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A NOTE ON 2-PRIME IDEALS

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Abstract. Let R be a commutative ring with identity. In this paper, we study 2-prime ideals of a Dedekind domain and a Prüfer domain. We prove that a nonzero ideal I of a Dedekind domain R is 2-prime if and only if $I = P^{\alpha}$, for some maximal ideal P of R and positive integer α. We give some results of ring R in which every ideal I is 2-prime. Finally, we define almost 2-prime, almost 2-primary and weakly 2-primary ideals, and investigate some properties of these ideals.

Keywords: 2-prime ideal, almost 2-prime ideal, almost 2-primary ideal, weakly 2-primary ideal.
2020 MSC: 13C05, 13C13,13C15.

1. Introduction

In this paper, we focus on commutative rings with an identity $1 \neq 0$. Throughout the paper, R always denotes a ring, and I denotes an ideal. By a proper ideal I of R we mean an ideal with $I \neq R$. For any proper ideal I of R, the radical $\sqrt{I}$ is defined as $\sqrt{I} := \{a \in R : a^n \in I \text{ for some } n \in \mathbb{N}\}$.

Prime ideals play a central role in commutative ring theory, and so this notion has been generalized and studied in several directions. A proper ideal P of R is said to be a prime ideal if whenever $xy \in P$ for some $x, y \in R$, then $x \in P$ or $y \in P$ [3]. The set of all prime ideals of a ring R is denoted by $Spec(R)$ and for a ring R, set N(R)=$\{a \in R : a^n = 0$ for some positive integer $n\} = \bigcap\{P : P$ is a prime ideal of $R\}$ [12]. The importance of some of these generalizations is as important as prime ideals. Let I be an ideal of a commutative ring R. We say that I is a primary ideal of R when I is a proper ideal of R and whenever $a, b \in R$ with $ab \in I$ but $a \notin I$, then there exists $n \in \mathbb{N}$ such that $b^n \in I$ [11]. An ideal I of a ring R will be called semiprimary if it's radical, $\sqrt{I}$, is prime [7]. It is clear that every primary ideal is semiprimary.

In 2003, Anderson and Smith [1] introduced the notion of a weakly prime ideal. That is, a proper ideal I of R with the property that for $a, b \in R$, $0 \neq ab \in I$ implies $a \in I$ or $b \in I$. In 2005, Bhatwadekar and sharma [6] introduced the notion of an almost prime ideal, which is also a generalization of a prime ideal. A proper ideal I of a ring R is said to be almost prime if for $a, b \in R$ with $ab \in I - I^2$, then either $a \in I$ or $b \in I$. It is clear that every prime

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ideal is a weakly prime ideal and an almost prime ideal. In 2007, Badawi [4] introduced and investigated the notion of 2-absorbing ideals. A nonzero proper ideal I of R is called a 2-absorbing ideal if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. In 2016, Beddani and Messirdi [5] defined the concept of 2-prime ideals, and they characterized valuation domains in terms of this concept. A proper ideal I of R is said to be a 2-prime ideal if whenever $xy \in I$ for some $x, y \in R$, then $x^2 \in I$ or $y^2 \in I$. Note that every prime ideal is 2-prime, but the converse is not true. By [12] a domain R is called a valuation domain if, given two nonzero elements $a, b \in R$, either $(a) \subseteq (b)$ or $(b) \subseteq (a)$ and a domain R is called a Dedekind domain if and only if every nonzero proper ideal of R is a product of finitely many prime ideals.

The purpose of this paper is to investigate 2-prime ideals in commutative rings. Among other results, we check some relations between 2-prime ideal and other classical ideals such as prime ideal, semiprimary ideal, and primary ideal. Also, we characterize 2-prime ideals in a Dedekind domain (see Theorem 3.7). In Proposition 3.11, we investigate the properties of a ring in which every proper ideal is 2-prime. Also, we define 2-N(R) to be the intersection of all 2-prime ideals of R and investigate 2-N(R) in different rings. In section 3, we define almost 2-prime and almost 2-primary ideals. Let I be a proper ideal of a ring R. We say that I is almost 2-prime if for all $x, y \in R$ such that $xy \in I - I^2$, then either x^2 or y^2 lies in I, and we say that I is almost 2-primary if for all $x, y \in R$ such that $xy \in I - I^2$, then $x^2 \in I$ or $y^n \in I$ for some $n \in \mathbb{N}$. Also, we define weakly 2-primary ideal. Let I be a proper ideal of a ring R. We say that I is weakly 2-primary if for all $x, y \in R$ such that $0 \neq xy \in I$, then $x^2 \in I$ or $y^n \in I$, for some $n \in \mathbb{N}$. In section 3, we give some basic properties of these ideals.

2. Preliminaries

Definition 2.1 ([5]). Let I be a proper ideal of a ring R. We say that I is 2-prime if for all $x, y \in R$ such that $xy \in I$, then either x^2 or y^2 lies in I.

Example 2.2. *Let $R = \mathbb{Z}_{12}$ and let $I = (\bar{4}) = \{\bar{0}, \bar{4}, \bar{8}\}$. For every $a, b \in R$ such that $ab \in I$, we have $a^2 \in I$ or $b^2 \in R$. So I is a 2-prime ideal of R.*

Definition 2.3 ([13]). R is a Boolean ring if $a^2 = a$ for all $a \in R$.

Definition 2.4 ([8]). A ring R is called a von Neumann regular ring if for every $a \in R$, there exists $x \in R$ such that $a = a^2x$.

Definition 2.5 ([9]). If R denotes a commutative ring with unit in which the elements 0 and 1 are distinct and F denotes the total quotient ring of R, then for an ideal A of R, let A^{-1} denote the set $\{x \in F \mid xA \subset R\}$. An ideal A is called invertible if $AA^{-1} = R$.

Definition 2.6. An integral domain R is a Prüfer domain if each nonzero finitely generated ideal of R is invertible.

Proposition 2.7. *If R is an integral domain, then the following statements are equivalent:*

(1) *R is a Prüfer domain.*
(2) *For every prime ideal P of R the ring of quotients R_P is a valuation domain.*
(3) *For every maximal ideal P of R, the ring of quotients R_P is a valuation domain.*

Proof. See [8, Theorem 6.6 and Corollary 6.7]. □

Proposition 2.8. *Let I be an ideal of a ring R, then the following properties hold:*

(1) *If I is 2-prime ideal, then it's radical $\sqrt{I}$ is a prime ideal.*
(2) *Let S be a multiplicatively closed subset of R. If I is a 2-prime ideal of R, then the ideal IR_S is also a 2-prime ideal of R_S.*
(3) *If I is a p-primary ideal, then I is 2-prime if and only if IR_p is 2-prime.*

Proof. See [5, Proposition 1.3]. □

Proposition 2.9. *Let R be an integral domain. The following statements are equivalent:*

(1) *R is a valuation domain.*
(2) *Every principal ideal of R is 2-prime.*
(3) *Every ideal of R is 2-prime.*

Proof. See [5, Theorem 2.4]. □

3. 2-prime ideals

In this section, we investigate some properties of 2-prime ideals in different rings.

Proposition 3.1. *Let I be an ideal of a ring R.*

(1) *If I is a prime ideal of R and J is an ideal of R containing I, then IJ is a 2-prime ideal of R.*
(2) *If I is a weakly prime and 2-prime ideal of R, then I^2 is a 2-prime ideal.*
(3) *If I is an almost prime and 2-prime ideal, then I^2 is a 2-prime ideal.*
(4) *If 0 is a 2-prime ideal of R, then $N(R)$ is a prime ideal.*

Proof.
1) Let $xy \in IJ \subseteq I$. Then $x \in I$ or $y \in I$. Since $I \subseteq J$, so $x \in J$ or $y \in J$. Therefore $x^2 \in IJ$ or $y^2 \in IJ$. As a result, IJ is a 2-prime ideal of R.
2) Let $xy \in I^2 \subseteq I$ for some $x, y \in R$. Since I is 2-prime, $x^2 \in I$ or $y^2 \in I$. Without loss of generality, let $x^2 \in I$. If $0 \neq x^2$, then since I is weakly prime, $x \in I$. Therefore $x^2 \in I^2$. If $0 = x^2$, then it is clear that $x^2 \in I^2$.
3) Let $xy \in I^2 \subseteq I$ for some $x, y \in R$. Since I is 2-prime, $x^2 \in I$ or $y^2 \in I$.

Without loss of generality, let $x^2 \in I$. If $x^2 \in I^2$, we are done. If $x^2 \in I \setminus I^2$, since I is almost prime, then $x \in I$. Thus $x^2 \in I^2$.
4) Let $a, b \in R$ with $ab \in$ N(R). Then there exists a positive integer n such that $a^n b^n = 0$. Since 0 is 2-prime hence $(a^n)^2 = 0$ or $(b^n)^2 = 0$. Therefore $a \in$ N(R) or $b \in$N(R).

□

Example 3.2. *Let $R = K[x, y]$ be a polynomial ring in two variables x and y over a field K and let $I = (x^2, xy) = (x)(x, y)$. Then Proposition 3.1(1) shows that I is a 2-prime ideal of R.*
This example shows that in a UFD, 2-prime ideals aren't necessarily principal.

Proposition 3.3. *Let R be a unique factorization domain, and let p be an irreducible element of R. Then for every positive integer α, the ideal (p^α) is 2-prime.*

Proof. Let $xy \in P = (p^\alpha)$, where $x = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ and $y = p_1^{\gamma_1} \cdots p_k^{\gamma_k}$, where p_i's are distinct irreducible elements of R. Then $xy = p_1^{\alpha_1+\gamma_1} \cdots$ $p_k^{\alpha_k+\gamma_k}$. Since $p^\alpha | xy$, there exists $i \in \{1, ..., k\}$ such that $p^\alpha | p_i^{\alpha_i+\gamma_i}$. Therefore $p_i = p$ and $\alpha \leq \alpha_i + \gamma_i$. Moreover, $\alpha_i \geq \frac{\alpha}{2}$ or $\gamma_i \geq \frac{\alpha}{2}$. If $\alpha_i \geq \frac{\alpha}{2}$, then $x^2 = p_1^{2\alpha_1} \cdots p_k^{2\alpha_k} \in (p^\alpha)$, and if $\gamma_i \geq \frac{\alpha}{2}$, then $y^2 = p_1^{2\gamma_1} \cdots p_k^{2\gamma_k} \in (p^\alpha)$. Finally, (p^α) is 2-prime. □

Example 3.2 shows that every 2-prime ideal of a unique factorization domain is not, in general, a power of a prime ideal.

Theorem 3.4. *Let R be a von Neumann regular ring. If 0 is a 2-prime ideal of R, then R is a field.*

Proof. Let $0 \neq a \in R$. Since R is a von Neumann regular ring, there exists an element $x \in R$ such that $a = a^2 x$. So $a(1 - ax) = 0$. Since 0 is a 2-prime ideal of R, hence $a^2 = 0$ or $(1 - ax)^2 = 0$. If $a^2 = 0$, then $a^2 x = 0$. Therefore $a = 0$, a contradiction. Hence $(1 - ax)^2 = 0$, which implies $1 + a^2x^2 - 2ax = 0$. So $1 = 2ax - a^2x^2 = a(2x - ax^2)$. This implies that a is a unit element of R. Hence R is a field. □

Proposition 3.5. *If R is a Noetherian integral domain, then the following statements are equivalent:*
1. R is a Dedekind domain.
2. R is integrally closed and every nonzero prime ideal of R is maximal.
3. For every maximal ideal P of R, the ring of quotients R_P is a valuation domain.

Proof. See [8, Theorem 6.20]. □

Lemma 3.6. *Let I and J be some proper comaximal ideals of a ring R. Then IJ is not a 2-prime ideal of R.*

Proof. Let IJ be a 2-prime ideal of R. Since I and J are comaximal ideals of R, there exist $x \in I$, $y \in J$ such that $x + y = 1$. We have $xy \in IJ$, so $x^2 \in IJ$ or $y^2 \in IJ$. If $x^2 \in IJ$, since $x + y = 1$ then $x^2 + xy = x \in IJ \subseteq J$. Hence $1 \in J$, a contradiction. Similarly if $y^2 \in IJ$, then $I = R$, a contradiction. So IJ is not 2-prime. □

Theorem 3.7. *Let R be a Dedekind domain and let I be a nonzero ideal of R. Then the following statements are equivalent:*

(1) *$I = P^\alpha$, for some maximal ideal P of R and positive integer α.*
(2) *I is a semiprimary ideal of R.*
(3) *I is a 2-prime ideal of R.*
(4) *I is a primary ideal of R.*

Proof. $(1 \Rightarrow 2)$ It is obvious.

$(3 \Rightarrow 1)$ Let I be a 2-prime ideal. Since R is a Dedekind domain, there exist some distinct prime ideals $P_1, \dots, P_n$ of R such that $I = P_1^{i_1} P_2^{i_2} \cdots P_n^{i_n}$, for some positive integers i_j, $1 \leq j \leq n$. Since $I \neq 0, P_j \neq 0$, for every $j = 1, \dots, n$, and since R is a Dedekind domain, P_j is a maximal ideal for every $j = 1, \dots, n$. Let $n \geq 2$. Since $P_1^{i_1} + P_2^{i_2} \cdots P_n^{i_n} = R$, there exist $x \in P_1^{i_1}$ and $y \in P_2^{i_2} \cdots P_n^{i_n}$ with $x + y = 1$. Since $xy \in I$ and I is a 2-prime ideal, $x^2 \in I$ or $y^2 \in I$. If $x^2 \in I$, since $x + y = 1$ then $x^2 + xy = x \in I \subseteq P_2^{i_2} \cdots P_n^{i_n}$. Hence $1 \in P_2^{i_2} \cdots P_n^{i_n}$. Therefore $P_j = R$, for $j = 2, \dots, n$, a contradiction. Thus $n = 1$ and $I = P_1^{i_1}$.

$(2 \Rightarrow 4)$ Let I be a semiprimary ideal of R. Then $\sqrt{I} = P$ and P is prime. If $P = 0$, then $I = 0$. So I is primary. If $P \neq 0$, then P is a maximal ideal, and so I is primary by [11, Proposition 4.9].

$(4 \Rightarrow 3)$ Let Q be a P-primary ideal of R. By Proposition 3.5, R_P is a valuation domain. Then by Proposition 2.9, QR_P is a 2-prime ideal of R. Consequently, Q is a 2-prime ideal by Proposition 2.8.

□

Example 3.8. *Let $n = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$ be the factorization of positive integer n into powers of distinct primes. Then 2-prime ideals of $R = \mathbb{Z}_n$ are in the form $(\overline{p_i^\beta})$ for all $\beta = 1, ..., \alpha_i$ and $i = 1, ..., t$.*

Because by [5, Proposition 1.3(8)] every 2-prime ideal of $\mathbb{Z}_n \cong \dfrac{\mathbb{Z}}{n\mathbb{Z}}$ is in the form of $\dfrac{I}{n\mathbb{Z}}$, where I is a 2-prime ideal of $\mathbb{Z}$ containing $n\mathbb{Z}$. Since $\mathbb{Z}$ is a Dedekind domain and $n\mathbb{Z} \subseteq I$, by Proposition 3.7, $I = (p_i^\beta)$ for some $\beta = 1, ..., \alpha_i$ and $i = 1, ..., t$. So every 2-prime ideal of $\mathbb{Z}_n$ is in the form $(\overline{p_i^\beta})$ for some $\beta = 1, ..., \alpha_i$ and $i = 1, ..., t$.

Proposition 3.9. *Every ideal of R is semiprimary if and only if $Spec(R)$ is totally ordered by inclusion.*

Proof. $\Leftarrow$) Let $Spec(R)$ be totally ordered by inclusion and let I be an ideal of R. Then $\sqrt{I} \in Spec(R)$. Therefore I is semiprimary.
$\Rightarrow$) let every ideal of R is semiprimary and let $P, Q \in Spec(R)$. Then $P \cap Q$ is a semiprimary ideal. Let $P \nsubseteq Q$. Then there exists an element $x \in P \setminus Q$. Assume that $q \in Q$. Hence $xq \in P \cap Q$. Since $P \cap Q$ is semiprimary, $x \in \sqrt{P \cap Q}$ or $q \in \sqrt{P \cap Q}$. If $x \in \sqrt{P \cap Q}$, then $x \in Q$, a contradiction. Thus $q \in \sqrt{P \cap Q} \subseteq P$. Therefore $Q \subseteq P$. □

Proposition 3.10. *Let R be a Prüfer domain. Then every primary ideal is 2-prime.*

Proof. Let Q be a P-primary ideal. Proposition 2.7 implies that R_P is a valuation domain. Now, it follows from Proposition 2.9 that QR_P is a 2-prime ideal. So Proposition 2.8 implies that Q is 2-prime. □

Proposition 3.11. *Let R be a ring such that every ideal of R is 2-prime. Then the following properties hold:*

(1) *$Spec(R)$ is totally ordered by inclusion.*
(2) *$\frac{R}{N(R)}$ is a valuation domain.*

Proof. (1) By Proposition 2.8(1), every 2-prime ideal is semiprimary. So by Proposition 3.9, we are done.
(2) N(R) is a prime ideal and $\frac{R}{N(R)}$ is an integral domain. On the other hand, every ideal of $\frac{R}{N(R)}$ is in the form of $\frac{I}{N(R)}$, where I is an ideal of R. By hypothesis I is a 2-prime ideal. So by [5, Proposition 1.3(8)], $\frac{I}{N(R)}$ is 2-prime. Now, by Proposition 2.8 and Proposition 2.9($1 \Leftrightarrow 3$), $\frac{R}{N(R)}$ is a valuation domain.

□

Definition 3.12. Let R be a ring. We define 2-N(R) to be the intersection of all 2-prime ideals of R.

By Proposition 3.1(1), for every $P \in Spec(R)$, P^2 is a 2-prime ideal. So 2-N(R) $\subseteq \bigcap\{P^2 \mid P$ is prime$\}$. In the following, we investigate some cases in which equality holds.

Proposition 3.13. *If R is a ring such that for every 2-prime ideal I of R, $(\sqrt{I})^2 \subseteq I$. Then 2-N(R)$=\bigcap\{P^2 \mid P$ is prime$\}$.*

Proof. It is clear that 2-N(R) $\subseteq \bigcap\{P^2 \mid P$ is prime$\}$. Conversely, let I be a 2-prime ideal of R. Then $\sqrt{I}$ is prime. So $\bigcap\{P^2 \mid P$ is prime$\} \subseteq (\sqrt{I})^2 \subseteq I$. Then $\bigcap\{P^2 \mid P$ is prime$\} \subseteq$2-N(R). □

Corollary 3.14. *Let R be a ring such that every 2-prime ideal of R is prime. Then $2 - N(R) = \bigcap\{P^2 \mid P$ is prime$\}$.*

Proof. Since every 2-prime ideal I is prime, $\sqrt{I} = I$. Therefore $(\sqrt{I})^2 = I^2 \subseteq I$. So by Proposition 3.13, 2-N(R)$=\bigcap\{P^2 \mid P$ is prime$\}$. □

Example 3.15. *In a Boolean ring, 2-N(R)=N(R), because every 2-prime ideal is prime.*

Example 3.16. *Let* $R = \mathbb{Z}_n$ *and let* $n = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$ *for some distinct prime integers* p_i *and* , $1 \leq i \leq t$. *By Example 3.8, every 2-prime ideal of R is in the form of* $(\overline{p_i^{\beta}})$ *for all* $\beta = 1, ..., \alpha_i$ *and* $i = 1, ..., t$. *Thus 2-N(R)=*$\bigcap\{P \mid$ *P is 2-prime*$\} = \bigcap_{i=1}^{t} (\overline{p_i^{\alpha_i}}) = \prod_{i=1}^{t} (\overline{p_i^{\alpha_i}}) = 0$.

4. Almost 2-prime and almost 2-primary ideals

In this section, we give the definitions of almost 2-prime, almost 2-primary, and weakly 2-primary ideals, and investigate their properties.

Definition 4.1. Let I be a proper ideal of a ring R. We say that I is almost 2-prime if for all $x, y \in R$ such that $xy \in I - I^2$, then either x^2 or y^2 lies in I.

Example 4.2. *It is clear that every idempotent ideal is almost 2-prime. Also, every 2-prime ideal and every almost prime ideal is almost 2-prime ideal.*

Example 4.3. *Let* $R = \mathbb{Z}_6$ *and let* $I = (\bar{0})$ *be an ideal of R. Since I is idempotent, it is an almost 2-prime ideal of R. But I is not 2-prime.*

Note that if I and J are prime ideals of R, then $I \cap J$ need not be almost 2-prime ideal of R; see the following example.

Example 4.4. *Let* $R = K[x, y]$ *be the polynomial ring in two variables x and y over a field K, and set* $I = (x)$ *and* $J = (y)$. *Then I and J are prime ideals of R and* $I \cap J = (xy)$. *It is clear that* $xy \in (I \cap J) - (I \cap J)^2$, *but* $x^2 \notin I \cap J$ *and* $y^2 \notin I \cap J$.

Definition 4.5. Let I be a proper ideal of a ring R. We say that I is almost 2-primary if for all $x, y \in R$ such that $xy \in I - I^2$, it holds that $x^2 \in I$ or $y^n \in I$ for some $n \in \mathbb{N}$.

Example 4.6. *It is obvious that every primary ideal, every almost prime ideal, every almost 2-prime ideal, and every idempotent ideal of R are almost 2-primary. Also, proper ideals of fully idempotent rings and of Boolean rings are almost 2-primary. Recall that R is said to be a fully idempotent ring if every ideal of R is idempotent [13].*

Proposition 4.7. *Let I be an ideal of R. I is almost 2-primary if and only if* $(I : x) \subseteq (I^2 : x) \cup \sqrt{I}$ *for all* $x \in R$ *such that* $x^2 \notin I$.

Proof. $\Rightarrow$) Let I be an almost 2-primary ideal of R and $y \in (I : x)$. Then $xy \in I$. If $xy \in I^2$, then $y \in (I^2 : x)$. If $xy \notin I^2$, then $xy \in I - I^2$, and so $x^2 \in I$ or $y^n \in I$, for some $n \in \mathbb{N}$. Since $x^2 \notin I$, hence $y^n \in I$ and $y \in \sqrt{I}$. Finally, $(I : x) \subseteq (I^2 : x) \bigcup \sqrt{I}$.
$\Leftarrow$) Let $xy \in I - I^2$. Then $y \in (I : x)$. Since $(I : x) \subseteq (I^2 : x) \cup \sqrt{I}$, hence $y \in (I^2 : x)$ or $y \in \sqrt{I}$. Thus $xy \in I^2$ or $y^n \in I$. Since $xy \notin I^2$ hence $y^n \in I$ for some $n \in \mathbb{N}$. Therefore I is almost 2-primary. □

Proposition 4.8. *Let I be an ideal of R such that $I = \sqrt{I}$. Then I is almost 2-primary if and only if I is almost prime.*

Proof. Suppose that I is almost 2-primary and that $a, b \in R$ with $ab \in I - I^2$. Assume that $a \notin I$. If $a^2 \in I$, then $a \in \sqrt{I} = I$. So $a^2 \notin I$ implies that $b^n \in I$ for some $n \in \mathbb{N}$, and hence $b \in \sqrt{I} = I$. Thus I is an almost prime ideal. The converse is trivial. □

Weakly primary ideals have been introduced and studied in [2]. In the following, we define the concept of weakly 2-primary ideal which is a mild generalization of the notion of weakly primary ideal.

Definition 4.9. Let I be a proper ideal of a ring R. We say that I is weakly 2-primary if for all $x, y \in R$ such that $0 \neq xy \in I$, then $x^2 \in I$ or $y^n \in I$, for some $n \in \mathbb{N}$.

Example 4.10. *Let $R = \dfrac{\mathbb{Z}_2[X,Y,Z,T]}{(X^2, Z^3, ZT, XYZ, XYT)}$ and let $x, y, z,$ and t be the cosets of the ideal (X^2, Z^3, ZT, XYZ, XYT) with representatives $X, Y, Z,$ and T, respectively. So we have $R = \mathbb{Z}_2[x, y, z, t]$ where $x^2 = z^3 = zt = xyz = xyt = 0$. Let $I = (xy)$ be an ideal of R. As $zt = 0 \in I$ but $z^2 \notin I$ and $t \notin \sqrt{I}$, I is not a 2-primary ideal, and since $0 \neq xy \in I$ but $x \notin I$ and $y \notin \sqrt{I}$, I is not a weakly primary ideal of R. Now we show that I is a weakly 2-primary ideal of R. Suppose that $f, g \in R$ are such that $0 \neq fg \in I$. By the relations $x^2 = z^3 = 0$ we have*

$$f = a_0 + a_1x + a_2z + a_3xz + a_4z^2 + a_5xz^2$$
and
$$g = b_0 + b_1x + b_2z + b_3xz + b_4z^2 + b_5xz^2$$
where $a_0, a_1, b_0, b_1 \in \mathbb{Z}_2[y, t]$, and $a_i, b_i \in \mathbb{Z}_2[y]$ for $i = 2, 3, 4, 5$. Then

$$fg = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_2b_0)z + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)xz + (a_0b_4 + a_2b_2 + a_4b_0)z^2 + (a_0b_5 + a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1 + a_5b_0)xz^2.$$
Now, $fg \in I$ implies that
$a_0b_0 = 0$ *(1)*
$a_0b_1 + a_1b_0 = yc_0$ *(2)*
$a_0b_2 + a_2b_0 = tc_1$ *(3)*
for some $c_0, c_1 \in \mathbb{Z}_2[y, t]$. Note that $fg = a_0b_0 + (a_0b_1 + a_1b_0)x$ because $zt = xyz = xyt = 0$. Since $fg \neq 0$, (1) implies that just one of the a_0 or b_0 is 0. Let $a_0 = 0$ and $b_0 \neq 0$. We show that $a_2 = 0$. If $a_2 \neq 0$, as $a_2 \in \mathbb{Z}_2[y]$ then by (3), $b_0 = tc_2$ for some $c_2 \in \mathbb{Z}_2[y, t]$. So by (2), $a_1tc_2 = yc_0 \in \mathbb{Z}_2[y, t]$. Since $\mathbb{Z}_2[y, t]$ is a UFD, $c_0 = tc_3$ for some $c_3 \in \mathbb{Z}_2[y, t]$. Thus $fg = a_1b_0x = yc_0x = ytc_3x = 0$, a contradiction. It follows that $a_2 = 0$ and then $f^2 = a_0^2 + a_2^2z^2 = 0 \in I$. By symmetry, if $a_0 \neq 0$ and $b_0 = 0$, then $g^2 = 0 \in I$. Therefore, I is a weakly 2-primary ideal of R.

Proposition 4.11. *Let I and P be ideals of R with $I \subseteq P$.*

(1) *If P is an almost 2-primary ideal of R, then $\frac{P}{I}$ is an almost 2-primary ideal of $\frac{R}{I}$.*
(2) *If I is an almost 2-primary ideal of R and $\frac{P}{I}$ is a weakly 2-primary ideal of $\frac{R}{I}$, then P is an almost 2-primary ideal of R.*

Proof. (1) Let $(a+I)(b+I) \in \dfrac{P}{I} - (\dfrac{P}{I})^2$ and let $(a^2+I) \notin \dfrac{P}{I}$. Then $ab \in P \setminus P^2$, so $a^2 \in P$ or $b^n \in P$. Since $(a^2+I) \notin \dfrac{P}{I}$, hence $a^2 \notin P$. Then $b^n + I = (b+I)^n \in \dfrac{P}{I}$ and $\dfrac{P}{I}$ is almost 2-primary

(2) Let $a, b \in R$ be such that $ab \in P - P^2$. We have the following two cases:
Case (1) If $ab \in I$, then we get either $a^2 \in I$ or $b^n \in I$ for some $n \in \mathbb{N}$. Since $I \subseteq P$, we have either $a^2 \in P$ or $b^n \in P$.
Case (2) If $ab \notin I$, then $0 \neq (a+I)(b+I) \in \frac{P}{I}$. Since $\frac{P}{I}$ is a weakly 2-primary ideal of $\frac{R}{I}$, we get either $(a^2+I) \in \frac{P}{I}$ or $(b^n+I) \in \frac{P}{I}$, for some $n \in \mathbb{N}$, which gives $a^2 \in P$ or $b^n \in P$. Hence P is almost 2-primary.

□

Proposition 4.12. *A proper ideal I of R is almost 2-primary if and only if $\frac{I}{I^2}$ is a weakly 2-primary ideal of $\frac{R}{I^2}$.*

Proof. (⇒) Let I be almost 2-primary and let $I^2 \neq (a+I^2)(b+I^2) \in \frac{I}{I^2}$, where $a, b \in R$. Then $ab \in I$ and $ab \notin I^2$. Since I is almost 2-primary, so either $a^2 \in I$ or $b^n \in I$ for some $n \in \mathbb{N}$. If $a^2 \in I$, then $a^2 + I^2 \in \frac{I}{I^2}$, and if $b^n \in I$, then $(b^n + I^2) = (b+I^2)^n \in \frac{I}{I^2}$.
(⇐) Let $\frac{I}{I^2}$ be a weakly 2-primary ideal of $\frac{R}{I^2}$ and let $ab \in I - I^2$, where $a, b \in R$. Then $ab + I^2 \in \frac{I}{I^2}$ and $ab + I^2 \neq I^2$. From this, we get $I^2 \neq (a+I^2)(b+I^2) \in \frac{I}{I^2}$, so either $(a^2+I^2) \in \frac{I}{I^2}$ or $(b^n+I^2) \in \frac{I}{I^2}$, for some $n \in \mathbb{N}$, which gives either $a^2 \in I$ or $b^n \in I$. □

We conclude our discussion with the following, which are slight modifications of some results in [2].

Proposition 4.13. *Let R be a ring, and let P be a weakly 2-primary ideal of R that is not semiprimary. Then $P^2 = 0$. In particular, $\sqrt{P} = \sqrt{0}$.*

Proof. See [2, Theorem 2.2] □

Proposition 4.14. *Let R be a ring, and let $\{P_i\}_{i \in I}$ be a family of weakly 2-primary ideal of R that are not semiprimary. Then $P = \bigcap_{i \in I} P_i$ is a weakly 2-primary ideal of R.*

Proof. See [2, Theorem 2.3] □

Proposition 4.15. *Let $I \subseteq P$ be proper ideals of a ring R. Then the following assertions hold:*

(1) *If P is weakly 2-primary, then $\frac{P}{I}$ is weakly 2-primary.*
(2) *If I and $\frac{P}{I}$ are weakly 2-primary, then P is weakly 2-primary.*

Proof. See [2, Proposition 2.10] □

Proposition 4.16. *Let P and Q be weakly 2-primary ideals of a ring R that are not semiprimary. Then $P+Q$ is a weakly 2-primary ideal of R. In particular, $\sqrt{P+Q} = \sqrt{P}$.*

Proof. See [2, Theorem 2.11].

□

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