# FRAÏSSÉ LIMIT VIA FORCING

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ABSTRACT. Given a Fraïssé class $\mathcal{K}$ and an infinite cardinal $\kappa$, we define a forcing notion which adds a structure of size $\kappa$ using elements of the class $\mathcal{K}$. We show that our construction extends the Fraïssé construction in the case $\kappa = \omega$.

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## 1. introduction

The Fraïssé construction is a basic and important tool in model theory that allows to pass from a suitable class of finite structures to a canonical countable structures, called the Fraïssé limit of that class.

Suppose $\mathcal{L}$ is a finite relational language and $\mathcal{K}$ is a class of finite $\mathcal{L}$-structures closed under substructures and isomorphisms. It is called a Fraïssé class if it satisfies Joint Embedding Property (JEP) and Amalgamation Property (AP). A Fraïssé limit, denoted $\mathrm{Flim}(\mathcal{K})$, of a Fraïssé class $\mathcal{K}$ is the unique[1] countable ultrahomogeneous (every isomorphism of finitely-generated substructures extends to an automorphism of $\mathrm{Flim}(\mathcal{K})$) structure into which every member of $\mathcal{K}$ embeds.

Given a Fraïssé class $\mathcal{K}$ and an infinite cardinal $\kappa$, we would like to force a structure of size $\kappa$ which shares many properties with the Fraïssé construction.

We first consider, as an example, the special cases of linear orders, and then discuss the more general case. We also show that our construction gives the original Fraïssé construction in the case $\kappa = \omega$. The results of this paper are further extended in [1], [3] and [4],

## 2. Fraïssé limit of linear orders

Let $\mathcal{K}$ be the class of finite linear orders and let $\kappa$ be an infinite cardinal.

**Definition 2.1.** A condition in $\mathbb{P}_{\kappa,\mathcal{K}}$ is of the form $p = (A_p, \leq_p)$, where

    (1) $A_p$ is a finite subset of $\kappa$.

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[1]The existence and uniqueness follows from Fraïssé's theorem, See [2].

(2) $(A_p, \leq_p) \in \mathcal{K}$, i.e., it is a finite linear order.

The order on $\mathbb{P}_{\kappa,\mathcal{K}}$ is defined in the natural way.

**Definition 2.2.** Suppose $p, q \in \mathbb{P}_{\kappa,\mathcal{K}}$. Then $p \leq q$ if and only if

(1) $A_p \supseteq A_q$,
(2) $\leq_q = \leq_p \cap (A_q \times A_q)$.

**Lemma 2.3.** $\mathbb{P}_{\kappa,\mathcal{K}}$ *is c.c.c. (and in fact $\aleph_1$-Knaster).*

*Proof.* Suppose $\{p_\alpha = (A_\alpha, <_\alpha) : \alpha < \omega_1\} \subseteq \mathbb{P}_{\kappa,\mathcal{K}}$. By $\Delta$-system lemma, we may assume that $(A_\alpha : \alpha < \omega_1)$ forms a $\Delta$-system with root $d$. Since there are only finitely many different orders on $d$, so we can find $I \subseteq \omega_1$ of size $\aleph_1$ such that for all $\alpha < \beta$ in $I$, we have $<_a \cap d \times d = <_\beta \cap d \times d$. But then $q_{\alpha,\beta} = (A_\alpha \cup A_\beta, <_\alpha \cup <_\beta)$ is a condition extending both of $p_\alpha$ and $p_\beta$. $\square$

It follows that forcing with $\mathbb{P}_{\kappa,\mathcal{K}}$ preserves cardinals and cofinalities. Let $G$ be $\mathbb{P}_{\kappa,\mathcal{K}}$-generic over $V$. The next lemma follows by a simple density argument.

**Lemma 2.4.** $\kappa = \bigcup_{p \in G} A_p$.

Let $\leq_G = \bigcup_{p \in G} \leq_p$. As $G$ is a filter, $\leq_G$ is well-defined and it is a linear order on $\kappa$.

**Lemma 2.5.** $(\kappa, \leq_G)$ *is $\kappa$-dense.*

*Proof.* Suppose $\alpha <_G \beta$ and $p \in \mathbb{P}_{\kappa,\mathcal{K}}$. Suppose $p \Vdash$ "$\{\gamma \in \kappa : \alpha <_G \gamma <_G \beta\}$ has size $< \kappa$". By Lemma 2.3, we can find $I \in V$ of size less than $\kappa$ such that $p \Vdash$ "$\check{I} \supseteq \{\gamma \in \kappa : \alpha <_G \gamma <_G \beta\}$". Let $q \leq p$ be such that $\alpha, \beta \in A_q$ and for some $\gamma \in A_q \setminus I, \alpha <_q \gamma <_q \beta$. Then $q \Vdash$ "$\gamma \notin I$ and $\alpha \lesssim_G \gamma \lesssim_G \beta$", which is a contradiction. $\square$

**Lemma 2.6.** *Suppose $I \in V$ is an infinite subset of $\kappa$. Then $(I, <_G)$ is dense in $(\kappa, <_G)$.*

*Proof.* Suppose $p \in \mathbb{P}_{\kappa,\mathcal{K}}$ and $p \Vdash$ "$\alpha \lesssim_G \beta$". Let $\gamma \in I \setminus A_p$ and let $q = (A_q, \leq_q)$, where $A_q = A_p \cup \{\gamma\}$ and $\alpha <_q \gamma <_q \beta$. Then $q \Vdash$ "$\gamma \in I$ and $\alpha \lesssim_G \gamma \lesssim_G \beta$". $\square$

*Remark* 2.7. Work in $V[G]$. Let $\overline{(\kappa, <_G)}$ denote the completion of $(\kappa, <_G)$. Then $\overline{(\kappa, <_G)} \cong (\mathbb{R}^{V[G]}, <)$, where $(\mathbb{R}^{V[G]}, <)$ denotes the set of real numbers in $V[G]$.

**Lemma 2.8.** *Suppose $I, J \in V$ are non-empty subsets of $\kappa$ and $I <_G J$ (i.e., $\alpha <_G \beta$ for all $\alpha \in I$ and $\beta \in J$). Then $I$ and $J$ are finite, in particular, there exists $\gamma \in \kappa$ such that $I <_G \gamma <_G J$.*

As the next lemma shows, the structure $(\kappa, <_G)$ is rigid if $\kappa$ is uncountable.

**Lemma 2.9.** *( [4, Theorem 2]) Suppose $\kappa$ is an uncountable cardinal. Then $(\kappa, <_G)$ is rigid.*

### 3. Fraïssé limit–The general case

Suppose $\mathcal{L}$ is a finite relational language and $\mathcal{K}$ is a Fraïssé class. For any relation symbol $R \in \mathcal{L}$, let $n_R$ denote its arity.

**Definition 3.1.** A condition $p$ is in $\mathbb{P}_{\kappa,\mathcal{K}}$ if and only if

(1) $p \in \mathcal{K}$.
(2) $A_p$, the universe of the structure $p$, is a subset of $\kappa$.

The order on $\mathbb{P}_{\kappa,\mathcal{K}}$ is defined in the natural way.

**Definition 3.2.** Suppose $p, q \in \mathbb{P}_{\kappa,\mathcal{K}}$. Then $p \leq q$ if and only if

(1) $A_p \supseteq A_q$,
(2) $q = p \upharpoonright A_q$, .i.e., for any relational symbol $R \in \mathcal{L}, R^q = R^p \cap A_q^{n_R}$.

**Lemma 3.3.** $\mathbb{P}_{\kappa,\mathcal{K}}$ *is c.c.c. (and in fact $\aleph_1$-Knaster).*

*Proof.* Suppose $\{p_\alpha = (A_\alpha, <_\alpha) : \alpha < \omega_1\} \subseteq \mathbb{P}_{\kappa,\mathcal{K}}$. By $\Delta$-system lemma, we may assume that $(A_\alpha : \alpha < \omega_1)$ forms a $\Delta$-system with root $d$. Since there are only countably many $\mathcal{K}$-structures with universe $d$, so we can find $I \subseteq \omega_1$ of size $\aleph_1$ such that for all $\alpha < \beta$ in $I$, we have $p \upharpoonright d = q \upharpoonright d$. But then, using the amalgamation property, we can find $q_{\alpha,\beta} \in \mathcal{K}$ which extends both of $p_\alpha$ and $q_\alpha$. $\square$

It follows that forcing with $\mathbb{P}_{\kappa,\mathcal{K}}$ preserves cardinals and cofinalities. Let $G$ be $\mathbb{P}_{\kappa,\mathcal{K}}$-generic over $V$. The next lemma follows by a simple density argument.

**Lemma 3.4.** $\kappa = \bigcup_{p \in G} A_p$.

For any relational symbol $R \in \mathcal{L}$ let $R^G = \bigcup_{p \in G} R^p$, where $R^p$ is the interpretation of $R$ in $p$. As $G$ is a filter, $R^G$ is a well-defined $n_R$-ary relation on $\kappa$. Consider the structure

$$\mathcal{M}_G = (\kappa, R^G)_{R \in \mathcal{L}}.$$

Then $\mathcal{M}_G$ is an $\mathcal{L}$-structure with universe $\kappa$.

**Lemma 3.5.** *Each element of $\mathcal{K}$ embeds into $\mathcal{M}_G$.*

*Proof.* Suppose $p \in \mathcal{K}$. We may assume that $A_p$, the universe of the structure $p$, is a subset of $\kappa$. Then the set

$$D = \{q \in \mathcal{K} : p \text{ embeds into } q\}$$

is dense, from which the lemma follows. $\square$

The next lemma can be proved by a simple density argument.

**Lemma 3.6.** *Suppose $I \in V$ is an infinite subset of $\kappa$. Then for any $R \in \mathcal{L}$ and any $(\alpha_1, \ldots, \alpha_i) \in \kappa^i, i < n_R$ the sets*

$$\{(a_{i+1}, \ldots, a_{n_R}) \in I^{n_R - i} : R^G(a_1, \ldots, a_{n_R})\}$$

*and*

$$\{(a_{i+1}, \ldots, a_{n_R}) \in I^{n_R - i} : \neg R^G(a_1, \ldots, a_{n_R})\}$$

*are infinite; and in fact they have the same size as $I$.*

## 4. **Connection with the original Fraïssé construction**

In this section, we show that the Fraïssé construction can be obtained from the above results for the case $\kappa = \omega$.

Thus fix the language $\mathcal{L}$ and the class $\mathcal{K}$ as before. We may suppose that each $p \in \mathcal{K}$ has a domain subset of $\omega$, so in particular $\mathcal{K}$, is countable. Let $\mathbb{P} = \mathbb{P}_{\omega, \mathcal{K}}$. We define the following dense subsets of $\mathbb{P}$:

(1) $D_n = \{p \in \mathbb{P} : n \in A_p\}$, for $n < \omega$.

(2) $D_{(\mathcal{A}, \mathcal{A}'), \mathcal{B}, f}^{\rightarrow} = \{p \in \mathbb{P} : p \upharpoonright A = \mathcal{A}, A' \subseteq A_p$ and $\exists\, q$ such that $q \upharpoonright B = \mathcal{B}$ and $f$ extends to some $\bar{f} : p \cong q\}$, where $\mathcal{A} = (A, \dots), \mathcal{B} = (B, \dots), \mathcal{A}' = (A', \dots) \in \mathcal{K}, \mathcal{A}$ is a substructure of $\mathcal{A}'$ and $f : \mathcal{A} \cong \mathcal{B}$.

(3) $D_{\mathcal{A}, (\mathcal{B}, \mathcal{B}'), f}^{\leftarrow} = \{q \in \mathbb{P} : q \upharpoonright B = \mathcal{B}, B' \subseteq A_q$ and $\exists\, p$ such that $p \upharpoonright A = \mathcal{A}$ and $f$ extends to some $\bar{f} : p \cong q\}$, where $\mathcal{A} = (A, \dots), \mathcal{B} = (B, \dots), \mathcal{B}' = (B', \dots) \in \mathcal{K}, \mathcal{B}$ is a substructure of $\mathcal{B}'$ and $f : \mathcal{A} \cong \mathcal{B}$.

Note that

$$\{D_n\} \cup \{D_{(\mathcal{A}, \mathcal{A}'), \mathcal{B}, f}^{\rightarrow}\} \cup \{D_{\mathcal{A}, (\mathcal{B}, \mathcal{B}'), f}^{\leftarrow}\}$$

is countable and hence by the Rasiowa-Sikorski lemma, we can find a filter $G \subseteq \mathbb{P}$ which meets all the above dense sets. Then the resulting structure $\mathcal{M}_G$ is isomorphic to the Fraïssé limit of the class $\mathcal{K}$.

## 5. **Conclusion**

We studied the Fraïssé construction using the method of forcing and showed that it gives a new approach to it. It also allowed us to extend the construction to higher cardinals, though in this case, the construction behaves differently.

## 6. **Aknowledgement**

## **References**

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