Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505
JMMR 2012

ORTHOGONAL BASES IN SPECIFIC GENERALIZED SYMMETRY CLASSES OF TENSORS

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Article type: Research Article

(Received: 03 October 2023, Received in revised form 12 January 2024)

(Accepted: 28 January 2024, Published Online: 29 January 2024)

Abstract. Let V be a unitary vector space. Suppose G is a permutation group of degree m and Λ is an irreducible unitary representation of G. We denote by $V_\Lambda(G)$ the generalized symmetry class of tensors associated with G and Λ. In this paper, we prove the existence of orthogonal bases consisting of generalized decomposable symmetrized tensors for the generalized symmetry classes of tensors associated with unitary irreducible representations of group U_{6n}, as well as dihedral and dicyclic groups.

Keywords: Generalized symmetry classes of tensors, orthogonal basis, the group U_{6n}, dihedral group, dicyclic group.
2020 MSC: Primary 20C15, 15A69.

1. Introduction

The study of symmetry classes of tensors finds its motivation in various branches of both pure and applied mathematics, including matrix theory, operator theory, combinatorial theory, differential geometry, group representation theory, partial differential equations, quantum mechanics, and other related fields (refer to [14, 15] for further details). In this research, we focus on generalized symmetry classes of tensors.

In this section, we provide an overview of the generalized symmetry classes of tensors. For a more comprehensive introduction, readers are encouraged to refer to [13, 16, 17].

Let V represent an n-dimensional inner product space and $V^{\otimes m}$ denotes the tensor product of m copies of V. Consider a permutation group G acting on m elements. Let Λ be an irreducible unitary representation of G over an inner product space U with dimension r, which affords the character λ of G. For any $\sigma \in G$, we define the permutation operator as follows:

$$P(\sigma) : V^{\otimes m} \to V^{\otimes m}$$

$$P(\sigma)(v_1 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}.$$

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https://doi.org/10.22103/jmmr.2024.22282.1519
Publisher: Shahid Bahonar University of Kerman
How to cite: G. Rafatneshan, Y. Zamani, *Orthogonal bases in specific generalized symmetry classes of tensors*, J. Mahani Math. Res. 2024; 13(2): 209 - 223.

The "generalized symmetry class of tensors" associated with G and Λ is the range of the projection operator:

$$S_\Lambda = \frac{1}{|G|}\sum_{\sigma\in G}\Lambda(\sigma)\otimes P(\sigma).$$

This class is denoted as $V_\Lambda(G)$. When dim $U=1$, $V_\Lambda(G)$ reduces to $V_\lambda(G)$, which is the symmetry class of tensors associated with G and λ (for more details, refer to [6,10,11,14,15]). The elements of $V_\Lambda(G)$ in the form of $S_\Lambda(u\otimes x^{\otimes})$ are referred to as "generalized decomposable symmetrized tensors" and are denoted as $u\circledast x^{\circledast}$.

The inner product on $U\otimes V^{\otimes m}$ induces an inner product on $V_\Lambda(G)$ that satisfies:

$$(u\circledast x^{\circledast}, v\circledast y^{\circledast}) = \frac{1}{|G|}(D_\Lambda(A)u,v).$$

Here, $A=[(x_i,y_j)]$ and $D_\Lambda:\mathbb{C}_{m\times m}\to \mathrm{End}(U)$ is the "generalized Schur function":

$$D_\Lambda(A)=\sum_{\sigma\in G}\Lambda(\sigma)\prod_{i=1}^{m}a_{i\sigma(i)}.$$

Let $\Gamma_{m,n}$ denote the set of all sequences $\alpha=(\alpha(1),\dots,\alpha(m))$ with $1\le\alpha(i)\le n$ for $1\le i\le m$. We define the action of G on $\Gamma_{m,n}$ as follows:

$$\alpha\sigma=(\alpha(\sigma(1)),\dots,\alpha(\sigma(m))).$$

We denote $\alpha\sim\beta$ if α and β belong to the same orbit in $\Gamma_{m,n}$. Let Δ represent a system of distinct representatives of these orbits. For clarity, note that each sequence in Δ is selected as the first element in its orbit, based on the lexicographic order. We use G_α to denote the stabilizer subgroup of α.

For any $\alpha\in\Gamma_{m,n}$, we define the linear map $T_\alpha:U\to U$ as follows:

$$T_\alpha=\frac{1}{|G_\alpha|}\sum_{\sigma\in G_\alpha}\Lambda(\sigma).$$

It is a well-established result that T_α represents an orthogonal projection on U and $T_\alpha\neq 0$ if and only if $\sum_{\sigma\in G_\alpha}\lambda(\sigma)\neq 0$. We define the set Ω as

$$\Omega=\{\alpha\in\Gamma_{m,n}\mid \sum_{\sigma\in G_\alpha}\lambda(\sigma)\neq 0\}$$

and denote the intersection of Δ and Ω as $\bar{\Delta}=\Delta\cap\Omega$.

Let $\{u_1,\dots,u_f\}$ and $\{e_1,\dots,e_n\}$ be orthonormal bases for U and V, respectively. Then for each $1\leqslant i,j\leqslant f$ and $\alpha,\beta\in\Gamma_{m,n}$, we have

$$(u_i\circledast e_\alpha^{\circledast}, u_j\circledast e_\beta^{\circledast})=\begin{cases}0 & \text{if } \alpha\nsim\beta\\ \dfrac{1}{[G:G_\alpha]}(T_\alpha u_i,u_j) & \text{if } \alpha=\beta\end{cases}.$$

In particular, $u_j \circledast e_\alpha^\circledast = 0$ if and only if $T_\alpha u_j = 0$.

For $\alpha \in \Gamma_{m,n}$, the subspace

$$V_\alpha^\circledast = \langle\ u_i \circledast e_\alpha^\circledast\ \mid\ 1 \le i \le f\ \rangle = \langle u_1 \circledast e_{\alpha\sigma}^\circledast \mid \sigma \in G\rangle$$

of $V_\Lambda(G)$ is called the *generalized orbital subspace* corresponding to α. It is proved that

$$\dim V_\alpha^\circledast = \frac{1}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \lambda(\sigma).$$

Also,

$$V_\Lambda(G) = \bigoplus_{\alpha \in \bar{\Delta}} V_\alpha^\circledast$$

is an orthogonal direct sum. So, if Λ is a linear representation of G then $\dim V_\alpha^\circledast = 1$. In this case, the set

$$\{u_1 \circledast e_\alpha^\circledast \mid \alpha \in \bar{\Delta}\}$$

forms an orthogonal basis for $V_\Lambda(G)$. We refer to a basis comprising of generalized decomposable symmetrized tensors $u_1 \circledast e_\alpha^\circledast$ as an orthogonal $\circledast$-basis. However, if Λ is not a linear representation, it is possible that $V_\Lambda(G)$ does not possess an orthogonal $\circledast$-basis. In [16, Theorem 2.3], a necessary condition for the existence of an orthogonal $\circledast$-basis is provided. In this paper, we prove the existence of orthogonal $\circledast$-bases for the generalized symmetry classes of tensors associated with unitary irreducible representations of group U_{6n}, as well as dihedral and dicyclic groups. The existence of orthogonal bases in other types of symmetry classes has been explored by various authors (refer to [1–3, 8, 9, 22]).

2. The dihedral group

The dihedral group of order $2k$ ($k \ge 3$) is defined by

$$D_{2k} = \langle a, b \mid a^k = b^2 = 1, b^{-1}ab = a^{-1}\rangle.$$

In particular,

$$D_{2k} = \{a^r, a^r b \mid 0 \le r \le k-1\}.$$

In D_{2k}, there exist two types of subgroups:

(i) Subgroups of the form $\langle a^r\rangle$, where r divides k. For each r that divides k, there exists exactly one such subgroup. The order of these subgroups is given by $|\langle a^r\rangle| = k/\gcd(k,r)$, where $\gcd(k,r)$ denotes the greatest common divisor of k and r.

(ii) Subgroups of the form $\langle a^r, a^s b\rangle$, where r divides k and $0 \le s < r$. The order of these subgroups is given by $|\langle a^r, a^s b\rangle| = 2k/\gcd(k,r)$.

For each integer $0 < h < \frac{k}{2}$, the group D_{2k} has an irreducible unitary representation of degree 2 given by

$$\Lambda_h(a) = \begin{pmatrix} \omega^h & 0 \\ 0 & \omega^{-h} \end{pmatrix} \text{ and } \Lambda_h(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\omega = e^{\frac{2\pi i}{k}}$ is a primitive kth root of unity (see [12]). Note that here i is the imaginary unit. Let λ_h be the corresponding character of the representation Λ_h. Then, $\lambda_h(a^r) = 2\cos(2\pi rh/k)$ and $\lambda_h(a^r b) = 0$. The other irreducible representations of D_{2k} are linear. We first prove the following lemma.

Lemma 2.1. *Let $r, s, j, n \in \mathbb{N}$, $0 \le k \in \mathbb{Z}$, $\omega = e^{\frac{2\pi i}{n}}$ and l is the order of ω^k. Then*

$$\sum_{t=1}^{r} \omega^{kt} = \begin{cases} r & \text{if } \ n \mid k \\ 0 & \text{if } \ n \nmid k, \ r = jl \end{cases}$$

Also if $n \nmid k$, $s < l$ and $l = rs$ then $\sum_{t=1}^{jr} \omega^{kst} = 0$.

Proof. We know that $n = o(\omega)$ and $l = o(\omega^k) = n/gcd(n,k)$. Therefore if $n \mid k$ then $\omega^k = 1$ and so $\sum_{t=1}^{r} \omega^{kt} = r$. Now suppose that $n \nmid k$. Thus $\omega^k \neq 1$. If $r = jl$ then

$$0 = \omega^k(\omega^{kr} - 1) = (\omega^k - 1)\sum_{t=1}^{r} \omega^{kt}$$

and so $\sum_{t=1}^{r} \omega^{kt} = 0$. In particular, we have $\sum_{t=1}^{l} \omega^{kt} = 0$. Also if $s < l$ and $l = sr$ then

$$0 = \omega^{k(s-1)} \sum_{t=1}^{jl} \omega^{kt} = (\sum_{t=1}^{jr} \omega^{kst})(\sum_{t=0}^{s-1} \omega^{kt})$$

and so $\sum_{t=1}^{jr} \omega^{kst} = 0$, because $\sum_{t=0}^{s-1} \omega^{kt} \neq 0$ □

Lemma 2.2. *Let $G = D_{2k}$ $(k \ge 3)$ and $\lambda = \lambda_h$ $(0 < h < \frac{k}{2})$. Assume $\alpha \in \Gamma_{m,n}$ and G_α is of the form $G_\alpha = \langle a^r \rangle$, where $r \mid k$. Then,*

$$\sum_{g \in G_\alpha} \lambda(g) = \begin{cases} 2|G_\alpha| & \text{if k} \mid \text{rh} \\ 0 & \text{if k} \nmid \text{rh} \end{cases}.$$

Proof. Let $|G_\alpha| = l = k/\gcd(k,r)$. Applying Lemma 2.1, we have

$$\begin{aligned} \sum_{g \in G_\alpha} \lambda(g) &= \sum_{t=1}^{l} \lambda(a^{rt}) \\ &= 2\sum_{t=1}^{l} \cos\left(\frac{2\pi rht}{k}\right) \\ &= \begin{cases} 2l & \text{if } k \mid rh \\ 0 & \text{if } k \nmid rh \end{cases}. \end{aligned}$$

So the result holds. □

Lemma 2.3. *Let $G = D_{2k}$ $(k \geq 3)$ and $\lambda = \lambda_h$ $(0 < h < \frac{k}{2})$. Assume $\alpha \in \Gamma_{m,n}$ and G_α is of the form $G_\alpha = \langle a^r, a^s b\rangle$, where $r \mid k$, $0 \leq s < r$. Then,*

$$\sum_{g \in G_\alpha} \lambda(g) = \begin{cases} |G_\alpha| & \text{if } \ k \mid rh \\ 0 & \text{if } \ k \nmid rh \end{cases}.$$

Proof. By assumption, $|G_\alpha| = 2l$, where $l = k/\gcd(k, r)$. Using Lemma 2.1,

$$\begin{aligned}\sum_{g \in G_\alpha} \lambda(g) &= \sum_{t=1}^{l} \lambda(a^{rt}) + \sum_{t=1}^{l} \lambda(a^{rt+s}b) \\ &= 2\sum_{t=1}^{l} \cos\left(\frac{2\pi rht}{k}\right) + 0 \\ &= \begin{cases} 2l & \text{if } k \mid rh \\ 0 & \text{if } k \nmid rh \end{cases}.\end{aligned}$$

So the result holds. □

Theorem 2.4. *Let $G = D_{2k}$ be a subgroup of S_m and $\Lambda = \Lambda_h$, where $k \geq 3$ and $0 < h < k/2$. Then $V_\Lambda(G)$ has an orthogonal $\circledast$-basis.*

Proof. Since $V_\Lambda(G) = \bigoplus_{\alpha \in \bar{\Delta}} V_\alpha^{\circledast}$, it is sufficient to prove that, for every $\alpha \in \bar{\Delta}$, the generalized orbital subspace $V_\alpha^{\circledast}$ has an orthogonal $\circledast$-basis.

Take $\alpha \in \bar{\Delta}$. Then, $\sum_{\sigma \in G_\alpha} \lambda_h(\sigma) \neq 0$. By Lemmas 2.2 and 2.3, we have $\dim V_\alpha^{\circledast} = (1/|G_\alpha|) \sum_{g \in G_\alpha} \lambda(g) = 1$ or 2. If $\dim V_\alpha^{\circledast} = 1$ then there is no problem with the existence of an orthogonal $\circledast$-basis. Therefore, we assume that $\dim V_\alpha^{\circledast} = 2$. Lemma 2.2 implies that $G_\alpha = \langle a^r\rangle$, where $r \mid k$ and $k \mid rh$. Thus, $\omega^{rht} = e^{\frac{2\pi i rh}{k}} = 1$. Therefore, the orthogonal projection T_α is the identity. Since Λ is two-dimensional, let $\{u_1, u_2\}$ be an orthonormal basis for U. Then,

$$V_\alpha^{\circledast} = \langle u_1 \circledast e_\alpha^{\circledast}, u_2 \circledast e_\alpha^{\circledast}\rangle.$$

Now,

$$(u_1 \circledast e_\alpha^{\circledast}, u_2 \circledast e_\alpha^{\circledast}) = \frac{1}{[G : G_\alpha]}(T_\alpha u_1, u_2) = \frac{l}{2k}(u_1, u_2) = 0,$$

and the set $\{u_1 \circledast e_\alpha^{\circledast}, u_2 \circledast e_\alpha^{\circledast}\}$ is an orthogonal basis for $V_\alpha^{\circledast}$. This completes the proof of the Theorem. □

Example 2.5. *Let $G = D_6 = \langle a, b \mid a^3 = b^2 = 1, b^{-1}ab = a^{-1}\rangle \cong S_3$. Then G has only one non-linear irreducible unitary representation given by*

$$\Lambda(a) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \text{ and } \Lambda(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\omega = e^{\frac{2\pi i}{3}}$ is a primitive 3th root of unity. Let λ be the corresponding character of the representation Λ. Then,

$$\lambda(1) = 2,\ \lambda((132)) = \lambda((123)) = -1,\ \lambda((23)) = \lambda((12)) = \lambda((13)) = 0.$$

Let $\dim V = n = 2$. *Then,*

$$\Delta = \{\alpha = (1,1,1), \beta = (1,1,2), \gamma = (1,2,2), \delta = (2,2,2)\}.$$

Clearly, $G_\alpha = G_\delta = G$, $G_\beta = \{(1),(12)\}$ *and* $G_\gamma = \{(1),(23)\}$. *It is easy to see that* $\bar{\Delta} = \{\beta, \gamma\}$ *and* $\dim V_\beta^\circledast = \dim V_\gamma^\circledast = 1$. *Since* $V_\Lambda(G) = V_\beta^\circledast \bigoplus V_\gamma^\circledast$, *so the set* $\{u_1 \circledast e_\beta^\circledast, u_1 \circledast e_\gamma^\circledast\}$ *is an orthogonal basis for* $V_\Lambda(G)$.

3. The dicyclic group

The group T_{4k} $(k \geq 2)$, generated by the elements a, b such that

$$a^{2k} = 1,\ a^k = b^2,\ b^{-1}ab = a^{-1},$$

is called the dicyclic group of degree k. This group is of order $4k$ and

$$T_{4k} = \{a^i, a^i b \mid 0 \leq i < 2k\}.$$

For each integer $0 < j < k$, the group T_{4k} has an irreducible unitary representation of degree 2 given by

$$\Lambda_j(a) = \begin{pmatrix} \xi^j & 0 \\ 0 & \xi^{-j} \end{pmatrix} \text{ and } \Lambda_j(b) = \begin{pmatrix} 0 & 1 \\ \xi^{jk} & 0 \end{pmatrix},$$

where $\xi = e^{\frac{\pi i}{k}}$ is a primitive $(2k)$th root of unity. Let λ_j be the corresponding character Λ_j. The other irreducible representations of T_{4k} are linear (see [12]).

Lemma 3.1. *[4, Lemma 1]*
Let H *be a subgroup of* T_{4k}. *Then, there exists an integer* r, $0 \leq r < 2k$, *such that* $H = \langle a^r \rangle$ *or* $\langle a^r \rangle \subsetneq H$ *with* $H \cap \langle a \rangle = \langle a^r \rangle$. *In the second case, we have* $|H| \geqslant 2|\langle a^r \rangle|$.

Lemma 3.2. *[4, Lemma 3]*
Suppose $G = T_{4k}$ $(k \geqslant 2)$ *and the representation* Λ_j $(0 < j < k)$ *affords character* λ_j *of* G. *Let* H *be a subgroup of* G, *i.e.* $H = \langle a^r \rangle$ *or* $\langle a^r \rangle \subsetneq H$ *with* $H \cap \langle a \rangle = \langle a^r \rangle$, *for some* $0 \leq r < 2k$. *If* $l = 2k/gcd(2k, r)$, *then we have*

$$\sum_{\sigma \in H} \lambda_j(\sigma) = \begin{cases} 2l & \text{if } 2k \mid rj \\ 0 & \text{if } 2k \nmid rj \end{cases}.$$

Theorem 3.3. *Let* $G = T_{4k}$ $(k \geq 2)$ *be a subgroup of* S_m, $\Lambda = \Lambda_j$ *and* $\lambda = \lambda_j$, *where* $0 < j < k$. *Then,* $V_\Lambda(G)$ *has an orthogonal* $\circledast$*-basis.*

Proof. To establish the desired result, it is sufficient to demonstrate that for every $\alpha \in \bar{\Delta}$, the generalized orbital subspace $V_\alpha^\circledast$ possesses an orthogonal $\circledast$-basis.

Let $\alpha \in \bar{\Delta}$. Hence, $\sum_{\sigma \in G_\alpha} \lambda(\sigma) \neq 0$. According to Lemma 3.1, $G_\alpha = \langle a^r \rangle$ or $\langle a^r \rangle \subsetneq G_\alpha$ with $G_\alpha \cap \langle a \rangle = \langle a^r \rangle$. We will consider two cases:

(i) $G_\alpha = \langle a^r \rangle$. Then, $|G_\alpha| = 2k/gcd(2k, r)$. By Lemma 3.2, $2k \mid rj$ and we have $\dim V_\alpha^{\circledast} = (1/|G_\alpha|) \sum_{\sigma \in G_\alpha} \lambda(\sigma) = 2$. Suppose $\{u_1, u_2\}$ is an orthonormal basis for U. Therefore,

$$V_\alpha^{\circledast} = \langle u_1 \circledast e_\alpha^{\circledast}, u_2 \circledast e_\alpha^{\circledast} \rangle.$$

One can easily see that the orthogonal projection T_α is identity. Thus,

$$\left(u_1 \circledast e_\alpha^{\circledast}, u_2 \circledast e_\alpha^{\circledast}\right) = \frac{1}{[G : G_\alpha]}(T_\alpha u_1, u_2) = 0$$

and $\{u_1 \circledast e_\alpha^{\circledast}, u_2 \circledast e_\alpha^{\circledast}\}$ is an orthogonal $\circledast$-basis for $V_\alpha^{\circledast}$.

(ii) $\langle a^r \rangle \subsetneq G_\alpha$. Then, $G_\alpha \cap \langle a \rangle = \langle a^r \rangle$ and $|G_\alpha| \geqslant 2|\langle a^r \rangle|$. By Lemma 3.2, we still have $2k \mid rj$. Consequently,

$$\dim V_\alpha^{\circledast} = \frac{1}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \lambda(\sigma) = \frac{1}{|G_\alpha|}\, 2l \leqslant 1,$$

which establishes the desired result.

□

Example 3.4. *Let $G = T_8 = \langle a, b \mid a^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^{-1} \rangle$. Then $G \cong Q_8$, the quaternion group of order 8. By classical Cayley Theorem, we can embed T_8 in S_8 and so we assume that T_8 is a subgroup of S_8. The group T_8 has an irreducible unitary representation of degree 2 given by*

$$\Lambda(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ and } \Lambda(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where i is the imaginary unit.
Let λ be the corresponding character Λ. Then,

$$\lambda(1) = 2,\ \lambda(a) = \lambda(a^3) = 0,\ \lambda(a^2) = -2,$$

$$\lambda(a^k b) = 0,\ 0 \leq k < 4.$$

The subgroups of G are one of the following forms:

$$H = \langle a^r \rangle (0 \leq r \leq 2), H = \langle b \rangle \text{ and } H = \langle ab \rangle.$$

It is easy to see that $\sum_{\sigma \in H} \lambda(\sigma) \neq 0$ if and only if $H = \{1\}$. So, if $\alpha \in \bar{\Delta}$, then $G_\alpha = \{1\}$. Consequently, $\dim V_\alpha^{\circledast} = 2$.
Suppose $\{u_1, u_2\}$ is an orthonormal basis for U. Assume that $\bar{\Delta} = \{\alpha, \beta, \ldots\}$. As in the proof of Theorem 3.3, we can easily see that the set

$$\{u_1 \circledast e_\alpha^{\circledast}, u_2 \circledast e_\alpha^{\circledast}; u_1 \circledast e_\beta^{\circledast}, u_2 \circledast e_\beta^{\circledast}; \ldots\}$$

is an orthogonal basis for $V_\Lambda(G)$.

4. The group U_{6n}

The group U_{6n} is generated by two elements, a and b, with the orders $2n$ and 3, respectively, satisfying the condition $a^{-1}ba = b^{-1}$. In other words,

$$U_{6n} = \langle a, b \mid a^{2n} = b^3 = 1,\ a^{-1}ba = b^{-1}\rangle.$$

The relation $a^{-1}ba = b^{-1}$ implies:

$$a^{2k}\, b = b\, a^{2k},\ a^{2k-1}\, b^{-1} = b\, a^{2k-1}.$$

Hence, we deduce that all members of U_{6n} can be expressed in the form of $a^i b^j$, where i ranges from 0 to $2n-1$ and j ranges from 0 to 2:

$$U_{6n} = \{a^i b^j \mid 0 \leq i \leq 2n-1,\ 0 \leq j \leq 2\}.$$

The group $G = U_{6n}$ has conjugacy classes given by:

$$\{a^{2k}\},\quad \{a^{2k}b, a^{2k}b^2\},\quad \{a^{2k-1}, a^{2k-1}b, a^{2k-1}b^2\}.$$

where $0 \leq k \leq n-1$. Additionally, $G' = \langle b\rangle$, and G/G' is isomorphic to C_{2n}, where C_{2n} is a cyclic group of order $2n$.

Let's denote $\varepsilon = e^{\frac{2\pi i}{2n}}$ and $\omega = e^{\frac{2\pi i}{3}}$ as the $2n$-th and 3-th primitive roots of unity, respectively. For each $1 \leq j \leq n$, there exists an irreducible representation of G of degree 2 as follows:

$$\Lambda_j(a) = \begin{pmatrix} 0 & \varepsilon^j \\ \varepsilon^j & 0 \end{pmatrix},\qquad \Lambda_j(b) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}.$$

Then,

$$\Lambda_j(a^{2k}) = \varepsilon^{2kj}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \Lambda_j(a^{2k}b) = \varepsilon^{2kj}\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix},\ \Lambda_j(a^{2k-1}) = \varepsilon^{(2k-1)j}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let's denote χ_j as the corresponding character of Λ_j. In this case,

$$\chi_j(a^{2k}) = 2\varepsilon^{2kj},\ \chi_j(a^{2k-1}) = 0,\ \chi_j(a^{2k}b) = -\varepsilon^{2kj}.$$

The other irreducible characters of G are all linear (see [12]). The subgroups of U_{6n} can take one of the following forms:

$$(i)\ H = \langle a^k\rangle,\qquad (ii)\ H = \langle a^k, b\rangle,$$
$$(iii)\ H = \langle a^k b\rangle,\qquad (iv)\ H = \langle a^k b^2\rangle,$$

where $0 \leq k \leq 2n$ and k divides $2n$ (see [20, Table 4]). These subgroups are not necessarily distinct and under some conditions they might be even similar. Let

$$l = o(a^k) = \frac{o(a)}{gcd(k, o(a))} = \frac{2n}{gcd(k, 2n)} = \frac{2n}{k}.$$

(i) Let's assume $H = \langle a^k\rangle$. In this case, H is a cyclic subgroup of order l.
(ii) If $H = \langle a^k, b\rangle$ then the order of H is equal to $3l$, and it can be expressed as

$$H = \{1, a^k, \ldots, a^{k(l-1)}, b, a^k b, \ldots, a^{k(l-1)}b, b^2, a^k b^2, \ldots, a^{k(l-1)}b^2\}.$$

(iii) If $H = \langle a^k b \rangle$ then let's consider two cases as below:

Case 1 : k is odd. Then, for all integers r, we have

$$(a^k b)^r = \begin{cases} a^{kr} b, & \text{if } r \text{ is odd} \\ a^{kr}, & \text{if } r \text{ is even} \end{cases}.$$

Since k is odd, l will be even and

$$H = \{1, a^k b, a^{2k}, a^{3k} b, \ldots, a^{k(l-1)} b\}.$$

In this case, H is a subgroup of the order l.

Case 2 : k is even. Then,

$$(a^k b)^r = \begin{cases} a^{kr}, & \text{if } r \equiv 0 \pmod 3 \\ a^{kr} b, & \text{if } r \equiv 1 \pmod 3 \\ a^{kr} b^2, & \text{if } r \equiv 2 \pmod 3 \end{cases}.$$

If $l \equiv 1 \pmod 3$ then $(a^k b)^l = a^{kl} b = b$, which means b is an element of H. Therefore, $H = \langle a^k, b \rangle$.
If $l \equiv 2 \pmod 3$ then $(a^k b)^l = a^{kl} b^2 = b^2$, implying that b^2 is in H and thus, b is in H. Consequently, $H = \langle a^k, b \rangle$.
If $l \equiv 0 \pmod 3$ then $(a^k b)^l = a^{kl} = 1$, meaning that H is a cyclic subgroup of order l.

(iv) If $H = \langle a^k b^2 \rangle$, let's consider two cases:

Case 1 : k is odd. In this case,

$$(a^k b^2)^r = \begin{cases} a^{kr} b^2, & \text{if } r \text{ is odd} \\ a^{kr}, & \text{if } r \text{ is even} \end{cases}.$$

Since k is odd, l will be even and we can express H as follows:

$$H = \{1, a^k b^2, a^{2k}, a^{3k} b^2, \ldots, a^{k(l-1)} b^2\}.$$

In this case, H is a subgroup of order l.

Case 2 : Let's consider that k is even. In this case,

$$(a^k b^2)^r = \begin{cases} a^{kr}, & \text{if } r \equiv 0 \pmod 3 \\ a^{kr} b^2, & \text{if } r \equiv 1 \pmod 3 \\ a^{kr} b, & \text{if } r \equiv 2 \pmod 3 \end{cases}.$$

If $l \equiv 1 \pmod 3$ then $(a^k b^2)^l = a^{kl} b^2 = b^2$. This implies that b^2 is in H, which in turn means that b is in H. Therefore, $H = \langle a^k, b \rangle$.
If $l \equiv 2 \pmod 3$ then $(a^k b)^l = a^{kl} b = b$. Thus, b is in H and again, $H = \langle a^k, b \rangle$. If $l \equiv 0 \pmod 3$ then $(a^k b)^l = a^{kl} = 1$. This implies that H is a cyclic subgroup of order l.

Lemma 4.1. *Let $H = \langle a^k \rangle$, where k divides $2n$ and $1 \leq j \leq n$. Then,*

$$\sum_{\sigma \in H} \chi_j(\sigma) = \begin{cases} 2l & \text{if } 2n \mid jk \\ 0 & \text{otherwise} \end{cases}$$

if k is even, and

$$\sum_{\sigma \in H} \chi_j(\sigma) = \begin{cases} l & \text{if } 2n \mid jk \\ 0 & \text{otherwise} \end{cases}$$

if k is odd.

Proof. First note that $l = 2n/gcd(2n, k) = 2n/k$ is an integer multiple of $o(\varepsilon^{jk}) = 2n/gcd(2n, jk)$. Using Lemma 2.1, if k is even, then

$$\begin{aligned} \sum_{\sigma \in H} \chi_j(\sigma) &= \sum_{t=1}^{l} \chi_j(a^{kt}) = \sum_{t=1}^{l} 2\varepsilon^{jkt} = 2\sum_{t=1}^{l} (e^{\frac{2\pi jki}{2n}})^t \\ &= \begin{cases} 2l & \text{if } 2n \mid jk \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and if k is odd, then l is even and so

$$\begin{aligned} \sum_{\sigma \in H} \chi_j(\sigma) &= \sum_{t=1}^{l} \chi_j(a^{kt}) = \sum_{t=1}^{\frac{l}{2}} 2\varepsilon^{2jkt} = 2\sum_{t=1}^{\frac{l}{2}} (e^{\frac{2\pi jki}{2n}})^{2t} \\ &= \begin{cases} l & \text{if } 2n \mid jk \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

□

Lemma 4.2. *Consider $H = \langle a^k, b \rangle$, where k divides $2n$ and $1 \leq j \leq n$. Then,*

$$\sum_{\sigma \in H} \chi_j(\sigma) = 0.$$

Proof. Let's consider two cases:

(i) If k is odd then l is even. Let's suppose $l = 2s$. In this case,

$$\begin{aligned} \sum_{\sigma \in H} \chi_j(\sigma) &= \sum_{t=1}^{l} \chi_j(a^{kt}) + \sum_{t=1}^{l} \chi_j(a^{kt}b) + \sum_{t=1}^{l} \chi_j(a^{kt}b^2) \\ &= \sum_{t=1}^{s} \chi_j(a^{2kt}) + \sum_{t=1}^{s} \chi_j(a^{2kt}b) + \sum_{t=1}^{s} \chi_j(a^{2kt}b^2) \\ &= \sum_{t=1}^{s} 2\varepsilon^{2ktj} - \sum_{t=1}^{s} \varepsilon^{2ktj} - \sum_{t=1}^{s} \varepsilon^{2ktj} = 0. \end{aligned}$$

(ii) If k is even, then

$$\begin{aligned}\sum_{\sigma\in H}\chi_j(\sigma) &= \sum_{t=1}^{l}\chi_j(a^{kt}) + \sum_{t=1}^{l}\chi_j(a^{kt}b) + \sum_{t=1}^{l}\chi_j(a^{kt}b^2)\\ &= \sum_{t=1}^{l}2\varepsilon^{ktj} - \sum_{t=1}^{l}\varepsilon^{ktj} - \sum_{t=1}^{l}\varepsilon^{ktj} = 0.\end{aligned}$$

So, the result holds. □

Lemma 4.3. *Consider $H = \langle a^k b^s\rangle$, where $1 \le s \le 2$, $1 \le j \le n$, and k is an odd number. Then,*

$$\sum_{\sigma\in H}\chi_j(\sigma) = \begin{cases} l & \text{if } n \mid kj \\ 0 & \text{otherwise} \end{cases}.$$

Proof. Suppose $2n = kl$. Since k is odd, let's say $l = 2r$. Thus

$$H = \{1, a^k b^s, a^{2k}, a^{3k}b^s, \dots, a^{k(l-1)}b^s\}$$

and

$$\begin{aligned}\sum_{\sigma\in H}\chi_j(\sigma) &= \sum_{t=1}^{l}\chi_j((a^k b^s)^t) = \sum_{t=1}^{r}\chi_j(a^{2tk})\\ &= 2\sum_{t=1}^{r}\varepsilon^{2jkt} = 2\sum_{t=1}^{r}e^{\frac{4\pi jkti}{2n}}.\end{aligned}$$

Now, by using Lemma 2.1, the result is obtained. □

Lemma 4.4. *Suppose $H = \langle a^k b^s\rangle$, where $1 \le s \le 2$, $1 \le j \le n$, k is even, and $l \equiv 0 \pmod 3$. Then,*

$$\sum_{\sigma\in H}\chi_j(\sigma) = \begin{cases} 0 & \text{if } 2n \nmid 3jk \\ 0 & \text{if } 2n \mid jk \\ l & \text{if } 2n \mid 3jk \text{ and } 2n \nmid jk \end{cases}.$$

Proof. Let us assume $s = 1$ and $l = 3r$. Then,

$$\begin{aligned}
\sum_{\sigma\in H}\chi_j(\sigma) &= \sum_{t=1}^{l}\chi_j((a^kb)^t)\\
&= \sum_{t=1}^{r}\chi_j((a^kb)^{3t}) + \sum_{t=1}^{r}\chi_j((a^kb)^{(3t-1)}) + \sum_{t=1}^{r}\chi_j((a^kb)^{(3t-2)})\\
&= \sum_{t=1}^{r}\chi_j(a^{3tk}) + \sum_{t=1}^{r}\chi_j(a^{(3t-1)k}b^2) + \sum_{t=1}^{r}\chi_j(a^{(3t-2)k}b)\\
&= \sum_{t=1}^{r}2\varepsilon^{3kjt} - \sum_{t=1}^{r}\varepsilon^{(3t-1)kj} - \sum_{t=1}^{r}\varepsilon^{(3t-2)kj}\\
&= 2\sum_{t=1}^{r}\varepsilon^{3kjt} - \varepsilon^{-kj}\sum_{t=1}^{r}\varepsilon^{3tkj} - \varepsilon^{-2kj}\sum_{t=1}^{r}\varepsilon^{3tkj}\\
&= \left(2-\varepsilon^{-kj}-\varepsilon^{-2kj}\right)\sum_{t=1}^{r}\varepsilon^{3kjt}\\
&= \left(2-\varepsilon^{-kj}-\varepsilon^{-2kj}\right)\sum_{t=1}^{r}e^{\frac{6\pi jkt}{2n}i}.
\end{aligned}$$

If $2n \nmid 3kj$ then, by Lemma 2.1, $\sum_{\sigma\in H}\chi_j(\sigma) = 0$. Now, let's consider the case where $2n \mid 3kj$, and we'll set $3jk = 2nd$. Then,

$$A = 2-\varepsilon^{-kj}-\varepsilon^{-2kj} = 2 - e^{-\frac{2\pi d}{3}i} - e^{-\frac{4\pi d}{3}i} = \overline{2-\omega^d-\omega^{2d}}.$$

Using the division algorithm, we can express d as $d = 3q+q'$, where $q' = 0, 1, 2$. If $q' = 0$ then $d = 3q$ and therefore, $3jk = 2n(3q)$, implying that $2n \mid jk$. In this case, we find that $A = 0$. However, when $q' = 1$ or $q' = 2$, we get

$$A = \overline{2-\omega^{q'}-\omega^{2q'}} = \overline{2-\omega^1-\omega^2} = \overline{2-(\omega+\omega^2)} = 3.$$

This is because $\omega+\omega^2 = -1$. Therefore, in these cases, we conclude that the result holds. Similarly, for $s = 2$, the proof follows a similar pattern. □

Notice that if $H = \langle a^kb^s\rangle$, where $1 \le s \le 2$, $1 \le j \le n$, k is even and $l \equiv 1, 2 \pmod 3$ then $H = \langle a^k, b\rangle$. By Lemma 4.2, we have $\sum_{\sigma\in H}\chi_j(\sigma) = 0$.

Theorem 4.5. *Let $G = U_{6n}$ $(n \ge 1)$ and Λ be a unitary irreducible representation of G. Then, $V_\Lambda(G)$ has an orthogonal $\circledast$-basis.*

Proof. If the degree of Λ is equal to 1, there is nothing to prove. Suppose Λ has a degree greater than 1 and let $\Lambda = \Lambda_j$ for $1 \le j \le n$. Take an arbitrary $\alpha \in \bar{\Delta}$. Then, $\sum_{\sigma\in G_\alpha}\chi_j(\sigma) \neq 0$. According to the four Lemmas 4.1-4.4, G_α falls into one of the following three cases:

(a) $G_\alpha = \langle a^k\rangle$, where k divides $2n$.
(b) $G_\alpha = \langle a^kb^s\rangle$, where $s = 1, 2$, k divides $2n$ and k is odd.

(c) $G_\alpha = \langle a^k b^s \rangle$, where $s = 1, 2$, k divides $2n$, k is even and $l \equiv 0 \pmod 3$.

In all three cases, $|G_\alpha| = l$ and $\sum_{\sigma \in G_\alpha} \chi(\sigma) = l$ or $2l$. So,

$$\dim V_\alpha^\circledast = \frac{1}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma) = 1 \text{ or } 2.$$

If $\dim V_\alpha^\circledast = 1$, then the set $\{u_1 \circledast e_\alpha^\circledast \mid \alpha \in \bar{\Delta}\}$ forms an orthogonal basis for the symmetry class of $V_\Lambda(G)$ and the result is established. Therefore, we assume that $\dim V_\alpha^\circledast = 2$. Then $G_\alpha = \langle a^k \rangle$, where k is even. In this case,

$$T_\alpha = \frac{1}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \Lambda(\sigma) = \frac{1}{l} \sum_{t=1}^{l} \Lambda_j(a^{kt}) = (\frac{1}{l} \sum_{t=1}^{l} \varepsilon^{jkt})\ I_2$$
$$= (\frac{1}{2|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma))\ I_2 = I_2,$$

where I_2 is the identity matrix of order 2×2. Let $\{u_1, u_2\}$ be an orthonormal basis for U. Then,

$$V_\alpha^\circledast = \langle u_1 \circledast e_\alpha^\circledast, u_2 \circledast e_\alpha^\circledast \rangle.$$

Now,

$$(u_1 \circledast e_\alpha^\circledast, u_2 \circledast e_\alpha^\circledast) = \frac{1}{[G : G_\alpha]} (T_\alpha u_1, u_2) = 0,$$

and the set $\{u_1 \circledast e_\alpha^\circledast, u_2 \circledast e_\alpha^\circledast\}$ is an orthogonal basis for $V_\alpha^\circledast$. □

Remark 4.6. By utilizing the Theorems 2.4, 3.3 and 4.5, it becomes evident that the existence of an orthogonal ⊛-basis within the generalized symmetry classes of tensors associated with the group U_{6n}, the dihedral group and the dicyclic group is not contingent upon the permutation structure of the respective groups.

Finally, we propose the following conjecture.

Conjecture 4.7. Let V be a unitary vector space. Consider G as a subgroup of S_m and let Λ be an irreducible unitary representation of G over an inner product space U. Then, $V_\Lambda(G)$ always possesses an orthogonal ⊛-basis.

5. Conclusion

Consider a unitary complex vector space V. Let G be a permutation group that acts on m elements. Let Λ be an irreducible unitary representation of G over an inner product space U, which affords the character λ of G. We define $V_\Lambda(G)$ as the generalized symmetry class of tensors associated with G and Λ. When the dimension of U equals 1, $V_\Lambda(G)$ reduces to $V_\lambda(G)$, which represents the symmetry class of tensors associated with G and λ. In the case where λ is a linear character of G, $V_\lambda(G)$ possesses an orthogonal $*$-basis.
The existence of an orthogonal basis consisting of decomposable symmetrized

tensors for symmetry class $V_\lambda(G)$ associated with G and a non-linear irreducible character λ of G has been investigated by many researchers (see [6, 18, 19, 21, 23]). In particular, in [4, 10], a necessary and sufficient condition for the existence of an orthogonal $*$-basis for the symmetry class $V_\lambda(G)$ is given when G is the dicyclic or dihedral group. Additionally, [5, 7] prove that there is no orthogonal $*$-basis for the symmetry class $V_\lambda(U_{6n})$.
In this paper, we proved that for the generalized symmetry class $V_\Lambda(G)$, there is always an orthogonal $\circledast$-basis, where G is the group U_{6n}, dihedral or dicyclic group.

6. Acknowledgment

The authors would like to thank the referees and the editor for their comprehensive and useful comments which helped the improvement of this work to the present form.

7. Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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