Journal of Mahani Mathematical Research
Shahid Bahonar University of Kerman
Print ISSN: 2251-7952 Online ISSN: 2645-4505

APPLICATION OF SIGMOID FUNCTION IN THE SPACE OF UNIVALENT FUNCTIONS BASED ON SUBORDINATION

F. Madadi Tamrin, Sh. Najafzadeh ✉, and M.R. Foroutan

Article type: Research Article
(Received: 15 March 2023, Received in revised form 04 November 2023)
(Accepted: 01 December 2023, Published Online: 02 December 2023)

Abstract. In the present paper, we introduce a new subclass of normalized analytic and univalent functions in the open unit disk associated with Sigmoid function. Coefficient estimates, convolution conditions, convexity and some other geometric properties for functions in this class are investigated. Also, subordination and inclusion results are obtained.

Keywords: Univalent function, Sigmoid function, Convolution, Subordination, Coefficient bound, Convex set.
2020 MSC: 30C45, 30C50.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions of the type:

$$f(x) = x + \sum_{k=2}^{\infty} \alpha_k x^k, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{x \in \mathbb{C} : |x| < 1\}$. Also, suppose that $\mathcal{N}$ denotes the subclass of $\mathcal{A}$ consisting of analytic functions of the form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad (a_k > 0, z \in \mathbb{U}). \tag{2}$$

The convolution of f given by (2) and $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$ is defined by:

$$(f * g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{3}$$

Further, let $\mathcal{P}$ be the class of functions:

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k, \tag{4}$$

which are analytic and convex in $\mathbb{U}$.

✉ shnajafzadeh44@pnu.ac.ir, ORCID: 0000-0002-8124-8344
https://doi.org/10.22103/jmmr.2023.21245.1421
Publisher: Shahid Bahonar University of Kerman
How to cite: F. Madadi Tamrin, Sh. Najafzadeh, M.R. Foroutan, *Application of Sigmoid function in the space of univalent functions based on subordination*, J. Mahani Math. Res. 2024; 13(2): 169 - 177.

The logistic Sigmoid function is given by

$$L(z) = \frac{1}{1-e^{-z}}, \tag{5}$$

which is differentiable. For the purpose of our results, the following lemma shall be necessary.

Lemma 1.1. *Let $L(z)$ be a Sigmoid function and*

$$\Phi(z) = 2L(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \Big(\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} z^k \Big)^m, \tag{6}$$

then $|\Phi(z)| < 2$, $|z| < 1$, where $\Phi(z)$ is a modified Sigmoid function.

Setting $m = 1$, Fadipe-Joseph et al. [4] remarked that

$$\Phi(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2(k!)} z^k. \tag{7}$$

For more details see also [2], [5], [6], [1], [8], [10] and [11–14].
By applying the convolution structure, we consider the function

$$X_f(z) = \big((F * F) * f\big)(z), \tag{8}$$

where f is given by (2) and

$$F(z) = 1 + \frac{3}{2}z - \Phi(z).$$

With a simple calculation, we conclude that:

$$X_f(z) = z - \sum_{k=2}^{\infty} \frac{1}{4(k!)^2} a_k z^k. \tag{9}$$

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then $f(z)$ is said to be subordinate to $g(z)$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a function w analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$, see [3] and [7].

2. Main Results

In this section, first we define a new subclass of univalent functions. Then we obtain the sharp coefficient bounds for functions in this subclass. Also, convolution preserving property with some restrictions on parameters is investigated. Finally, we introduce the integral representation for the functions defined by (9). By using this class of functions, we can find many interesting geometric properties.

Definition 2.1. For $-1 \leqslant B < A \leqslant 1$, $0 \leqslant t \leqslant 1$, let $Y_t(A, B)$ denotes the class of functions $f \in \mathcal{N}$ for which

$$\frac{z\big(X_f(z)\big)'}{f_t(z)} \prec \frac{1+Az}{1+Bz}, \tag{10}$$

or equivalently

$$\left|\frac{z\big(X_f(z)\big)' - f_t(z)}{Af_t(z) - Bz\big(X_f(z)\big)'}\right| < 1, \tag{11}$$

where $X_f(z)$ is given by (9) and

$$f_t(z) = (1-t) + tf(z), \qquad f(z) \in \mathcal{N}.$$

For defining this class, we take an idea from [9].

Theorem 2.2. *Let* $f(z) = z - \sum\limits_{k=2}^{\infty} a_k z^k$ *be analytic in* $\mathbb{U}$*. Then* $f \in Y_t(A,B)$ *if and only if:*

$$\sum_{k=2}^{\infty}\Big[\Big(\frac{k}{4(k!)^2} - t\Big)(1-B) + t(A-B)\Big]a_k \leqslant A - B. \tag{12}$$

Proof. Let (12) hold true. We have to show that (10) or equivalently (11) is satisfied. But we have

$$\begin{aligned}
&\Big|z\big(X_f(z)\big)' - f_t(z)\Big| - \Big|Af_t(z) - Bz\big(X_f(z)\big)'\Big| \\
&= \left|z - \sum_{k=2}^{\infty}\frac{k}{4(k!)^2}a_k z^k - (1-t)z - tf(z)\right| \\
&- \left|A(1-t)z + Atf(z) - Bz + \sum_{k=2}^{\infty}\frac{Bk}{4(k!)^2}a_k z^k\right| \\
&= \left|-\sum_{k=2}^{\infty}\Big(\frac{k}{4(k!)^2} - t\Big)a_k z^k\right| - \left|(A-B)z - \sum_{k=2}^{\infty}\Big(At - \frac{Bk}{4(k!)^2}\Big)a_k z^k\right|.
\end{aligned}$$

But, putting

$$At - \frac{Bk}{4(k!)^2} = t(A-B) - \Big(\frac{k}{4(k!)^2} - t\Big)B,$$

letting $z \to 1$ and applying (12), the above expression is less than or equal to zero, so (11) holds true and hence $f \in Y_t(A,B)$.

To prove the converse, let $f \in Y_t(A,B)$, then

$$\left|\frac{z\big(X_f(z)\big)' - f_t(z)}{Af_t(z) - Bz\big(X_f(z)\big)'}\right| = \frac{\left|\sum\limits_{k=2}^{\infty}\Big(\frac{k}{4(k!)^2} - t\Big)a_k z^k\right|}{\left|(A-B)z - \sum\limits_{k=2}^{\infty}\Big(At - \frac{Bk}{4(k!)^2}\Big)a_k z^k\right|} < 1.$$

But $\mathrm{Re}(z) \leqslant |z|$ for all z, we have:

$$\mathrm{Re}\left\{\frac{\sum\limits_{k=2}^{\infty}\Big(\frac{k}{4(k!)^2} - t\Big)a_k z^k}{(A-B)z - \sum\limits_{k=2}^{\infty}\Big(At - \frac{Bk}{4(k!)^2}\Big)a_k z^k}\right\} < 1.$$

By letting $z \to 1$ through positive values and choosing the values of z such that $\frac{z\left(X_f(z)\right)'}{f_t(z)}$ is real, we get the required result, so the proof is complete. □

Remark 2.3. (Sharpness of inequality (12)): We note that the function

$$G(z) = z - \frac{A-B}{\left(\frac{1}{8}-t\right)(1-B)+t(A-B)}z^2, \tag{13}$$

shows that the inequality (12) is sharp.

Theorem 2.4. *Let the functions* $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ *and* $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$ *be in the class* $Y_t(A,B)$*, then* $(f*g)(z)$ *belongs to* $Y_t(A,B_0)$*, where:*

$$B_0 \leqslant \frac{\left[\left(\frac{V}{A-B}\right)^2 - t\right]A - U}{\left(\frac{V}{A-B}\right)^2 - t - U}, \qquad U = \frac{k}{4(k!)^2} - t$$

and

$$V = U(1-B) + t(A-B). \tag{14}$$

Proof. It is sufficient to show that

$$\sum_{k=2}^{\infty}\left[\left(\frac{k}{4(k!)^2}-t\right)\left(\frac{1-B_0}{A-B_0}\right)+t\right]a_k b_k \leqslant 1.$$

By using Cauchy-Schwarz inequality, from (12), we obtain:

$$\sum_{k=2}^{\infty}\frac{\left(\frac{k}{4(k!)^2}-t\right)(1-B)+t(A-B)}{A-B}\sqrt{a_k b_k} \leqslant 1.$$

Hence, we find the largest B_0 such that:

$$\sum_{k=2}^{\infty}\frac{\left(\frac{k}{4(k!)^2}-t\right)(1-B_0)+t(A-B_0)}{A-B_0}a_k b_k$$
$$\leqslant \sum_{k=2}^{\infty}\frac{\left(\frac{k}{4(k!)^2}-t\right)(1-B)+t(A-B)}{A-B}\sqrt{a_k b_k} \leqslant 1,$$

or equivalently

$$\sqrt{a_k b_k} \leqslant \frac{\left[\left(\frac{k}{4(k!)^2}-t\right)(1-B)+t(A-B)\right](A-B_0)}{\left[\left(\frac{k}{4(k!)^2}-t\right)(1-B_0)+t(A-B_0)\right](A-B)}.$$

This inequality holds if

$$\frac{A-B}{\left(\frac{k}{4(k!)^2}-t\right)(1-B)+t(A-B)} \leqslant \frac{\left[\left(\frac{k}{4(k!)^2}-t\right)(1-B)+t(A-B)\right](A-B_0)}{\left[\left(\frac{k}{4(k!)^2}-t\right)(1-B_0)+t(A-B_0)\right](A-B)}.$$

After a simple algebraic manipulation, we conclude the required result. □

Theorem 2.5. *Let $f \in Y_t(A,B)$, then:*

$$X_f(z) = \int_0^z \frac{1+AW(s)}{s(1+BW(s))} f_t(s)ds, \qquad (|W(z)|<1). \tag{15}$$

Proof. Since $f(z) \in Y_t(A,B)$, so (11) holds. Hence

$$\frac{z\big(X_f(z)\big)' - f_t(z)}{Af_t(z) - Bz\big(X_f(z)\big)'} = W(z), \qquad (|W(z)|<1).$$

Therefore, we can write

$$\big(X_f(z)\big)' = \frac{\big(1+AW(z)\big)f_t(z)}{z\big(1+BW(z)\big)}.$$

After integration, we obtain the required result. □

3. Geometric properties of subfamilies of $Y_t(A,B)$

In this section, we introduce two subclasses of $Y_t(A,B)$ and obtain some Geometric properties of functions in these subclasses.

Let $\mathcal{P}(C,D)$ consist of all analytic functions $g(z)$ in $\mathbb{U}$ for which $g(0)=1$ and

$$g(z) \prec \frac{1+Cz}{1+Dz}, \tag{16}$$

where $-1 \leqslant C < D \leqslant 1$ and $0 < D \leqslant 1$.

Furthermore, suppose that $\mathcal{Q}(C,D)$ denote the class of all functions $f(z) \in Y_t(A,B)$ for which

$$\frac{z\big(X_f(z)\big)'}{X_f(z)} \in \mathcal{P}(C,D). \tag{17}$$

Theorem 3.1. $f(z) \in \mathcal{Q}(C,D)$ *if and only if*

$$\sum_{k=2}^{\infty} \frac{1}{4(k!)^2}\Big[1 + \frac{(D+1)(k-1)}{D-C}\Big] a_k < 1. \tag{18}$$

Proof. Let $f(z) \in \mathcal{Q}(C,D)$ then by (10), (16) and (17) we have:

$$\left| \frac{z - \sum_{k=2}^{\infty} \frac{1}{4(k!)^2} a_k z^k - z + \sum_{k=2}^{\infty} \frac{k}{4(k!)^2} a_k z^k}{Dz\Big(1 - \sum_{k=2}^{\infty} \frac{k}{4(k!)^2} a_k z^{k-1}\Big) - C\Big(z - \sum_{k=2}^{\infty} \frac{1}{4(k!)^2} a_k z^k\Big)} \right| < 1,$$

which implies that

$$\mathrm{Re}\left\{ \frac{\sum_{k=2}^{\infty} (k-1)\big(\frac{1}{4(k!)^2}\big) a_k z^{k-1}}{(D-C) - \sum_{k=2}^{\infty} (Dk - C) \frac{1}{4(k!)^2} a_k z^{k-1}} \right\} < 1.$$

Now, by choosing the values of z on the real axis and letting $z \to 1^-$, we obtain:

$$\frac{\sum\limits_{k=2}^{\infty} \frac{k-1}{4(k!)^2} a_k}{(D-C) - \sum\limits_{k=2}^{\infty} (Dk-C) \frac{1}{4(k!)^2} a_k} < 1.$$

Then after a simple calculation, we conclude the result.

Conversely, assume that the relation (18) holds. We must show that $f(z) \in \mathcal{Q}(C,D)$, or equivalently

$$\mathcal{M}(z) = \left| \frac{X_f(z) - z\big(X_f(z)\big)'}{Dz\big(X_f(z)\big)' - CX_f(z)} \right| < 1.$$

But we have:

$$\begin{aligned} \mathcal{M}(z) &= \left| \frac{\sum\limits_{k=2}^{\infty} \frac{k-1}{4(k!)^2} a_k z^{k-1}}{(D-C) - \sum\limits_{k=2}^{\infty} (Dk-C) \frac{1}{4(k!)^2} a_k z^{k-1}} \right| \\ &< \frac{\sum\limits_{k=2}^{\infty} \frac{k-1}{4(k!)^2} a_k}{(D-C) - \sum\limits_{k=2}^{\infty} (Dk-C) \frac{1}{4(k!)^2} a_k}. \end{aligned}$$

By using (18), the last inequality is less than one, so the proof is complete. □

Theorem 3.2. *Let $f(z) \in \mathcal{Q}(C,D)$ and*

$$\frac{z\big(X_f(z)\big)'}{X_f(z)} = a + ib = \eta.$$

Then the values of η lie in the circle, with center at $\left(\frac{1-CD}{1-D^2}, 0\right)$ and radius $\frac{D-C}{1-D^2}$.

Proof. By (16) and (17), we have:

$$\eta = a + ib = \frac{1 + Cv(z)}{1 + Dv(z)}, \qquad (|v(z)| < 1).$$

Then $(a+ib)(1+Dv(z)) = 1 + Cv(z)$, or

$$(a-1) + ib = \big[(C - aD) - ibD\big] v(z),$$

and so

$$(a-1)^2 + b^2 < (C - aD)^2 + b^2 D^2.$$

After a simple calculation, we obtain:

$$\left(a - \frac{1-CD}{1-D^2}\right)^2 + b^2 < \left(\frac{D-C}{1-D^2}\right)^2.$$

Hence the values of η lie in the circle with center at $\left(\frac{1-CD}{1-D^2}, 0\right)$ and radius $\frac{D-C}{1-D^2}$. □

Theorem 3.3. *Let $0 \leqslant C_2 < C_1 < 1$, then $\mathcal{Q}(C_1, D) \subset \mathcal{Q}(C_2, D)$.*

Proof. Suppose that $f(z) \in \mathcal{Q}(C, D)$, then

$$\sum_{k=2}^{\infty} \frac{1}{4(k!)^2}\Big[1 + \frac{(D+1)(k-1)}{D-C}\Big] a_k < 1.$$

We have to prove that

$$\sum_{k=2}^{\infty} \frac{1}{4(k!)^2}\Big[1 + \frac{(D+1)(k-1)}{D-C_2}\Big] a_k < 1.$$

But the last inequality holds if

$$1 + \frac{(D+1)(k-1)}{D-C_2} \leqslant 1 + \frac{(D+1)(k-1)}{D-C_1},$$

and this by hypothesis definitely holds. □

Theorem 3.4. *The class $\mathcal{Q}(C, D)$ is a convex set.*

Proof. We must show that if $f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k$ $(j = 1, 2, \ldots, m)$ is in $\mathcal{Q}(C, D)$, then the function $F(z) = \sum_{j=1}^{m} \lambda_j f_j(z)$ where $\sum_{j=1}^{m} \lambda_j = 1$ is also in $\mathcal{Q}(C, D)$. But we have

$$\begin{aligned} F(z) &= \sum_{j=1}^{m} \lambda_j \Big(z - \sum_{k=2}^{\infty} a_{k,j} z^k\Big) \\ &= z - \sum_{j=1}^{m} \lambda_j \Big(\sum_{k=2}^{\infty} a_{k,j} z^k\Big) \\ &= z - \sum_{k=2}^{\infty} \Big(\sum_{j=1}^{m} \lambda_j a_{k,j}\Big) z^k. \end{aligned}$$

Hence

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{1}{4(k!)^2}\Big[1 + \frac{(D+1)(k-1)}{D-C}\Big]\Big(\sum_{j=1}^{m} \lambda_j a_{k,j}\Big) \\ &= \sum_{j=1}^{m} \Bigg[\sum_{k=2}^{\infty} \frac{1}{4(k!)^2}\Big[1 + \frac{(D+1)(k-1)}{D-C}\Big] a_{k,j}\Bigg] \lambda_j. \end{aligned}$$

Since $f_j(z) \in \mathcal{Q}(C, D)$, so by Theorem 3.1 (inequality (18)), we have

$$\sum_{k=2}^{\infty} \frac{1}{4(k!)^2}\Big(1 + \frac{(D+1)(k-1)}{D-C}\Big) a_{k,j} \leq 1.$$

Hence

$$\sum_{j=1}^{m}\left[\sum_{k=2}^{\infty}\frac{1}{4(k!)^2}\Big[1+\frac{(D+1)(k-1)}{D-C}\Big]a_{k,j}\right]\lambda_j \le \sum_{j=1}^{m}\lambda_j = 1.$$

So by Theorem 3.1, $F(z) \in \mathcal{Q}(C, D)$. □

Conclusion

In Geometric Function Theory, many authors have studied various coefficient estimates of other classes of univalent functions. By using the Sigmoid function, convolution structure and subordination, we achieved a new subclass of univalent functions, the sharp coefficient bounds, convolution preserving property, integral representation and many other geometric properties.

4. Aknowledgement

The authors would like to express their sincere thanks to anonymous referees for their valuable comments which improved the presentation of the manuscript.

References

[1] Arif, M., Marwa, S., Xin, Q., Tchier, F., Ayaz, M. and Malik, SN (2022). *Sharp Coefficient Problems of Functions with Bounded Turnings Subordinated by Sigmoid Function.* Mathematics, 10(20), 3862. https://doi.org/10.3390/math10203862

[2] Cağlar, M., and Orhan, H. (2019). *(θ, μ, τ)-neighborhood for analytic functions involving modified sigmoid function.* Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 68(2), 2161–2169. https://doi.org/10.31801/cfsuasmas.515557

[3] Duren., PL (1983). *Univalent functions, Grundlehren der mathematischen.* Wissenschaften 259, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo.

[4] Fadipe-Joseph, OA, Oluwayemi, MO, and Titiloye, EO (2021). *Subclasses of Univalent Functions Involving Modified Sigmoid Function.* Int. J. Differ. Equ., 16(1), 81–93. https://dx.doi.org/10.37622/IJDE/16.1.2021.81-93

[5] Hamzat, JO, Oladipo, AT, and Oros, GI (2022). *Bi-univalent problems involving certain new subclasses of generalized multiplier transform on analytic functions associated with modified sigmoid function.* Symmetry, 14(7), 1479. https://doi.org/10.3390/sym14071479

[6] Kamali, M.,Orhan, H., and CAĞLAR M. (2020). *The Fekete-Szegö inequality for subclasses of analytic functions related to modified Sigmoid functions.* Turk. J. Math., 44(3), 1016–1026. https://doi.org/10.3906/mat-1910-85

[7] Miller, SS, and Mocanu, PT (2000). *Differential subordinations: theory and applications.* CRC Press.

[8] Murugusundaramoorthy, G., and Janani, T. (2015). *Sigmoid function in the space of univalent λ-pseudo starlike functions.* Int. J. Pure Appl. Math., 101(1), 33–41. https://doi.org/10.12732/ijpam.v101i1.4

[9] Najafzadeh, Sh, and Kulkarni, SR (2006). *Note on Application of Fractional calculus and subordination to p-valent functions.* Mathematica (cluj), 48(71), No 2, 167–172.

[10] Olatunji, S., Gbolagade, A., Anake, T., and Fadipe-Joseph O. (2013). *Sigmoid function in the space of univalent function of Bazilevic type.* Scientia Magna, 9(3), 43-51.

[11] Orhan, H., Murugusundaramoorthy, G., and Caglar, M. (2022). *The Fekete-Szegö problem for subclass of bi-univalent functions associated with sigmoid function.* Facta Univ., Math. Inform., 495–506. https://doi.org/10.22190/FUMI201022034O

[12] Priyanka, G., and Sivaprasad Kumar, S. (2020). *Certain class of starlike functions associated with modified sigmoid function.* Bull. Malaysian Math. Sci. Soc., 43(1), 957–991. https://doi.org/10.1007/s40840-019-00784-y

[13] Sakar, FM, and Aydogan, SM (2023). *Inequalities of bi-starlike functions involving Sigmoid function and Bernoulli Lemniscate by subordination.* Int. J. Open Problems Compt. Math., 16(1), 71–82.

[14] Wang, X., and Wang, Z. (2018). *Coefficient inequality for a new subclass of analytic and univalent functions related to sigmoid function.* Int. J. Mod. Math. Sci., 16(1), 51–57.

Farideh Madadi Tamrin
Orcid number: 0009-0001-5766-7266
Department of Mathematics
Payame Noor University
Post Office Box: 19395–3697
Tehran, Iran
Email address: f.madadi1611@gmail.com

Shahram Najafzadeh
Orcid number: 0000-0002-8124-8344
Department of Mathematics
Payame Noor University
Post Office Box: 19395–3697
Tehran, Iran
Email address: shnajafzadeh44@pnu.ac.ir, najafzadeh1234@yahoo.ie

Mohammadreza Foroutan
Orcid number: 0000-0002-7373-617X
Department of Mathematics
Payame Noor University
Post Office Box: 19395–3697
Tehran, Iran
Email address: mr_forootan@pnu.ac.ir, foroutan_mohammadreza@yahoo.com