7). □

In the following, we can provide the invariant solutions for some of the obtained reduced equations in Table (4). For example, the invariant solution of Eq.(10) becomes:

$$h(p,q) = u(x,y,t) = C_1 x + F_3(-\ln(y) + t),$$

where $C_1 \in \mathbb{R}$. Using a similar argument, for the vector ϑ^3, the Eq. (2) is reduced as:

$$\begin{aligned}(11) \qquad 2h_{qp}^2 - h_p + 8\alpha h_{q^4}^4 - 24\alpha h_{q^3} + 22\alpha h_{q^2} - 6\alpha h_q + 12\beta h_q h_{q^2}^2 \\ -10\beta h_{q^2}^2 - 4\beta h_{pq} h_{q^2}^2 + 4\beta h_q = 0,\end{aligned}$$

where the independent variables are as $p = y, q = 2Ln(x)+t$ and the dependent function is as $u = h(p,q)/x$. The invariant solution of Eq. (11) becomes:

$$xh(p,q) = u(x,y,t) = F_2(y)\mathrm{e}^{\frac{1}{2}t},$$

where F_2 represents an arbitrary function. Also, for the vector ϑ^5, Eq. (2) is reduced as:

$$(12) \qquad h_{qp}^2 = 0,$$

where the independent variables are as $p = x, q = y$, and the dependent function is as $u = h(p,q)$. The equivalent solution of Eq. (12) is derived as:

$$h(p,q) = u(x,y,t) = F_3(x) + F_2(y),$$

where F_2 and F_3 are arbitrary functions. For case (8) in Tables (3) and (4), the corresponding solution is derived as:

$$\frac{x^2 + 2\beta h(p,q)}{2x\beta} = u(x,y,t) = \frac{1}{2}\frac{x\sqrt{y} + 2F_2\left(\frac{1}{2}\mathrm{e}^t - \frac{1}{2}y\right)\beta}{\sqrt{y}\beta},$$

where F_2 represents an arbitrary function. For case (9) in Tables (3) and (4), the corresponding solution is derived as:

$$\frac{h(p,q)}{x} = u(x,y,t) = F_2(y)\sqrt{\mathrm{e}^t + 1},$$

again, where F_2 represents an arbitrary function. For case (10) in Tables (3) and (4), the corresponding solution is derived as:

$$\frac{-x^2 + 2\beta h(p,q)}{2x\beta} = u(x,y,t) = \frac{1}{2}\frac{-x^2 + 2C_1\beta}{x\beta},$$

where $C_1 \in R$.

TABLE 3. Similarity solution and Lie invariants.

i	H_i	p_i	q_i	w_i	u_i
1	ϑ^1	$\frac{y}{x^2}$	t	xu	$\frac{h(p,q)}{x}$
2	ϑ^2	x	t	u	$h(p,q)$
3	ϑ^3	y	$2\ln(x)+t$	xu	$\frac{h(p,q)}{x}$
4	ϑ^4	$\frac{y}{x^2}$	$-2\ln(x)+t$	$\frac{2y\beta xu - x^2e^t}{2y\beta}$	$\frac{1}{2}\frac{x^2e^t + 2h(p,q)y\beta}{y\beta x}$
5	ϑ^5	x	y	u	$h(p,q)$
6	ϑ^6	x	y	$\frac{2y\beta u - xe^t}{2y\beta}$	$\frac{1}{2}\frac{xe^t + 2h(p,q)\beta}{x\beta}$
7	$\vartheta^1+\vartheta^3$	x	$-\ln(y)+t$	u	$h(p,q)$
8	$\vartheta^1+\vartheta^6$	$\frac{y}{x^2}$	$\frac{1}{2}\frac{(e^t-y)x^2}{y}$	$\frac{2\beta x - x^2}{2\beta}$	$\frac{1}{2}\frac{x^2+2h)p,q)\beta}{x\beta}$
9	$\vartheta^3+\vartheta^5$	y	$-\frac{1}{2}\ln(e^t+1)-\ln(x)$	xu	$\frac{h(p,q)}{x}$
10	$\vartheta^3+\vartheta^6$	y	$-\frac{1}{2}\ln(y+e^y)-\ln(x)$	$\frac{2\beta x - x^2}{2\beta}$	$h(p,q)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

TABLE 4. Reduced equations in terms of infinitesimal symmetries.

i	
1	$-2h_ph_{qp}^2 - 2h_ph_q - 8\alpha h_{q^3p}^4q^3 - 48\alpha h_{qp}^3q^2 - 54\alpha h_{qp}^2q - 6\alpha h_p + 8\beta h_qq^2h_{qp}^2 + 14\beta h_qqh_p + 4\beta h_{qp}^2q + 4\beta h_p + 4\beta h_ph_q^2q^2 = 0,$
2	$\alpha h_{q^3p}^4 + betah_qh_{qp} + \beta h_{q^2}^2h_p = 0,$
3	$2h_{qp}^2 - h_p + 8\alpha h_{q^4}^4 - 24\alpha h_{q^3} + 22\alpha h_{q^2} - 6\alpha h_q + 12\beta h_qh_{q^2}^2 - 10\beta h_{q^2}^2 - 4\beta h_{pq}h_{q^2}^2 + 4\beta h_q = 0,$
4	$-6h_pp^2\beta - 4h_{qp}^2p^2\beta - 4p^3\beta = 0,$
5	$h_{qp}^2 = 0,$
6	$-1 + 1 + 2h_{qp}^2q^2\beta + 2h_pq\beta + h_{p^2}^2pq\beta = 0,$
$1+3$	$-h_{qp}^2 + \alpha h_{qp^3}^4 + 2\beta h_ph_{qp}^2 + \beta h_{p^2}^2h_{q^2} = 0,$
$1+6$	$-2h_{qp}^2p^2 + h_qp + 2h_{q^2}^2pq = 0,$
$3+5$	$-2h_{qp}^2e^{q^2} - 2hpe^{q^2} = 0,$
$3+6$	$\alpha h_{q^4}^4e^{q^4} + 6\alpha h_{q^3}^3(e^q)^4 + 11\alpha h_{q^2}^2(e^q)^4 - 3\beta h_qh_{q^2}^2(e^q)^4 - 5\beta(hq)^2(e^q)^4 - 2\beta h_{pq}h_{q^2}^2(e^q)^4 - 4\beta h_q(e^q)^4 = 0,$
$\vdots$	$\vdots$

5. Conservation laws

Alternative methods, like the direct method and the Noether method, can be used to check the conservation laws. Here, the direct method is used to analyze conservation laws [1, 2]. Assume the differential equation in the form $\rho\{x,u\}$ with k order and independent variables of n, where x can be defined as $x = (x^1, \dots, x^n)$ and u can be one dependent variable, represented by

$$\rho[u] = \rho(x, u, \partial u, \dots, \partial^k u) = 0.$$

Considering a multiplier with $\Lambda(x, u, \partial u, \dots, \partial^t u)$ can give us a conservation law in the form of $\Lambda[u]\rho[u] = D_i\phi^i[u] = 0$ for the equation $\rho\{x,u\}$ contingent upon

$$E_U(\Lambda(x, U, \partial U, \dots, \partial^t U)\rho(x, U, \partial U, \dots, \partial^k U)) \equiv 0,$$

by taking $U(x)$ as an expression for an arbitrary function. The expression E_U which is the Euler operator acting on U, is defined as below:

$$E_U = \partial U - D_i \partial U + \dots + (-1)^s D_{i_1} \dots D_{i_s} \partial U_{i_1} \dots i_s$$

Since the $vnCBS$ equation is concerned about u, x, y, t, it results in multipliers that further provide locally configured conservation laws for Eq. (2) in the format of $\Lambda = \Lambda(x, y, t, U, U_x U_y, U_t)$. We can calculate all nontrivial local conservation laws related to Eq. (2) from multipliers. Accordingly, the expression $\Lambda = \Lambda(x, y, t, U, \partial_x U, \partial_y U, \partial_t U)$ is a conservation law multiplier regarding Eq. (2) contingent upon

$$E_U[\Lambda(x, y, t, U, \partial_x U, \partial_y U, \partial_t U)(U_{xt} + U_{xxxy} + 4U_x U_{xy} + 2U_{xx} U_y + U_{xy}))] \equiv 0$$

for $U(x, y, t)$ in the form of an arbitrary function. We then find all potential multipliers in the format of

$$\Lambda = \Lambda(x, y, t, U, \partial U_x, \partial U_y, \partial U_t)$$

for Eq. (2). Therefore, the Euler operator is determined to be as follows:

$$E_U = \partial U - D_i \partial U_i + \dots + (-1)^3 D_{i_1} \dots D_{i_3} \partial U_{i_1 \dots i_3}.$$

Moreover, the determining equations become.

$$E_U[\Lambda(x, y, t, U, \partial U_x, \partial U_y, \partial U_t)(U_{xt} + U_{xxxy} + 4U_x U_{xy} + 2U_{xy} U_y + U_{xy}))] \equiv 0$$

so that $U(x, y, t)$ represents an arbitrary function. The above equation can be separated with respect to U_x, U_y, U_t to provide the over-determined equations:

$$\Lambda_{U,y,y} = 0, \Lambda_{x,x} = 0, \Lambda_{x,y} = \Lambda_{U,y} U_x, \Lambda_{t,x} = -\frac{\Lambda_{U,y} g}{\beta}, \Lambda_{U,x} = 0,$$
$$\Lambda_{U_x,x} = \Lambda_U, \Lambda_{U_y,x} = 0, \Lambda_{Ut,x} = 0, \Lambda_{U_x,y} = -\frac{g\Lambda_t \beta - 2\beta U_t \Lambda_U g + \beta U_t \Lambda_{Ut} g_t}{g^2},$$
$$\Lambda_{U_y,y} = 4\Lambda_U, \Lambda_{Ut,y} = \frac{2\beta\,(-\Lambda_x + \Lambda_U U_x)}{g},$$
$$\Lambda_{t,t} = -\frac{-g g_t \Lambda_t + 4gUt\Lambda_U g_t + gU_t \Lambda_{U_t} g_{t,t} - 3g_t^2 U_t \Lambda_{U_t}}{g^2}, \Lambda_{U,t} = 0, \Lambda_{U_x,t} = 0,$$
$$\Lambda_{U_y,t} = 0, \Lambda_{Ut,t} = \frac{2\Lambda_U g - \Lambda_{U_t} g_t}{g}, \Lambda_{U,U} = 0, \Lambda_{U,U_x} = 0, \Lambda_{U,U_y} = 0, \Lambda_{U,U_t} = 0,$$
$$\Lambda_{U_x,U_x} = 0, \Lambda_{Ux,Uy} = 0, \Lambda_{U_t,U_x} = 0, \Lambda_{U_y,U_y} = 0, \Lambda_{U_t,U_y} = 0, \Lambda_{Ut,Ut} = 0.$$

Solving the above equations, we get the following results:

Theorem 5.1. *For Eq.* (2)*, we have the infinite set of local multipliers:*

$$\Lambda_(x, y, t, U, U_x, U_y, U_t) = C_1 U_x + C_2 U_y + F(y),$$

where C_1 and C_2 are real constants, and F is a real function. Meanwhile, Eq. (2) *possesses global conservation laws.*

Proof. Using the Anco & Bluman ($A\&B$) homotopy formula [1,2], we determine the conserved elements of ϕ^t, ϕ^x and ϕ^y corresponding to Λ in the following format. In particular, we have solved for

$[U_{txy}, U_{xxy}, U_{xyy}, U_{ttxy}, U_{txxy}, U_{txyy}, U_{xxxy}, U_{xxyy}, U_{xyyy}, U_{tttxy}, U_{ttxxy}, U_{ttxyy}, U_{txxxy}, U_{txxyy}, U_{txyyy}, U_{xxxxy}, U_{xxxyy}, U_{xxyyy}, U_{xyyyy}]$.

Thus, we obtain the following fluxes:

$$\begin{aligned}\phi^t =&\frac{2}{3}U_{x^2}\left(C_1U_x+C_2U_y\right)\beta+\frac{1}{3}U\left(C_1U_x+C_2U_y\right)\beta U_{xx}+\\&\frac{1}{2}U_{xxx}\left(C_1U_x+C_2U_y\right)\alpha+U_{x^2}F(y)\beta+\frac{1}{2}UF(y)\beta U_{xx}+U_{xxx}F(y)\alpha.\end{aligned}$$

$$\begin{aligned}\phi^x =&\frac{2}{3}U\left(C_1Ux+C_2Uy\right)\beta Utx-\frac{1}{3}U\left(\left(2C_1\beta U_x\ \ +2\left(C_1U_x+C_2U_y\right)\beta\right)U_{tx}\right.\\&\left.+2C_2\beta U_xU_{ty}\right)+\frac{1}{3}U_x\left(C_1U_x+C_2U_y\right)\beta U_t-\frac{1}{3}U\left(\left(C_1U_x\ \ +C_2U_y\right)\beta U_{tx}\right.\\&\left.+C_1\beta U_tU_{xx}+C_2\beta U_tU_{xy}\right)+\frac{1}{3}U(2\beta U_xU_{tx}+\beta U_{xx}U_t)C_1-\frac{1}{2}UF(y)\beta U_{tx}\\&+\frac{1}{2}U_xF(y)\beta U_t+\frac{1}{2}U(gU_{xy}+\alpha U_{txxx})C_1-\frac{1}{2}U\left(C_1\alpha U_{txxx}+C_2\alpha U_{txxy}\right)\\&+\frac{1}{2}U_x\left(C_1\alpha U_{txx}\ \ +C_2\alpha U_{txy}\right)-\frac{1}{2}U_{xx}\left(C_1\alpha U_{tx}+C_2\alpha U_{ty}\right)+\frac{1}{2}U_y\left(C_1U_x\right.\\&\left.+C_2U_y\right)g+U_yF(y)g.\end{aligned}$$

$$\begin{aligned}\phi^y =&\frac{1}{3}U(2\beta U_xU_{tx}+\beta U_{xx}U_t)C_2-\frac{1}{2}U\left(C_1gU_{xx}\ \ +C_2gU_{xy}\right)\\&+\frac{1}{2}U(gU_{xy}+\alpha U_{txxx})C_2.\end{aligned}$$

Case 1: By setting $C_1 = 1$ into the equation, we have:

$$\Lambda(x, y, t, U, U_x, U_y, U_t) = U_x$$

$$\begin{aligned}\phi^t =&\frac{2}{3}U_{x^3}\beta+\frac{1}{3}UU_x\beta U_{xx}+\frac{1}{2}U_{xxx}U_x\alpha,\\\phi^x =&-\frac{1}{3}UU_x\beta U_{tx}+\frac{1}{3}U_{x^2}\beta U_t+\frac{1}{2}UgU_{xy}+\frac{1}{2}U_x\alpha U_{txx}-\frac{1}{2}U_{xx}\alpha U_{tx}+\frac{1}{2}U_yU_xg,\\\phi^y =&-\frac{1}{2}U_gU_{xx}.\end{aligned}$$

Case 2: Set $C_2 = 1$:

$$\Lambda(x, y, t, U, U_x, U_y, U_t) = U_y,$$

$$\begin{aligned}
\phi^t =& \frac{2}{3}U_{x^2}U_y\beta + \frac{1}{3}UU_y\beta U_{xx} + \frac{1}{2}U_{xxx}U_y\alpha, \\
\phi^x =& -\frac{1}{3}UU_y\beta U_{tx} - \frac{2}{3}U\beta U_x U_{ty} + \frac{1}{3}U_x U_y \beta U_t - \frac{1}{3}U\beta U_t U_{xy} \\
& -\frac{1}{2}U\alpha U_{txxy} + \frac{1}{2}U_x\alpha U_{txy} - \frac{1}{2}U_{xx}\alpha U_{ty} + \frac{1}{2}U_{y^2}g, \\
\phi^y =& \frac{2}{3}UU_x\beta U_{tx} + \frac{1}{3}U\beta U_{xx}U_t + \frac{1}{2}U\alpha U_{txxx}
\end{aligned}$$

Case 3: F is an arbitrary function:

$$\Lambda(x, y, t, U, U_x, U_y, U_t) = F(y),$$

$$\begin{aligned}
\phi^t =& U_{x^2}F(y)\beta + \frac{1}{2}UF(y)\beta U_{xx} + U_{xxx}F(y)\alpha, \\
\phi^x =& -\frac{1}{2}UF(y)\beta U_{tx} + \frac{1}{2}U_x F(y)\beta U_t + U_y F(y) g, \\
\phi^y =& 0.
\end{aligned}$$

So, for all cases, we can see these fluxes yield the following conservation laws:

$$D_t\phi^t + D_x\phi^x + D_y\phi^y = 0.$$

□

Conclusion

Here, Infinitesimal generators and different vector fields for the vnCBS equation were constructed by applying the basic Lie symmetry group method. Applying the invariance condition of Eq. (2) under the infinitesimal prolongation, we earn the Lie symmetries group of the vnCBS equation and the similarity solution has an essential role in reducing the equation. We earned the conservation laws of Eq. (2) by adding the Bluman-Anco homotopy formula to the direct method. This paper is important because the obtained invariant solutions and conservation laws of Eq. (2) should be applicable in other applied sciences such as field theory, fluid dynamics, plasma physics and nonlinear optics. Also, in classical physics, laws of this type govern energy, momentum, angular momentum, mass, and electric charge. Therefore, in future physical articles, the obtained invariant solutions and conservation laws of Eq. (2) can be used.

References

[1] Anco, S.C. and Bluman, G.W. (1997), Direct construction of conservation laws. Phys. Rev. Lett, 78(15), 2869–2873.

[2] Anco, S.C. and Bluman, G.W. (2002), Direct construction method for conservation laws of partial differential equations part II: general treatment. Eur J Appl Math, 13, 545–566.

[3] AryaNejad, Y. (2021), Symmetry Analysis of Wave Equation on Conformally Flat Spaces, J. Geom. Phys. 161, 104029.

[4] AryaNejad, Y. (2022), Exact solutions of diffusion equation on sphere, Comput. Methods Differ. Equ., 10(3), 789-798.

[5] AryaNejad, Y. and Mirzavand, R. (2022), Conservation laws and invariant solutions of time-dependent Calogero-Bogoyavlenskii-Schiff equation, J. Linear. Topological. Algebra. 11 (4), 243-252.

[6] Gandarias, M.L., Torrisi, M. and Valenti, A. (2004), Symmetry classification and optimal systems of a non-linear wave equation, Internat. J. Non-Linear Mech. 39, 389–398.

[7] Jafari, M., Zaeim, A. and Tanhaeivash, A. (2020), Symmetry group analysis and conservation laws of the potential modified KdV equation using the scaling method, Int. J. Geometric Methods in Modern.

[8] Hashemi, M.S., Haji-Badali, A. and Alizadeh, F. (2021), Nonclassical Lie symmetry and conservation laws of the nonlinear time-fractional Korteweg–de Vries equation , Commun. Theor. Phys. 73 095006 .

[9] Hashemi, M.S., Haji-Badali, A. and Alizadeh, F. (2021), Non-classical Lie symmetry and conservation laws of the nonlinear time-fractional Kundu–Eckhaus (KE) equation, Pramana- 95 (3), 107.

[10] Nadjafikhah, M. and Ahangari, F. (2013), Symmetry Analysis and Conservation Laws for the Hunter-Saxton Equation, Commun. Theor. Phys. 59 (3) .

[11] Nadjafikhah, M. and Shagholi, S. (2017), Mathematical modeling of optimized SIRS epidemic model and some dynamical behavior of the solution, Int. J. Nonlinear Anal. Appl. 8 (2), 125-134.

[12] Olver, P.J. (1986), Applications of Lie Groups to Differential Equations, Springer, New York.

[13] Olver, P.J. (1995), Equivalence, Invariants and Symmetry, Cambridge University Press, Cambridge.

[14] Rashidi, S, and Hejazi, S.R. (2020), Self-adjointness, conservation laws and invariant solutions of the Buckmaster equation, Comput. Methods Differ. Equ. 8(1) 85-98.

[15] Wazwaz, A.M. (2012), A study on two extensions of the Bogoyavlenskii–Schieff equation, Commun. Nonlinear Sci. Numer. Simul. 17(4), 1500-1505.

[16] Wazwaz, A.M. (2020), A variety of completely integrable Calogero–Bogoyavlenskii–Schiff equations with time-dependent coefficients, Int. J. Numer. Methods Heat Fluid Flow, 31(1), 174-185.

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