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ON THE POWER OF GINI INDEX-BASED GOODNESS-OF-FIT TEST FOR THE INVERSE GAUSSIAN DISTRIBUTION

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Abstract. The Inverse Gaussian distribution finds application in various fields, such as finance, survival analysis, psychology, engineering, physics, and quality control. Its capability to model skewed distributions and non-constant hazard rates make it a valuable tool for understanding a wide range of phenomena. In this paper, we present a goodness-of-fit test specifically designed for the Inverse Gaussian distribution. Our test uses an estimate of the Gini index, a statistical measure of inequality. We provide comprehensive details on the exact and asymptotic distributions of the newly developed test statistic. To facilitate the application of the test, we estimate the unknown parameters of the Inverse Gaussian distribution using maximum likelihood estimators. Monte Carlo methods are utilized to determine the critical points and assess the actual sizes of the test. A power comparison study is conducted to evaluate the performance of existing tests. Comparing its powers with those of other tests, we demonstrate that the Gini index-based test performs favorably. Finally, we present a real data analysis for illustrative purposes.

Keywords: Gini index, Inverse Gaussian distribution, Goodness-of-fit tests, Type-I error, Critical points, Test power, Monte Carlo simulation.
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1. Introduction

Goodness-of-fit (GOF) tests are statistical procedures used to evaluate how well an observed dataset fits a specific theoretical distribution or model. These tests aim to assess the degree of compatibility between the observed data and the hypothesized distribution or model. The applications of GOF tests are diverse and span various fields:

(1) **Quality Control:** GOF tests are commonly used in quality control to verify if a manufacturing process follows a specified distribution. By comparing the observed process data with the assumed distribution, GOF tests can help identify deviations and detect potential issues affecting product quality.

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(2) **Model Selection:** GOF tests play a vital role in model selection by comparing different models to determine which one offers the best fit to the observed data. This process helps researchers or analysts choose the most appropriate model for their specific research question or predictive purposes.
(3) **Hypothesis Testing:** GOF tests are used for hypothesis testing to assess the GOF between the observed sample and the hypothesized distribution. This helps determine whether the observed data can be generated from the assumed distribution or if another distribution should be considered.
(4) **Predictive Modeling:** GOF tests are important in predictive modeling, where the objective is to develop models that accurately predict future outcomes. By evaluating the GOF, these tests help determine if the proposed model is suitable for making reliable predictions.
(5) **Risk Management:** GOF tests are invaluable in risk management, particularly in assessing the fit of statistical models used for estimating and managing risks. By evaluating the GOF, these tests provide insights into the accuracy and reliability of the risk estimates obtained from the model.

Overall, GOF tests provide a quantitative measure of the agreement between observed data and theoretical distributions or models. They serve as valuable tools in various applications such as quality control, model selection, hypothesis testing, predictive modeling, and risk management. By assessing the GOF, these tests help ensure the validity and reliability of statistical analyses, decision-making processes, and predictive models in numerous fields.
The Inverse Gaussian (IG) distribution is widely applied in various fields, including finance, survival analysis, psychology, engineering, physics, and quality control. Its capacity to model skewed distributions and non-constant hazard rates makes it a valuable tool for understanding diverse phenomena. Notably, researchers such as Folks and Chhikara (1978), Bardsley (1980), Chhikara and Folks (1989), Seshadri (1993, 1999), Johnson et al. (1994), and Barndorff-Nielsen (1994) have recognized its significance. Consequently, it is crucial to assess whether the IG model adequately represents the observed data.
In order to determine the appropriateness of approximating the unknown distribution of a sample, denoted as F, with an IG model, several GOF tests can be employed. These tests are specifically designed to measure the level of agreement between the observed sample data and the proposed IG model.
One widely utilized class of GOF tests involves evaluating the discrepancy between the empirical distribution function (EDF) derived from the sample data and the distribution function assumed by the IG model. Prominent tests belonging to this class include the Cramer-von Mises, Kolmogorov-Smirnov, Kuiper, Watson, and Anderson-Darling tests. These tests assess the level of agreement between the observed sample data and the theoretical distribution

function implied by the IG model. Each of these tests quantifies the distance or dissimilarity between the EDF and the assumed distribution. For further details on these tests, including their specific formulations and properties, D'Agostino and Stephens (1986) provide comprehensive information. These tests serve as valuable tools for evaluating the GOF between the observed data and the proposed IG model, offering insights into the adequacy of the IG distribution as an approximation for the unknown distribution.

By applying these GOF tests, researchers can gain insights into the extent to which the observed sample data aligns with the theoretical assumptions of the IG model. The examination of these tests allows for effective evaluation and determination of the degree of fit between the data and the IG model, facilitating informed decision-making in statistical analysis.

Recently, EDF-tests for the IG distribution are investigated by Alizadeh and Shafaei (2024a). They compared the power of the tests based on the empirical distribution function and discovered that the actual sizes of the tests based on Zhang's (2002) statistics (i.e., Z_A and Z_C) were deemed acceptable. As a result, they recommend employing the tests based on Z_A and Z_C statistics in practical applications. Their overall conclusion suggests that these tests, Z_A and Z_C, exhibit strong performance across a wide range of alternative hypotheses, making them reliable and suitable choices for hypothesis testing in practice. Following their work, Alizadeh and Shafaei (2024b) proposed some new tests for the IG distribution based on varentropy and obtained the power of the tests. These varentropy-based tests, denoted as TE and TA, were found to exhibit strong performance in terms of power. The power differences observed between these tests and the other competing tests were found to be significant.

The Gini index is a widely used statistical measure of inequality or diversity within a given dataset. Originally developed by the statistician Corrado Gini in 1912, it has become an invaluable tool for assessing and quantifying inequality in various fields.

In the realm of statistics, the Gini index is commonly applied to measure income or wealth inequality within populations. It provides a concise summary of the distribution of these resources by condensing the entire income or wealth distribution curve into a single numerical value. The Gini index ranges between 0 and 1, with 0 indicating perfect equality (all individuals have the same income or wealth) and 1 denoting extreme inequality (one individual possesses all the income or wealth).

Moreover, the Gini index finds applications beyond income and wealth inequality. It is also utilized in fields such as social sciences, healthcare, ecology, and even computer science. For instance, in social sciences, it can analyze educational attainment or life satisfaction disparities. In healthcare, it can assess disparities in disease prevalence or access to medical services. In ecology, it can determine species diversity and the evenness of distribution. In computer science, it can measure diversity in algorithms or recommenders.

The Gini index enables researchers to compare inequality among different populations or track changes in inequality over time. By capturing the essence of distribution patterns within a single number, it allows for straightforward comparisons, policy evaluations, and monitoring of social or economic conditions. In conclusion, the Gini index is a versatile statistical measure widely employed to quantify inequality or diversity. Its applications span various fields, providing valuable insights into income disparities, resource distribution, social disparities, ecological diversity, and beyond. The Gini index serves as a robust tool for researchers and policymakers seeking to understand, address, and monitor inequality in its diverse manifestations.
The Gini coefficient is defined as

$$G = 1 - 2\int_0^1 L(p)dp\,,$$

where L is the Lorenz function given by

$$L(p) = \frac{1}{E(X)}\int_0^p F^{-1}(t)dt\,.$$

Giles (2004) presents an alternative formulation of the Gini index, which can be expressed as

$$G = \frac{\int_m^M F(y)\,(1 - F(y))\,dy}{\mu}\,,$$

where the random variable Y is defined on a real interval (m, M), with $0 \leq m < M < \infty$. Additionally, μ represents the expected value of the variable Y within this interval.
Consider a random sample $X_1, \ldots, X_n$ drawn from a continuous probability distribution F with a corresponding probability density function $f(x)$, the standard estimator of the Gini index is typically calculated as follows:

$$\hat{G}_n = \frac{\int_m^M y\,(2F_n(y) - 1)\,dF_n(y)}{\bar{X}} = \sum_{i=1}^{n} \frac{(2i - n)X_{(i)}}{n\sum_{j=1}^{n} X_j}\,,$$

where $x_{(1)} \leq x_{(2)} \leq ... \leq x_{(n)}$ represent the order statistics derived from the sample, and F_n represents the empirical distribution function.
The Gail and Gastwirth study in 1978 made significant advancements in the field of GOF testing by introducing a test specifically for exponential distribution. They utilized the Gini statistic and provided evidence of its effectiveness in assessing the fit of data to the exponential distribution. Following their work, Jammalamadaka and Goria (2004) introduced a test using the Gini index of spacings, further expanding the applications of the Gini index in GOF testing. Recently, Qiu et al. (2022) and Pakyari (2023) constructed goodness of fit tests based on spacings for progressive Type-II censored data. Then, Alizadeh et al. (2014) suggested a general GOF test based on Gini index and constructed tests for normal, exponential, Laplace distributions. Building upon these foundational studies, Alizadeh Noughabi (2017) introduced a novel GOF test for

the logistic distribution by leveraging the Gini index. This approach proved valuable in evaluating the fit of observed data to the logistic distribution. In this paper, we contribute to this line of research by presenting a distinct GOF test specifically designed for the IG distribution. Our methodology revolves around estimating the Gini index, enabling us to assess the compatibility of the observed data with the IG distribution. To establish the effectiveness and practicality of our proposed test, we conduct empirical analysis and statistical evaluations. Through these empirical analyses and statistical evaluations, we demonstrate the efficacy and applicability of our proposed test for assessing the GOF of data to the IG distribution. Our research expands upon the existing body of literature and offers a valuable contribution to the field of GOF testing. Section 2 delves into a comprehensive examination of various properties associated with the IG distribution. Based on this analysis, we proceed to develop a novel GOF test specifically designed for the IG distribution, utilizing the Gini index as a key component. In Section 3, we employ Monte Carlo simulations to determine the critical points and actual sizes of the proposed test. These simulations enable us to generate synthetic data sets and evaluate the performance of the test under different scenarios. Additionally, we calculate the power values of our proposed test and compare them against those obtained from established competing tests. Moving forward, Section 4 presents a detailed illustrative example where we apply the proposed test to real-world data. Through an in-depth analysis of this example, we provide a comprehensive understanding of how the test performs in a practical context. Finally, in Section 5, we offer a concise conclusion that summarizes the key findings and contributions of our study. This section serves as a succinct summary, highlighting the significance and implications of our proposed test for the IG distribution.

2. The Inverse Gaussian Distribution and the Test Statistic

In this section, we present an outline of the properties of the IG distribution and introduce a novel GOF test statistic specifically designed for assessing the conformity of data to this distribution. By presenting these properties and developing a test statistic, we contribute to the field by offering a comprehensive approach to assessing the fit of data to the IG distribution.

2.1. **The Inverse Gaussian distribution.** Schrödinger (1915) was the first to derive the probability distribution of the first passage time in Brownian motion. Tweedie (1957) investigated the basic characteristics of this distribution and proposed the name inverse Gaussian, and it is also known as Wald's distribution. Folks and Chhikara (1978) gave a review of this distribution and mentioned numerous applications of it. An application of the IG distribution as a life time model is possibly the most appealing one; see Chhikara and Folks (1989), Gunes et al. (1997), and Seshadri (1993).

The probability density function for the IG distribution can be expressed as follows:

$$f(x;\mu,\lambda)=\left(\frac{\lambda}{2\pi x^3}\right)^{1/2}\exp\left\{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right\},\qquad x>0,$$

where $\mu>0$ and $\lambda>0$ are parameters. Specifically, the mean of the IG distribution is equal to μ, while the variance is given by μ^3/λ. The complete sufficient statistics for parameters μ and λ are (Folks and Chhikara, 1978, Seshadri, 1993) $\bar{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$ and $T=\sum_{i=1}^{n}\left(\frac{1}{X_i}-\frac{1}{\bar{X}}\right)$, respectively. Moreover, the MLEs of μ and λ are $\hat{\mu}=\bar{X}$ and $\hat{\lambda}=n/T$. This distribution is denoted as $IG(\mu,\lambda)$.

Figure 1 displays the density function of the IG for different values of parameters.

The IG distribution possesses several significant properties that contribute to its usefulness in diverse applications. A significant property of the IG distribution is its role as a conjugate prior distribution for the mean of a normal distribution with an unknown variance. This important property allows for the derivation of a posterior distribution, which provides updated information on the mean of the normal distribution utilizing the prior information from the IG distribution.

In survival analysis, the IG distribution finds extensive application in modeling the time until an event takes place. This is particularly relevant when studying events such as machine failures or patient deaths, where the distribution can effectively capture the underlying time-to-event patterns. Overall, the IG distribution's properties make it a valuable tool in various fields, enabling statistical analysis and modeling in scenarios involving unknown means, survival times, and more.

Indeed, the IG distribution holds significant applications in the field of finance. It is commonly employed to model the distribution of various financial variables, including stock prices, interest rates, and other relevant quantities. By capturing the specific characteristics of these variables, the IG distribution aids in understanding their behavior and making informed financial decisions.

Furthermore, the distribution finds utility in engineering and physics as well. In engineering, it can be used to model reaction times, particle sizes, and other physical variables encountered in various processes. In physics, the IG distribution helps describe the distribution of certain phenomena or measurements. Given its distinctive shape and valuable properties, the IG distribution serves as a versatile tool for modeling a broad spectrum of phenomena across numerous fields. For more in-depth exploration of its applications, the works of Folks and Chhikara (1978), Chhikara and Folks (1989), Seshadri (1999), and their respective references provide a wealth of insights.

In conclusion, the IG distribution finds extensive applications across various fields, including finance, survival analysis, psychology, engineering, physics, and

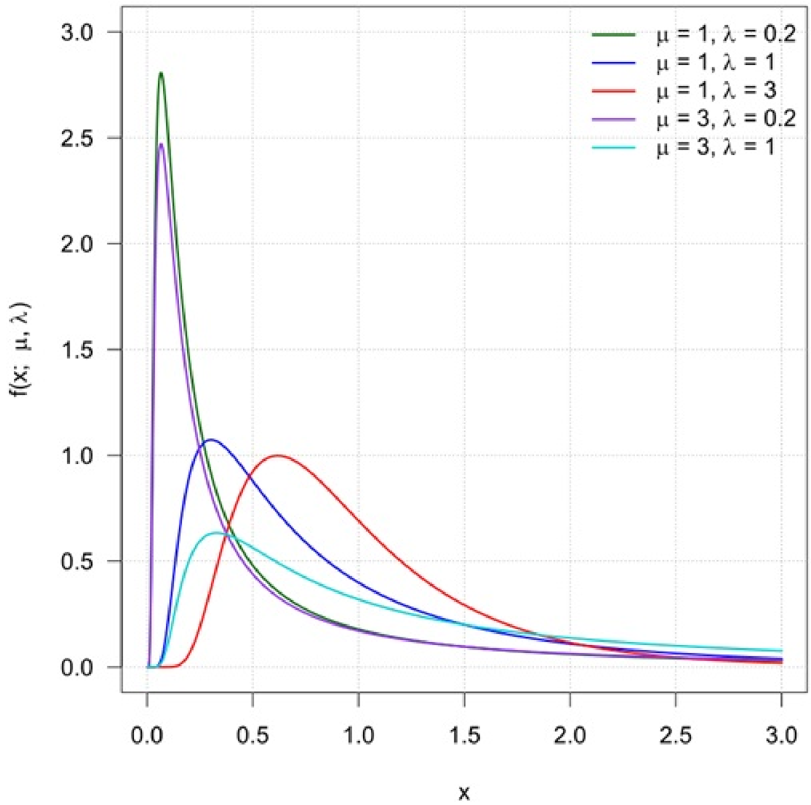

FIGURE 1. Probability density function of IG distribution for different parameters.

quality control. Its capability to model skewed distributions and non-constant hazard rates makes it a valuable tool for capturing a wide range of phenomena. Relevant references to explore further include Folks and Chhikara (1978), Bardsley (1980), Chhikara and Folks (1989), Seshadri (1993, 1999), Johnson et al. (1994), and Barndorff-Nielsen (1994).

The development of a reliable GOF test for the IG distribution holds substantial importance. In this article, we address this crucial issue by introducing a novel GOF test specifically tailored for the IG distribution. Our test statistic is based on the utilization of the Gini index, providing an accurate measure for evaluating the fit of observed data to the IG distribution. By incorporating the Gini index into our test, we enhance the effectiveness and precision of the GOF assessment for the IG distribution.

In order to estimate the unknown parameters μ and λ, of the IG distribution,

we will utilize the method of maximum likelihood estimation (MLE). This approach allows us to find the parameter values that maximize the likelihood of the observed data given the IG distribution.
Let us consider a random sample $X_1, ..., X_n$ drawn from the IG distribution. The ML estimators of the parameters μ and λ can be computed by maximizing the likelihood function, which is defined as the product of the probability density function of IG distribution for each observation. By maximizing the likelihood function with respect to μ and λ, we can find the parameter values that best fit the observed data to the IG distribution. The MLE of the parameters μ and λ are

$$\hat{\mu} = \bar{X} \qquad ; \qquad \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} \left(1/X_i - 1/\bar{X}\right)}.$$

2.2. **The test statistic.** Consider a random sample $X_1, ..., X_n$ drawn from a population with a cumulative distribution function F and its corresponding density function

$$H_0 : \; \{X_1, ..., X_n\} \;\; is\, a\, sample\;\, from\;\, IG(\mu, \lambda),$$

where μ and λ are unknown parameters. The alternative hypothesis is expressed as

$$H_1 : \; \{X_1, ..., X_n\} \;\; is\, not\, a\, sample\;\, from\;\, IG(\mu, \lambda).$$

If we denote the density function of the IG distribution as $f_0(x; \mu, \lambda)$, the hypothesis of interest can be stated as follows:

$$H_0 : \; f(x) = f_0(x; \mu, \lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right\}, \quad for\ some\ (\mu, \lambda) \in \Theta\,,$$

where $\Theta = R^+ \times R^+$. The alternative to H_0 is

$$H_1 : \; f(x) \neq f_0(x; \mu, \lambda), \qquad for\ any\ (\mu, \lambda) \in \Theta\,.$$

Let F_0 represent the IG distribution function. Without loss of generality, we can simplify the testing hypothesis by applying the probability integral transformation, $U = F_0(X)$, which allows us to test the hypothesis of uniformity on the interval (0,1). For more comprehensive information, please refer to the study by Ebner et al. (2022).
Suppose $U_i = F_0(X_i)$, $i = 1, 2, ..., n$ represents the transformed sample using the cumulative distribution function F_0. In this context, the hypothesis of interest can be stated as follows:

$$H_0 : \; f(u) = 1, \quad 0 < u < 1,$$

against

$$H_1 : \; f(u) \neq 1, \quad 0 < u < 1.$$

Consequently, the test for the IG distribution is transformed into a uniformity test on the interval (0,1). Under the null hypothesis, each U_i follows a uniform distribution. To assess the uniformity of the distribution of U_i's and, consequently, the IG assumption for the distribution of X_i's, we employ a test statistic based on the Gini index. Therefore, the proposed test can be expressed as:

$$\begin{aligned}\hat{G}_n &= \sum_{i=1}^{n} \frac{(2i-n)u_{(i)}}{n\sum_{j=1}^{n} u_j} \\ &= \sum_{i=1}^{n} \frac{(2i-n)F_0(x_{(i)};\hat{\mu},\hat{\lambda})}{n\sum_{j=1}^{n} F_0(x_i;\hat{\mu},\hat{\lambda})},\end{aligned}$$

where $u_{(1)} \leq u_{(2)} \leq ... \leq u_{(n)}$ denote the order statistics of transformed sample and $\hat{\mu}$ and $\hat{\lambda}$ are the MLEs of the parameters μ and λ, respectively.

$$\hat{\mu} = \bar{X} \quad ; \quad \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} \left(1/X_i - 1/\bar{X}\right)}.$$

The following theorem provides the exact distribution of the test statistic $\hat{G}_n$ under the null hypothesis.

Theorem 2.1. *Assuming $u_1, u_2, ..., u_n$ represents a random sample drawn from the uniform distribution, the following can be stated:*

$$F_{\hat{G}_n}(t) = P\left(\hat{G}_n \leq t\right) = \int_0^{\tau(a_n)} \int_0^{\tau(a_{n-1})} ... \int_0^{\tau(a_1)} e^{-\sum_{i=1}^{n} t_j} dt_1...dt_n,$$

where $a_i = (n+1-i)(i-nt)$ for $1 \leq i \leq n$ and

$$\tau(a_j) = \begin{cases} \infty & if\ a_j \leq 0, \\ -\sum_{i=j+1}^{n} a_i t_i / a_j & if\ a_j > 0\,. \end{cases}$$

Proof. The proof of this theorem follows a similar approach to the theorem mentioned in Martinez-Camblor and Correal's work in 2009 and Alizadeh Noughabi et al.'s work in 2014. For additional details and a more comprehensive understanding of the proof, it is recommended to refer to these sources. □

According to Martinez-Camblor and Correal (2009), the computation of the $F_{\hat{G}_n}$ is complicated for sample sizes $n \geq 5$. Consequently, the exact distribution of the test statistic under the null hypothesis is not applicable. However, the asymptotic distribution of the test statistic can be derived and is shown in the following theorem.

Theorem 2.2. *Assuming $u_1, u_2, ..., u_n$ represents a random sample drawn from the uniform distribution, then*

$$\sqrt{n}\frac{\hat{G}_n - 1/3}{\sqrt{8/135}} \to N(0,1)\,.$$

Proof. For the detailed proof of the theorem, we refer to the works by Martinez-Camblor and Correal in 2009, as well as the work by Alizadeh Noughabi et al. in 2014. These sources provide the necessary information and explanations to understand the proof in depth. □

Based on the aforementioned theorem, for sufficiently large sample sizes, we can obtain critical points for the proposed test statistic. These critical values are used to determine the rejection region and make decisions regarding the null hypothesis.

3. Simulation Study

3.1. **Critical points and actual sizes.** In order to determine the critical values of the proposed test statistic, Monte Carlo simulations are utilized, as the exact distribution of the test statistic is not readily derivable for different sample sizes. Monte Carlo simulations involve generating numerous random datasets that follow the assumed null hypothesis and calculating the test statistic for each simulated dataset. By repeating this process multiple times, critical values can be estimated based on the desired significance level and the distribution of the simulated test statistics. These simulations provide an empirical estimation of the critical values, enabling reliable inference and hypothesis testing in cases where an exact distribution is not analytically obtainable.
This approach involves conducting a large number of simulation runs, typically 100,000, for each sample size. The following steps outline the methodology for determining the critical values:

(1) Generate a random sample, denoted as $X_1, ..., X_n$, of size n, drawn from the $IG(1, 1)$ distribution.
(2) Calculate the proposed test statistic based on the sample $X_1, ..., X_n$.
(3) Repeat steps 1 and 2 a significant number of times, creating a large number of simulated datasets and corresponding test statistics.
(4) Determine the αth quantile of the test statistics obtained from the simulations. This quantile represents the critical value for the proposed test at the desired significance level α.

By generating multiple samples, calculating the test statistic, and repeating the process, we obtain an empirical distribution of the test statistic under the null hypothesis. The critical value, derived from this empirical distribution, allows us to make reliable decisions regarding the hypothesis test.
Table 1 contains the critical values of the proposed test statistic for various sample sizes, denoted as n. These critical values are essential in hypothesis testing, as they establish the threshold for accepting or rejecting the null hypothesis based on the observed test statistic. By referring to Table 1, one can determine the appropriate critical value corresponding to a specific sample size, facilitating the decision-making process in evaluating the GOF of the observed

data to the assumed distribution.

Table 1. Critical points of the proposed test statistic for different significance levels.

n	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
5	0.4380	0.4539	0.4700	0.4893	0.5605	0.5629	0.5644	0.5658
10	0.3484	0.3665	0.3808	0.3959	0.4590	0.4626	0.4649	0.4670
15	0.3306	0.3446	0.3558	0.3679	0.4232	0.4271	0.4298	0.4325
20	0.3230	0.3351	0.3449	0.3551	0.4045	0.4085	0.4113	0.4140
25	0.3204	0.3309	0.3393	0.3482	0.3931	0.3969	0.3997	0.4026
30	0.3187	0.3278	0.3355	0.3436	0.3850	0.3888	0.3916	0.3945
40	0.3178	0.3253	0.3316	0.3384	0.3748	0.3782	0.3810	0.3838
50	0.3174	0.3239	0.3294	0.3355	0.3684	0.3717	0.3743	0.3770

Table 2 presents the evaluation of the type I error of the proposed test for various sample sizes (n). The type I error refers to the proportion of times the null hypothesis is incorrectly rejected when it is actually true. By analyzing the actual size, which is the empirical estimation of the type I error rate, we gain insights into the performance of the test in controlling false positive errors. The results obtained from Table 2 offer valuable information about the behavior of the proposed test and its accuracy in maintaining the desired significance level. By examining the actual size, we can assess whether the test appropriately controls the probability of incorrectly rejecting the null hypothesis, even when it is true. This analysis enhances our understanding of the test's statistical properties and aids in determining its reliability and suitability for hypothesis testing.

Table 2. Type I error control of the tests for the nominal significance level $\alpha = 0.05$.

	n	$IG(0.5, 0.5)$	$IG(0.5, 1)$	$IG(0.5, 2)$	$IG(1, 0.5)$	$IG(1, 1)$	$IG(1, 2)$	$IG(2, 0.5)$	$IG(2, 1)$	$IG(2, 2)$
G_n^1	10	0.0501	0.0403	0.0351	0.0638	0.0518	0.0407	0.0778	0.0643	0.0509
	20	0.0507	0.0421	0.0379	0.0594	0.0501	0.0430	0.0680	0.0591	0.0502
	30	0.0481	0.0421	0.0361	0.0575	0.0490	0.0407	0.0658	0.0554	0.0481
	50	0.0513	0.0422	0.0371	0.0573	0.0482	0.0429	0.0645	0.0579	0.0508
G_n^2	10	0.0504	0.0485	0.0474	0.0632	0.0523	0.0489	0.0754	0.0618	0.0504
	20	0.0510	0.0459	0.0450	0.0602	0.0503	0.0472	0.0684	0.0611	0.0499
	30	0.0502	0.0459	0.0435	0.0589	0.0503	0.0451	0.0680	0.0572	0.0498
	50	0.0514	0.0470	0.0425	0.0569	0.0497	0.0465	0.0645	0.0569	0.0514

The findings from the analysis presented in Table 2 confirm that the empirical percentiles, as shown in Table 1, effectively control the type I error. This

indicates that the proposed test successfully maintains the desired significance level, ensuring accurate hypothesis testing. The results provide substantial evidence to support the validity and reliability of the proposed test, establishing its capability to control the probability of erroneously rejecting the null hypothesis when it is indeed true. These findings contribute to the confidence and trustworthiness of the proposed test procedure as a robust tool for hypothesis testing in the given context.
Considering the findings presented in Table 2, it is observed that the type I error increases as the value of μ/λ increases. Specifically, when $\mu/\lambda \approx 1$, the type I error (represented by α) is close to the nominal value. This suggests that the proposed test maintains its desired level of significance when $\mu/\lambda \approx 1$. Furthermore, when the parameters of the IG distribution are equal, the type I error remains within acceptable limits. This indicates that the proposed test performs well when the parameters are identical.
In general, the actual sizes of the proposed GOF test, as indicated by the type I error rates, are deemed acceptable. Therefore, based on these results, it can be concluded that the proposed test can be confidently utilized in practical applications.
In general, the actual sizes of the proposed GOF test, as indicated by the type I error rates, are deemed acceptable. Therefore, based on these results, it can be concluded that the proposed test can be confidently utilized in practical applications.

3.2. **Power study.** Alizadeh and Shafaei (2024a) examined the GOF tests utilizing the empirical distribution function for the IG distribution. They discovered that the actual sizes of the tests based on Z_A and Z_C statistics were deemed acceptable. As a result, they recommend employing the tests based on Z_A and Z_C statistics in practical applications. Their overall conclusion suggests that these tests, Z_A and Z_C, exhibit strong performance across a wide range of alternative hypotheses, making them reliable and suitable choices for hypothesis testing in practice. The statistics of these tests are as follows.

$$Z_A = -\sum_{i=1}^{n}\left(\frac{\log F_0(X_{(i)};\hat{\mu},\hat{\lambda})}{n-i+0.5} + \frac{\log\left[1 - F_0(X_{(i)};\hat{\mu},\hat{\lambda})\right]}{i-0.5}\right),$$

$$Z_C = \sum_{i=1}^{n}\left(\log\left\{\frac{F_0(X_{(i)};\hat{\mu},\hat{\lambda})^{-1} - 1}{(n-0.5)/(i-0.75)-1}\right\}\right)^2.$$

Furthermore, in a recent study by Alizadeh and Shafaei in 2024b, new GOF tests for the IG distribution were proposed based on varentropy estimators. These varentropy-based tests, denoted as TE and TA, were found to exhibit strong performance in terms of power. The power differences observed between these tests and the other competing tests were found to be significant. The statistics for the TE and TA tests are as follows:

$$TE = \frac{1}{n}\sum_{i=1}^{n}\log^2\left(\frac{c_i m/n}{U_{(i+m)}-U_{(i-m)}}\right) - \left[\frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{c_i m/n}{U_{(i+m)}-U_{(i-m)}}\right)\right]^2,$$

$$TA = \frac{1}{n}\sum_{i=1}^{n}\log^2\left\{\hat{f}(U_{(i+m)})+\hat{f}(U_{(i-m)})\right\} - \left[\frac{1}{n}\sum_{i=1}^{n}\log\left\{\hat{f}(U_{(i+m)})+\hat{f}(U_{(i-m)})\right\}\right]^2,$$

where

$$c_i = \begin{cases} 1+\frac{i-1}{m}, & 1 \le i \le m, \\ 2, & m+1 \le i \le n-m, \\ 1+\frac{n-i}{m}, & n-m+1 \le i \le n, \end{cases}$$

and

$$\hat{f}(X_i) = \frac{1}{nh}\sum_{j=1}^{n} k(\frac{X_i - X_j}{h}).$$

In their analysis, Alizadeh and Shafaei (2024a) selected the standard normal density function as the kernel function. Additionally, they chose the bandwidth h to be the normal optimal smoothing formula, which is defined as $h = 1.06sn^{-\frac{1}{5}}$, where s represents the sample standard deviation. Also, $U_i = F_0(X_i; \hat{\mu}, \hat{\lambda})$, $i = 1, 2, ..., n$, denote the transformed sample and $U_{(i)} = U_{(1)}$ if $i < 1$, $U_{(i)} = U_{(n)}$ if $i > n$.

These varentropy-based tests offer alternative approaches for assessing the GOF of the IG distribution, with promising power characteristics and the ability to detect deviations from the null hypothesis more effectively than other existing tests.

According to the studies conducted by Alizadeh and Shafaei in 2024a and 2024b, the GOF tests Z_A, Z_C, TE and TA demonstrate the highest power against a wide range of alternative hypotheses for the IG distribution. As a result, these tests have been selected as the competitor tests in their power comparison analysis. By comparing the power of the proposed test with these established tests, Alizadeh and Shafaei aim to assess the relative performance and efficacy of the proposed test in detecting deviations from the IG distribution.

To assess the power of the proposed test against various alternatives, Monte Carlo simulations are utilized. The power values indicate the test's ability to detect deviations from the null hypothesis. In the power comparison analysis, the following alternatives are considered:

- • the exponential distribution θ with density $\theta\exp(-\theta x)$;
- • the Weibull distribution with density $\theta x^{\theta-1}\exp\left(-x^{\theta}\right)$, denoted by $W(\theta)$;
- • the gamma distribution with density $\Gamma(\theta)^{-1}x^{\theta-1}\exp\left(-x\right)$, denoted by $\Gamma(\theta)$;
- • the half-normal HN distribution with density $\Gamma(2/\pi)^{1/2}\exp\left(-x^2/2\right)$;

- • the lognormal law $LN(\theta)$ with density $(\theta x)^{-1}(2\pi)^{-1/2}\exp\left(-(\log x)^2\Big/(2\theta^2)\right)$;
- • the Pareto distribution $Pa(\theta)$ with density $\theta/x^{\theta+1}$;
- • the uniform distribution U with density 1, $0 \le x \le 1$;
- • Chen's (2000) distribution $CH(\theta)$, with distribution function $1 - \exp\left(2\left(1 - e^{x^{\theta}}\right)\right)$;
- • the linear increasing failure rate law $LF(\theta)$ with density $(1+\theta x)\exp\left(-x - \theta x^2/2\right)$;
- • the modified extreme value $EV(\theta)$, with distribution function $1 - \exp\left(\theta^{-1}(1 - e^{x})\right)$;
- • Dhillon's (1981) law $DL(\theta)$ with distribution function $1-\exp\left(-(\log(x+1))^{\theta+1}\right)$;

By conducting Monte Carlo simulations under each of the specified alternative scenarios, we can calculate the power values of the proposed test. These simulations involve generating multiple datasets that follow each alternative hypothesis, applying the test to each dataset, and determining the proportion of times the test correctly detects deviations from the null hypothesis. The resulting power values provide a quantitative measure of the test's ability to detect and reject false null hypotheses under different alternative scenarios. These power values provide insights into the test's ability to detect deviations from the null hypothesis under different alternative hypotheses. Comparing the power across these alternatives allows for a comprehensive assessment of the test's performance and its sensitivity to various types of departures from the null hypothesis.
Tables 3-6 provide a comprehensive display and comparison of the power values of the tests, specifically at a significance level of $\alpha = 0.05$. In these tables, the $\hat{G}_n^1$ test refers to the two-sided test, while the $\hat{G}_n^2$ test corresponds to the one-sided test. These tables offer valuable insights into the comparative performance of the tests under different alternative hypotheses, allowing for an informed evaluation of their effectiveness in detecting deviations from the null hypothesis.

Upon careful examination of Tables 3-6, it becomes apparent that there is no single test that can be considered as the optimal choice across all alternative scenarios. Each test demonstrates varying levels of performance and power depending on the specific alternative hypothesis being considered. Therefore, the choice of the most suitable test should be based on the specific research question, the characteristics of the dataset, and the nature of the alternative hypothesis under investigation. It is important to carefully evaluate the strengths and limitations of each test in relation to the specific context in order to make an informed decision. Tables 3-6 likely provide the results of different GOF tests are applied to various alternative distributions. Each test may have its own strengths and weaknesses, and their performance varies depending on the specific characteristics of the data or the underlying distribution being tested. In terms of power, the tests based on the Z_A and $\hat{G}_n^2$ statistics demonstrate

TABLE 3. Power estimates of the considered tests for $n = 10$ at the level $\alpha = 0.05$.

Alternative	Z_A	Z_C	TE	TA	$\hat{G}_n^1$	$\hat{G}_n^2$
$Exp(1)$	0.3859	0.3646	0.3159	0.3633	0.3380	0.4000
$W(0.5)$	0.7622	0.7255	0.6713	0.6850	0.6088	0.6624
$W(2)$	0.2174	0.2144	0.1488	0.1900	0.1884	0.2495
$\Gamma(0.5)$	0.7106	0.6822	0.6485	0.6681	0.5967	0.6515
$\Gamma(2)$	0.1842	0.1772	0.1259	0.1626	0.1724	0.2252
HN	0.4134	0.4002	0.3528	0.3976	0.3602	0.4284
$LN(0, 0.5)$	0.0618	0.0624	0.0443	0.0453	0.0413	0.0579
$LN(0, 1)$	0.0973	0.0884	0.0669	0.0877	0.0954	0.1202
$LN(0, 2)$	0.4312	0.3696	0.2933	0.3352	0.3078	0.3551
$Pa(0.5)$	0.2363	0.2405	0.2235	0.2356	0.2004	0.2529
$Pa(1)$	0.3812	0.3908	0.4008	0.4308	0.0796	0.1066
$Pa(2)$	0.4551	0.4584	0.4281	0.4405	0.0682	0.0762
U	0.5633	0.5629	0.5545	0.5603	0.4353	0.5128
$CH(0.5)$	0.7337	0.7053	0.6715	0.6859	0.6056	0.6620
$CH(1)$	0.4191	0.4059	0.3589	0.4039	0.3614	0.4297
$CH(1.5)$	0.3201	0.3160	0.2507	0.2995	0.2712	0.3417
$LF(2)$	0.4020	0.3911	0.3374	0.3874	0.3568	0.4255
$LF(4)$	0.3911	0.3821	0.3238	0.3739	0.3461	0.4133
$EV(0.5)$	0.4202	0.4055	0.3618	0.4084	0.3595	0.4265
$EV(1.5)$	0.4464	0.4390	0.3912	0.4354	0.3793	0.4513
$DL(1)$	0.1627	0.1521	0.1059	0.1412	0.1539	0.1997
$DL(1.5)$	0.1405	0.1352	0.0914	0.1185	0.1299	0.1755

superior performance across a wide range of alternative hypotheses. Notably, the power differences between these two tests and the other tests are notable and substantial. Particularly for small sample sizes, the test based on the $\hat{G}_n^2$ statistic exhibits a stronger performance compared to the other tests. This finding showcases the efficacy of the $\hat{G}_n^2$ test in detecting deviations from the null hypothesis, particularly in scenarios with limited data. Overall, the results suggest that the Z_A and $\hat{G}_n^2$ tests consistently display better power characteristics than the alternative tests, with the $\hat{G}_n^2$ test excelling for smaller sample sizes.

Based on the data presented in Table 2, which shows that the actual sizes of the tests fall within acceptable ranges, we can confidently endorse the practical application of these tests. Moreover, the results demonstrate that as the sample sizes increase, the power values of the tests also increase. This underscores the effectiveness and reliability of the tests in detecting deviations from the null hypothesis. Therefore, we can conclude that these tests are not only suitable for practical use but also exhibit desirable statistical properties in terms

Table 4. Power estimates of the considered tests for $n = 20$ at the level $\alpha = 0.05$.

Alternative	Z_A	Z_C	TE	TA	$\hat{G}_n^1$	$\hat{G}_n^2$
$Exp(1)$	0.6390	0.6314	0.5670	0.6350	0.5935	0.6598
$W(0.5)$	0.9433	0.9355	0.9268	0.9313	0.8831	0.9109
$W(2)$	0.4220	0.4194	0.2754	0.3597	0.3440	0.4242
$\Gamma(0.5)$	0.9278	0.9228	0.9094	0.9196	0.8738	0.9035
$\Gamma(2)$	0.3335	0.3335	0.2220	0.2988	0.3079	0.3820
HN	0.6995	0.6954	0.6261	0.6872	0.6293	0.6943
$LN(0, 0.5)$	0.0691	0.0691	0.0404	0.0480	0.0491	0.0679
$LN(0, 1)$	0.1370	0.1306	0.0820	0.1279	0.1415	0.1890
$LN(0, 2)$	0.6368	0.5987	0.5393	0.5941	0.5277	0.5988
$Pa(0.5)$	0.4957	0.4716	0.5045	0.4340	0.2858	0.3633
$Pa(1)$	0.8236	0.7994	0.8038	0.7609	0.1334	0.1255
$Pa(2)$	0.8685	0.8454	0.8085	0.7500	0.1455	0.1489
U	0.9081	0.8971	0.8909	0.8532	0.7257	0.7871
$CH(0.5)$	0.9385	0.9336	0.9242	0.9290	0.8828	0.9110
$CH(1)$	0.7099	0.7061	0.6474	0.6993	0.6269	0.6930
$CH(1.5)$	0.6098	0.6029	0.4846	0.5559	0.4891	0.5704
$LF(2)$	0.6888	0.6854	0.6079	0.6738	0.6225	0.6892
$LF(4)$	0.6800	0.6754	0.5840	0.6537	0.6070	0.6758
$EV(0.5)$	0.7103	0.7059	0.6448	0.6943	0.6266	0.6928
$EV(1.5)$	0.7630	0.7568	0.6901	0.7362	0.6584	0.7248
$DL(1)$	0.2757	0.2722	0.1727	0.2510	0.2712	0.3375
$DL(1.5)$	0.2455	0.2456	0.1443	0.2082	0.2295	0.2963

of their power.
In general, based on the observed performance across different alternatives and sample sizes, we can conclude that the proposed test $\hat{G}_n^2$ demonstrates good performance, particularly for small sample sizes. This suggests that the test $\hat{G}_n^2$ is reliable and effective in practical applications, as it consistently exhibits favorable results when compared to alternative tests. Therefore, researchers and practitioners can confidently utilize the proposed test $\hat{G}_n^2$, especially when dealing with limited sample sizes.

4. An Illustrative Example

To demonstrate the practical implementation of the proposed procedure for evaluating the GOF of the IG distribution, let's consider the following example scenario.
Imagine we have collected a dataset containing n observations, and there is a suspicion that these observations may adhere to an IG distribution. Our

Table 5. Power estimates of the considered tests for $n = 30$ at the level $\alpha = 0.05$.

Alternative	Z_A	Z_C	TE	TA	$\hat{G}_n^1$	$\hat{G}_n^2$
$Exp(1)$	0.7968	0.7951	0.7305	0.7758	0.7538	0.8055
$W(0.5)$	0.9881	0.9864	0.9847	0.9836	0.9672	0.9774
$W(2)$	0.5835	0.5781	0.3907	0.4808	0.4739	0.5592
$\Gamma(0.5)$	0.9819	0.9810	0.9787	0.9768	0.9618	0.9734
$\Gamma(2)$	0.4658	0.4679	0.3085	0.3988	0.4213	0.5033
HN	0.8622	0.8594	0.7936	0.8224	0.7883	0.8383
$LN(0, 0.5)$	0.0726	0.0748	0.0442	0.0553	0.0534	0.0751
$LN(0, 1)$	0.1704	0.1693	0.0994	0.1613	0.1838	0.2465
$LN(0, 2)$	0.7705	0.7480	0.7026	0.7313	0.6921	0.7537
$Pa(0.5)$	0.7494	0.6964	0.7198	0.5293	0.3475	0.4432
$Pa(1)$	0.9712	0.9566	0.9493	0.8759	0.1691	0.1329
$Pa(2)$	0.9829	0.9712	0.9511	0.8680	0.1885	0.2017
U	0.9887	0.9833	0.9799	0.9413	0.8724	0.9096
$CH(0.5)$	0.9862	0.9854	0.9844	0.9817	0.9668	0.9765
$CH(1)$	0.8693	0.8650	0.8094	0.8271	0.7883	0.8369
$CH(1.5)$	0.7933	0.7843	0.6549	0.6949	0.6490	0.7207
$LF(2)$	0.8531	0.8488	0.7759	0.8112	0.7803	0.8299
$LF(4)$	0.8434	0.8398	0.7503	0.7947	0.7697	0.8220
$EV(0.5)$	0.8694	0.8651	0.8102	0.8276	0.7885	0.8375
$EV(1.5)$	0.9115	0.9056	0.8551	0.8646	0.8160	0.8620
$DL(1)$	0.3701	0.3735	0.2350	0.3318	0.3700	0.4476
$DL(1.5)$	0.3394	0.3439	0.1952	0.2765	0.3110	0.3893

objective is to assess the suitability of the IG distribution for this dataset by employing the proposed test methodology. To demonstrate the application of the proposed procedure, let's consider the following steps:

(1) **Data Collection:** Collect a dataset comprising n observations that you suspect may follow an IG distribution.
(2) **Parameter Estimation:** Calculate the MLE estimators $\hat{\mu}$ and $\hat{\lambda}$.
(3) **Hypotheses Formulation:** Formulate the null and alternative hypotheses for the GOF test. The null hypothesis (H_0) asserts that the data follows an IG distribution with parameters μ and λ. The alternative hypothesis (H_1) suggests that the data does not conform to an IG distribution.
(4) **Test Statistic Calculation:** Compute the proposed test statistic using the estimated parameters $\hat{\mu}$ and $\hat{\lambda}$ obtained in Step 2.
(5) **Critical Value Determination:** Determine the critical value corresponding to the chosen significance level α. The critical value establishes the threshold beyond which we reject the null hypothesis.

Table 6. Power estimates of the considered tests for $n = 50$ at the level $\alpha = 0.05$.

Alternative	Z_A	Z_C	TE	TA	$\hat{G}_n^1$	$\hat{G}_n^2$
$Exp(1)$	0.9412	0.9405	0.8933	0.9064	0.9154	0.9394
$W(0.5)$	0.9995	0.9994	0.9993	0.9987	0.9977	0.9987
$W(2)$	0.8014	0.7915	0.5649	0.6305	0.6756	0.7460
$\Gamma(0.5)$	0.9992	0.9992	0.9987	0.9979	0.9970	0.9980
$\Gamma(2)$	0.6626	0.6643	0.4481	0.5338	0.6026	0.6786
HN	0.9756	0.9732	0.9400	0.9410	0.9364	0.9558
$LN(0, 0.5)$	0.0800	0.0847	0.0423	0.0639	0.0599	0.0890
$LN(0, 1)$	0.2272	0.2366	0.1248	0.2132	0.2688	0.3473
$LN(0, 2)$	0.9115	0.9022	0.8736	0.8723	0.8747	0.9093
$Pa(0.5)$	0.9703	0.9388	0.9255	0.6636	0.4626	0.5590
$Pa(1)$	0.9998	0.9992	0.9980	0.9700	0.2304	0.2125
$Pa(2)$	0.9999	0.9997	0.9974	0.9619	0.2614	0.2912
U	0.9999	0.9998	0.9995	0.9900	0.9755	0.9848
$CH(0.5)$	0.9995	0.9995	0.9994	0.9984	0.9977	0.9985
$CH(1)$	0.9772	0.9748	0.9469	0.9400	0.9367	0.9562
$CH(1.5)$	0.9530	0.9448	0.8508	0.8484	0.8462	0.8886
$LF(2)$	0.9707	0.9683	0.9277	0.9320	0.9341	0.9539
$LF(4)$	0.9674	0.9650	0.9155	0.9257	0.9261	0.9478
$EV(0.5)$	0.9775	0.9754	0.9487	0.9433	0.9368	0.9559
$EV(1.5)$	0.9912	0.9890	0.9709	0.9625	0.9512	0.9669
$DL(1)$	0.5283	0.5372	0.3390	0.4520	0.5414	0.6186
$DL(1.5)$	0.4908	0.4985	0.2789	0.3817	0.4648	0.5481

(6) **Decision Making:** Compare the calculated test statistic from Step 4 with the critical value obtained in Step 5. If the test statistic falls outside the acceptance region, it indicates a rejection of the null hypothesis, suggesting that the data does not conform to an IG distribution. On the other hand, if the test statistic falls within the acceptance region, there is insufficient evidence to reject the null hypothesis, leading to the conclusion that the data is consistent with an IG distribution.

(7) **Interpretation:** Provide an interpretation of the test results within the context of the specific dataset and research question. Discuss the implications and draw conclusions regarding the GOF of the IG distribution to the observed data.

Example 4.1. *In a study conducted by Folks and Chhikara in 1989, a dataset consisting of 19 fracture toughness measurements of MIG (metal inert gas) welds was analyzed.*
54.4, 62.6, 63.2, 67.0, 70.2, 70.5, 70.6, 71.4, 71.8, 74.1, 74.1, 74.3, 78.8, 81.8, 83.0, 84.4, 85.3, 86.9, 87.3.

Based on the findings of Folks and Chhikara (1989), they concluded, using the Kolmogorov-Smirnov (KS) statistic, that the IG distribution provides a reasonable fit to the dataset they analyzed.
Figure 2 displays the histogram of the considered dataset, providing a visual representation of the data distribution. This histogram likely supports their conclusion by visually demonstrating the fit of the IG distribution to the observed data.

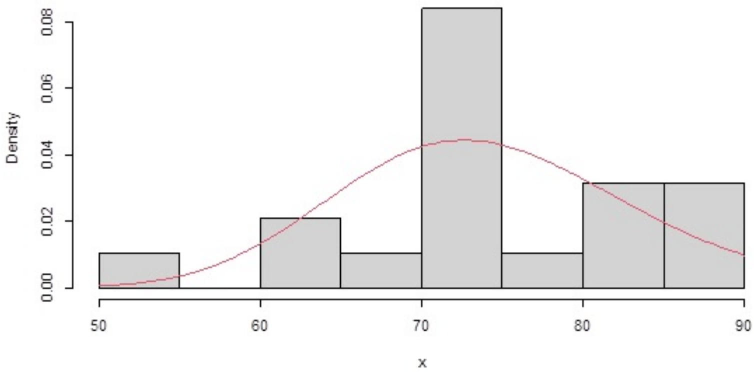

FIGURE 2. Histogram of data and a fitted IG density function.

TABLE 7. The value of the test statistics and critical points at significance level 5%.

Test	*Value of the test statistic*	*Critical point*	*Decision*
Z_A	*3.3847*	*3.4489*	*Not reject* H_0
Z_C	*5.6173*	*8.7055*	*Not reject* H_0
TE	*0.11087*	*0.20449*	*Not reject* H_0
TA	*0.00840*	*0.03523*	*Not reject* H_0
$\hat{G}_n^1$	*0.3729*	*0.3370, 0.4143*	*Not reject* H_0
$\hat{G}_n^2$	*0.3729*	*0.3469, 0.4115*	*Not reject* H_0

To apply the proposed test to the dataset from Folks and Chhikara (1989), we begin by computing the maximum likelihood (ML) estimates of the parameters μ *and* λ *for the IG distribution. These ML estimates can be obtained as follows:*

$$\hat{\mu} = \bar{X} = 74.3 \quad \textit{and} \quad \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} \left(1/X_i - 1/\bar{X}\right)} = 4923.952\,.$$

In accordance with the formula presented in Section 2, we can calculate the test statistic value for the dataset obtained from Folks and Chhikara's study in 1989. Furthermore, by conducting Monte Carlo simulations, critical values for the test statistic can be obtained specifically for a sample size of $n = 19$ *and at a significance level of 0.05. The outcomes of these calculations are succinctly presented in Table 7.*
By conducting a comparison between the computed test statistic and the critical value obtained from Table 7, we are able to make an informed decision regarding the GOF of the IG distribution to the dataset, following the prescribed test procedure.
Considering that the calculated test statistic for each individual data point falls within the acceptance region as determined by the provided critical values in Table 7, we fail to reject the null hypothesis of the IG distribution for the dataset at a significance level of 0.05. As a result, based on this analysis, we can conclude that the underlying distribution of the data from Folks and Chhikara's study in 1989 is consistent with an IG distribution. This observation indicates that the IG distribution offers a satisfactory fit for the dataset, corroborating the findings of Folks and Chhikara (1989), who also espoused that the IG distribution adequately represents the data based on the Kolmogorov-Smirnov statistic.
Therefore, the application of the proposed test in this case further confirms the appropriateness of employing the IG distribution to model fracture toughness measurements of MIG welds.

5. Conclusions

In this research paper, we have proposed and utilized the Gini index as a measure of fitness assessment for the IG distribution. Our proposed approach includes the development of a novel GOF test, leveraging the Gini index as its core metric for evaluation. Then, we have determined both the exact and asymptotic distributions of the test statistic. To further validate the performance and applicability of our proposed test, we have conducted extensive Monte Carlo simulations. Through these simulations, we were able to determine the critical points and actual sizes of the test, thereby ensuring its accuracy in practical scenarios.
Additionally, we have conducted a comprehensive comparative study to assess the performance of our proposed Gini index-based test in comparison to other existing methods. The results of this study provide compelling evidence of the superior performance of the Gini index-based test in detecting deviations from the IG distribution under certain alternative hypotheses. This highlights the effectiveness of the Gini index as a tool for assessing the GOF of the IG distribution.
To demonstrate the practical application and relevance of our proposed test, we have illustrated its usage using a real-world dataset. This practical example serves to showcase the test's ability to accurately assess the fit of the IG

distribution to observed data and reinforces the utility of the Gini index-based approach in real data analysis scenarios.
Overall, this comparative study and practical illustration enhance our understanding of the proposed Gini index-based test and its superiority in evaluating the GOF for the IG distribution. These findings contribute to the broader field of statistical analysis by providing a valuable alternative approach for assessing distributional fit.

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Conflict of interest

The authors declare no conflict of interest.

Data Availability Statement

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