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A CRITERION FOR p-SOLVABILITY OF FINITE GROUPS, WHERE $p = 7$ OR 11

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Abstract. For a finite group G, define $\psi''(G) = \psi(G)/|G|^2$, where $\psi(G) = \sum_{g\in G} o(g)$ and $o(g)$ denotes the order of $g \in G$. In this paper, we give a criterion for p-solvability by the function ψ'', where $p \in \{7, 11\}$. We prove that if G is a finite group and $\psi''(G) > \psi''(\mathrm{PSL}(2, p))$, where $p \in \{7, 11\}$, then G is a p-solvable group.

Keywords: Finite group, element order, p-solvability, .
2020 MSC: Primary 20D10, 20D15, 20D20.

1. Introduction

Let G be a finite group and $\psi(G) = \sum_{g\in G} o(g)$, where $o(g)$ denotes the order of $g \in G$, which was introduced by Amiri et al. (see [1]). They showed that C_n is the unique group of order n with the largest value of $\psi(G)$ for groups of that order. In [11], Herzog, Longobardi and Maj determined the exact upper bound for $\psi(G)$ for non-cyclic groups G. There are some applications for $\psi(G)$, for example, $\psi(G)$ is equal to the sum of the number of arcs and the number of vertices of a directed power graph [9].

A finite group G is a $\mathscr{B}_\psi$-group if $\psi(H) < |G|$ for all proper subgroups H of G. In [2], Baniasad Azad showed that if S is a finite simple group, such that $S \neq Alt(n)$ for any $n \geq 14$, then S is a $\mathscr{B}_\psi$-group. The function ψ has been considered in various works (see [7, 12]).

The functions $m(G) = \sum_{g\in G} 1/o(g)$, $l(G) = \sqrt[n]{\prod_{g\in G} o(g)}/|G|$ and $\psi'(G) = \psi(G)/\psi(C_n)$ were introduced in [5, 6, 12]. Many authors investigate the influence of these functions on the structure of a finite group G. For example, if $g \in \{\psi', l, m\}$, and $g(G) > g(C_2 \times C_2)$, $g(G) > g(S_3)$, $g(G) > g(A_4)$ or $g(G) > g(A_5)$, then G is cyclic, nilpotent, supersolvable or solvable, respectively (see [3, 5–8, 11, 12, 15]).

Tărnăuceanu in [14], introduced $\psi''(G) = \psi(G)/|G|^2$ and also proved the following theorem:

Theorem 1.1. *[14, Theorem 1.1] Let G be a finite group. Then the following holds:*

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(a) *If* $\psi''(G) > 7/16 = \psi''(C_2 \times C_2)$*, then G is cyclic.*
(b) *If* $\psi''(G) > 27/64 = \psi''(Q_8)$*, then G is abelian.*
(c) *If* $\psi''(G) > 13/36 = \psi''(S_3)$*, then G is nilpotent.*
(d) *If* $\psi''(G) > 31/144 = \psi''(A_4)$*, then G is supersolvable.*
(e) *If* $\psi''(G) > 211/3600 = \psi''(A_5)$*, then G is solvable.*

In [4], Baniasad Azad and Khosravi proved the following theorem:

Theorem 1.2. *[4, Main Theorem] Let G be a finite group such that* $\psi''(G) > \psi''(D_{2p})$*, where p is a prime number. Then* $G \cong O_p(G) \times O_{p'}(G)$ *and* $O_p(G)$ *is cyclic.*

In this paper, we focus on the function $\psi''(G)$. We give a criterion for p-solvability by the function ψ'', where $p \in \{7, 11\}$. We prove that if G is a finite group and $\psi''(G) > \psi''(\mathrm{PSL}(2,p))$, where $p \in \{7, 11\}$, then G is a p-solvable group.

2. A criterion for p-solvability, where $p = 7$ or 11

We need the following lemmas.

Lemma 2.1. *[16, Lemma 1] Let G be a non-solvable group. Then G has a normal series* $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ *such that* K/H *is a direct product of isomorphic non-abelian simple groups and* $|G/K| \mid |\mathrm{Out}(K/H)|$.

Lemma 2.2. *[13] Let A be a cyclic proper subgroup of a finite group G, and let* $K = \mathrm{core}_G(A)$*. Then* $|A : K| < |G : A|$*, and in particular, if* $|A| > |G : A|$*, then* $K > 1$.

Lemma 2.3. *[4, Lemma 2.1] If* $\psi''(G) > t$*, then G has an element x such that* $|G : \langle x \rangle| < 1/t$.

Lemma 2.4. *[14] Let H be a normal subgroup of the finite group G. Then* $\psi''(G) \leq \psi''(G/H)$.

Remark 2.5. By using GAP, we can conclude that the only non-solvable groups G with trivial Fitting subgroup of order at most 1482, which satisfy $\psi''(G) > \psi''(\mathrm{PSL}(2,7))$, are A_5 and S_5 (see Table 1).

Theorem 2.6. ***(a)*** *If G has no composition factor isomorphic to* A_5 *and* $\psi''(G) > \psi''(\mathrm{PSL}(2,7))$*, then G is a solvable group.*
(b) *If G is a finite group and* $\psi''(G) > \psi''(\mathrm{PSL}(2,7))$*, then G is a 7-solvable group.*

Proof. **(a)** We prove that G is solvable by induction on $|G|$. If $|G| \leq 59$, then G is a solvable group. If G has a non-trivial normal solvable subgroup N then, by Lemma 2.4,

$$\psi''(\mathrm{PSL}(2,7)) < \psi''(G) \leq \psi''(G/N).$$

Structure of G	IdGroup (G)	Out(G)	$\psi''(G)$	$\psi''(G) > \psi''(K)$	$\psi''(G) > \psi''(H)$
A_5	$(60, 5)$	C_2	211/3600	true	true
S_5	$(120, 34)$	1	157/4800	true	true
PSL(2, 7)	$(168, 42)$	C_2	715/28224	false	true
PSL(2, 7) : C_2	$(336, 208)$	1	593/37632	false	true
A_6	$(360, 118)$	$C_2 \times C_2$	1411/129600	false	true
PSL(2, 8)	$(504, 156)$	C_3	3319/254016	false	true
PSL(2, 11)	$(660, 13)$	C_2	1247/145200	false	false
S_6	$(720, 763)$	C_2	3271/518400	false	false
$A_6 : C_2$	$(720, 764)$	C_2	4363/518400	false	false
$H(9) = A_6 \cdot C_2$	$(720, 765)$	C_2	3571/518400	false	false
PSL(2, 13)	$(1092, 25)$	C_2	809/132496	false	false
PSL(2, 11) : C_2	$(1320, 133)$	1	9593/1742400	false	false
$(A6 \cdot C_2) : C_2$	$(1440, 5841)$	1	8383/2073600	false	false
A_7	NA	C_2	12601/6350400	false	false
PSL(3, 3)	NA	C_2	44539/31539456	false	false
PSU(3, 3)	NA	C_2	43639/36578304	false	false
PSU(3, 3) : C_2	NA	1	93535/146313216	false	false
M_{11}	NA	1	53131/62726400	false	false

Table 1. $K = \text{PSL}(2, 7)$ and $H = \text{PSL}(2, 11)$

By the inductive hypothesis, G/N is a solvable group and consequently, G is solvable. Now suppose that G has no non-trivial normal solvable subgroup. Since $\psi''(G) > \psi''(\text{PSL}(2, 7)) = 715/168^2$, Lemma 2.3 implies there exists an element $x \in G$ such that

(1) $$|G : \langle x\rangle| < 168^2/715 < 40.$$

Using Lemma 2.2, $|\langle x\rangle : \text{core}_G(\langle x\rangle)| \leqslant 38$. Therefore,

$$|G : \text{core}_G(\langle x\rangle)| = |G : \langle x\rangle| \cdot |\langle x\rangle : \text{core}_G(\langle x\rangle)| \leqslant 1482.$$

Since $\text{core}_G(\langle x\rangle) = 1$, $|G| \leqslant 1482$. Let G be a non-solvable group. By Lemma 2.1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of some isomorphic non-abelian simple groups and $|G/K| \bigm| |\text{Out}(K/H)|$. If H is non-solvable then $|K| = |K/H| \cdot |H|$ divides $|G|$. Therefore, $3600 \leq |G|$. This is a contradiction and H is solvable. So $H = 1$.

Since G has no composition factor isomorphic to A_5 and $|G| \leqslant 1482$, we have the following cases:

(1) Let $K \cong \text{PSL}(2, 7)$. Since $|\text{Out}(\text{PSL}(2, 7))| = 2$, it follows that $|G/K|$ is a divisor of 2. If $|G/K| = 1$ then $G \cong \text{PSL}(2, 7)$. Moreover since $\psi''(\text{PSL}(2, 7)) < \psi''(G)$, we get a contradiction. If $|G/K| = 2$ then G is a non-solvable group of order 336. By GAP, we can see that $\psi(G) \leqslant 2355$. Therefore,

$$\frac{715}{168^2} = \psi''(\text{PSL}(2, 7)) < \psi''(G) \leqslant \frac{2355}{336^2},$$

i.e. $2860 < 2355$, which is a contradiction.

(2) Let $K \cong A_6$. Then, $|G/K| \mid 4$ and we have the following cases:
- If $|G/K| = 1$ then $G \cong A_6$. By Lemma 2.4,

$$\frac{715}{168^2} = \psi''(\text{PSL}(2, 7)) < \psi''(G) = \psi''(A_6) = \frac{1411}{360^2},$$

which is a contradiction.

- If $|G/K| = 2$ then G is a non-solvable group of order 720. By GAP, we can see that $\psi(G) \leqslant 12557$. Therefore, $\dfrac{715}{168^2} < \dfrac{12557}{720^2}$, which is a contradiction.
- If $|G/K| = 4$ then $|G| = 1440$. Therefore, by (1), $|G : \langle x\rangle| \leqslant 38$. By Lemma 2.2, $|G| \leqslant 38 \cdot 37 = 1406$, which is a contradiction.

(3) Let $K \cong \mathrm{PSL}(2,8)$. Then, $|G/K| \mid 3$. If $|G/K| = 1$ then $G \cong \mathrm{PSL}(2,8)$, which is a contradiction since $\psi''(\mathrm{PSL}(2,7)) > \psi''(\mathrm{PSL}(2,8)) = 3319/504^2$. If $|G/K| = 3$ then $|G| \geqslant 3|\mathrm{PSL}(2,8)| = 1512$, which is a contradiction.

(4) Let $K \cong \mathrm{PSL}(2,11)$. Then, $|G/K|$ is a divisor of 2. Since $\psi''(\mathrm{PSL}(2,7)) > \psi''(\mathrm{PSL}(2,11))$, we get that $|G/K| = 2$. Therefore, G is a non-solvable group of order 1320. By GAP, we can see that $\psi(G) \leqslant 11993$. Therefore,

$$\frac{715}{168^2} < \frac{11993}{1320^2},$$

which is a contradiction.

(5) Let $K \cong \mathrm{PSL}(2,13)$. Then $|G/K|$ divides 2. Similar to the above, $|G/K| = 2$ which implies that $|G| = 2184$, which is a contradiction.

(b) Similarly to the above we get that $|G| \leqslant 1482$. Now suppose that G is not a 7-solvable group. Therefore, G is non-solvable, by Lemma 2.1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of some isomorphic non-abelian simple groups and $|G/K| \bigm| |\mathrm{Out}(K/H)|$. As we mentioned above, H is a solvable group.

Let $K/H \cong A_5$. If H is 7-solvable then G is 7-solvable. If H is not a 7-solvable group then $|H| \geq 168$ we have $|G| \geq 60 \cdot 168$ which is a contradiction.

The proof is now complete. □

Theorem 2.7. *If G is a finite group and $\psi''(G) > \psi''(\mathrm{PSL}(2,11))$, then G is an 11-solvable group.*

Proof. We prove that G is solvable by induction on $|G|$. If $|G| \leq 659$ or $11 \nmid |G|$ then G is an 11-solvable group. If G has a non-trivial normal 11-solvable subgroup N then, by Lemma 2.4,

$$\psi''(\mathrm{PSL}(2,11)) < \psi''(G) \leqslant \psi''(G/N),$$

So, by the inductive hypothesis, G/N is an 11-solvable group and consequently, G is 11-solvable. Therefore, suppose that G has no non-trivial normal 11-solvable subgroup. Since $\psi''(G) > \psi''(\mathrm{PSL}(2,11)) = 3741/660^2 = 1247/145200$, Lemma 2.3 implies there exists an element $x \in G$ such that $|G : \langle x\rangle| \leq 116$. Using Lemma 2.2, $|\langle x\rangle : \mathrm{core}_G(\langle x\rangle)| \leqslant 115$. Therefore,

$$|G : \mathrm{core}_G(\langle x\rangle)| = |G : \langle x\rangle| \cdot |\langle x\rangle : \mathrm{core}_G(\langle x\rangle)| \leqslant 116 \cdot 115 = 13340.$$

Since $\mathrm{core}_G(\langle x\rangle) = 1$, $|G| \leqslant 13340$ and G is not an 11-solvable group. By Lemma 2.1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is isomorphic to the direct product of some copies of a non-abelian simple group S and

$|G/K| \bigm| |\mathrm{Out}(K/H)|$. If H is not an 11-solvable group then $|K| = |K/H| \cdot |H|$ divides $|G|$. Therefore, $60 \cdot 660 \leq |G|$ which is a contradiction. Thus, H is 11-solvable and so, $H = 1$. By [10], we have

$$S \in \{\mathrm{PSL}(2,q) | q = 5,7,8,11,13,16,17,19,23,25,27,29\} \\ \cup \{A_6, A_7, \mathrm{PSL}(3,3), \mathrm{PSU}(3,3), M_{11}\}.$$

(1) Let $K \cong \mathrm{PSL}(2,11)$. Since $\psi''(\mathrm{PSL}(2,11)) < \psi''(G)$, we get $|G/K| = 2$ and so, G is a non-solvable group of order 1320.
$G \not\cong \mathrm{SL}(2,11)$, since $\mathrm{PSL}(2,11)$ is not subgroup of $\mathrm{SL}(2,11)$. Also $G \not\cong C_2 \times \mathrm{PSL}(2,11)$, since $\mathrm{F}(G) = 1$. Therefore, $G \cong \mathrm{PSL}(2,11) : C_2$ and by GAP, we have $\psi''(\mathrm{PSL}(2,11) : C_2) = 9593/1742400$. Therefore, we get a contradiction.
(2) Let $K \cong \mathrm{PSL}(2,23)$. Since $\psi''(\mathrm{PSL}(2,11)) > \psi''(\mathrm{PSL}(2,23))$, we get $|G/K| = 2$. Hence $|G| = 2|\mathrm{PSL}(2,23)|$.
We know that $|G| = |G : \langle x\rangle||\langle x\rangle|$, where $|\langle x\rangle| < |G : \langle x\rangle| < 117$. Therefore, $|G : \langle x\rangle| < 109$ and so, $|G| < 109 \cdot 108$ which is a contradiction.
(3) Let $K \cong M_{11}$. Since $\mathrm{Out}(M_{11}) = 1$, it follows that $G \cong M_{11}$ which is a contradiction since $\psi''(\mathrm{PSL}(2,11)) > \psi''(M_{11})$.
(4) Other cases, since $H = 1$, we have $|G| = |G/K| \cdot |K|$ and $|G/K| \mid |\mathrm{Out(K)}|$. Therefore, $11 \nmid |G|$ and so, G is 11-solvable.

The proof is now complete. □

Example. We note that using GAP, we have $\psi''(A_5 \times C_7) = 9073/176400 > 715/28224 = \psi''(\mathrm{PSL}(2,7))$. Therefore $A_5 \times C_7$ is a 7-solvable group but we know $A_5 \times C_7$ is not a solvable group.

Remark 2.8. We note that of using GAP, we have

$$\psi''(\mathrm{PSL}(2,31)) = \frac{\psi(\mathrm{PSL}(2,31))}{|\mathrm{PSL}(2,31)|^2} = \frac{181227}{14880^2} = \frac{60409}{73804800},$$

$$\psi''(\mathrm{PSL}(2,32)) = \frac{\psi(\mathrm{PSL}(2,32))}{|\mathrm{PSL}(2,32)|^2} = \frac{877983}{32736^2} = \frac{292661}{357215232},$$

and so $\psi''(\mathrm{PSL}(2,32)) > \psi''(\mathrm{PSL}(2,31))$, but $\mathrm{PSL}(2,32)$ is not a 31-solvable group.

Therefore $\psi''(G) > \psi''(\mathrm{PSL}(2,p))$ is not a sufficient condition for p-solvability of G. We believe that the following conjecture holds:
Conjecture. If G is a finite group and p is a prime such that

$$\psi''(G) > \max\{\psi''(S) : S \text{ is a simple group and } p \mid |S|\},$$

then G is a p-solvable group.

3. Conclusion

In this paper, we obtained a criterion for p-solvability by the function ψ'', where $p \in \{7, 11\}$. We proved that if G is a finite group and $\psi''(G) > \psi''(\mathrm{PSL}(2,p))$, where $p \in \{7, 11\}$, then G is a p-solvable group.

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