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IGMRES METHOD FOR LINEAR SYSTEMS

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Abstract. The Index Generalized Minimal RESidual (IGMRES) algorithm is designed to compute the Drazin-inverse solution of a linear system of equations $Ax = b$, where A is an arbitrary square matrix with index γ. If $\gamma = 0$, then the this method method coincide with Generalized Minimal RESidual (GMRES) method. Also, the k^{th} ideal index generalized minimal residual polynomial of A is introduced and the roots of these polynomials are studied. Moreover, by numerical results the convergence rate of these methods are compared by two examples.

Keywords: Singular systems, Drazin-inverse, GMRES, DGMRES.
2020 MSC: Primary 65F10, 65F20, 15A06.

1. Introduction

Let A be an $n \times n$ complex matrix. The smallest nonnegative integer γ such that $\text{rank}A^{\gamma+1} = \text{rank}A^{\gamma}$ is the index of A, and denoted by $\gamma = \text{ind}(A)$. The Drazin-inverse of A is the unique matrix called, A^D, satisfying the following relations

$$A^{\gamma+1}A^D = A^{\gamma},\ A^D A A^D = A^D,\ AA^D = A^D A.$$

If A is nonsingular, then $A^D = A^{-1}$. Let $b \in \mathbb{C}^n$. We consider the following linear system of equations

$$Ax = b. \tag{1}$$

This system is called singular if $\gamma > 0$. In [15] the author introduced the DGMRES method for singular system of equations (1). The Drazin-invese has many applications in the singular differential equations [2], theory of Markov chains [2], cryptographic system [10], dynamical systems [16] and iterative methods in numerical analysis [3, 4]. The DGMRES method is an iterative method to approximate the Drazin-inverse solution of the singular systems (1) by choosing an arbitrary initial guess x_0. For more details see [6, 9, 14, 15]. The k^{th} approximation solution x_k of the Drazin-inverse solution is derived by the Krylov subspace method. In the process of the DGMRES algorithm, the vectors $x_k = x_0 + \sum_{i=\gamma}^{k-1} e_i A^i r_0$ are generated, where the integer $k \geq \gamma + 1$, and

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$r_0 = b - Ax_0$. The k^{th} Krylov subspace is introduced in the following. For more details see [12].

$$\mathcal{K}_k(A,\ A^\gamma r_0) := \mathrm{Span}\{A^\gamma r_0,\ A^{\gamma+1} r_0, \ldots,\ A^{k-1} r_0\},\ k > \gamma. \tag{2}$$

Let $r_k = b - Ax_k$ be the the k^{th} residual. Then $r_k = r_0 - \sum_{i=\gamma+1}^{k} e_i A^i r_0$, and $A^\gamma r_k$ satisfies

$$A^\gamma r_k = (I - \sum_{i=\gamma+1}^{k} e_i A^i) A^\gamma r_0.$$

The coefficients $e_{\gamma+1}, e_{\gamma+2}, \ldots, e_k$ are choosen to minimize the norm of $A^\gamma r_k$. That is,

$$\|A^\gamma r_k\| = \min_{e_{\gamma+1}, e_{\gamma+2}, \ldots, e_k} \|(I - \sum_{i=\gamma+1}^{k} e_i A^i) A^\gamma r_0\|.$$

If $\gamma = 0$ the DGMRES method coincide to the GMRES method [13].

We know that the Drazin-inverse could be written as a polynomial of A, so the exact Drazin-inverse solution of $Ax = b$ is derived in at most $n - \gamma$ iterations [15].

By using the Jordan decomposition of an n by n matrix A with index $\gamma > 0$, we obtain the following decomposition.

$$A = P \begin{bmatrix} \mathbf{E} & 0 \\ 0 & \mathbf{N} \end{bmatrix} P^{-1}, \tag{3}$$

where $P \in M_n$ is a nonsingular matrix, $\mathbf{E} \in M_m$ is a nonsingular matrix, $0 \leq m \leq n$, and $\mathbf{N} \in M_{n-m}$ is a nilpotent of index γ [7, p. 185]. Then

$$A^D = P \begin{bmatrix} \mathbf{E}^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}. \tag{4}$$

We know that A^D can be written as a polynomial in A [7, p. 186]. Then

$$A^{\gamma+1} = P \begin{bmatrix} \mathbf{E}^{\gamma+1} & 0 \\ 0 & \mathbf{N}^{\gamma+1} \end{bmatrix} P^{-1} = P \begin{bmatrix} \mathbf{E}^{\gamma+1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

The matrix $\mathbf{E}^{\gamma+1}$ is invertible and hence by using characteristic polynomial of $\mathbf{E}^{\gamma+1}$, there exists a polynomial $q(x)$ with degree $r \leq m - 1$ such that $(\mathbf{E}^{\gamma+1})^{-1} = q(\mathbf{E}^{\gamma+1})$. Then

$$q(A^{\gamma+1}) = P \begin{bmatrix} \mathbf{E}^{-(\gamma+1)} & 0 \\ 0 & * \end{bmatrix} P^{-1}, \text{ and } A^\gamma = P \begin{bmatrix} \mathbf{E}^\gamma & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

Then for any $n \times n$ matrix A with index γ, there exists a polynomial $q(x)$ with degree $r \leq m - 1$

$$q(A^{\gamma+1}) A^\gamma = P \begin{bmatrix} \mathbf{E}^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1} = A^D. \tag{5}$$

Note that m is the size of the nonsingular part of A and A^D could be written as a multiplication of a polynomial of degree at most $m - 1$ of $A^{\gamma+1}$ by A^γ.

In Section 2, we introduce Index-GMRES (IGMRES) for solving singular linear systems and in Section 3, we study the k^{th} ideal IGMRES polynomial of A and In Section 4, we compare the residual errors in the DGMRES and IDGMRES methods by random matrices.

2. IGMRES

Let A be an arbitrary n by n matrix and $b \in \mathbb{C}^n$. In this section we introduce Index GMRES (IGMRES)method based on GMRES and DGMRES methods. The notion of index numerical range is also investigated in [14]. Note that the difference between DGMRES and IGMRES are the Krylov subspaces corresponding to them.

Proposition 2.1. *[8, Theorem 2] Let A be an arbitrary n by n matrix and $b \in \mathbb{C}^n$. A linear system of equations $Ax = b$ has a Krylov solution if and only if $b \in \mathcal{R}(A^\gamma)$, the range of A^γ.*

Note that we can write the above Krylov solution as the Drazin solution $x = A^D b$. In the following remark, we show that although the linear system $Ax = b$ may have no solution, but the linear system

$$A^{\gamma+1}x = A^\gamma b. \tag{6}$$

has a solution, where γ is the index of A.

Remark 2.2. Let A be an arbitrary n by n matrix with index γ and $b \in \mathbb{C}^n$. Since $A^{\gamma+1}A^D = A^\gamma$, the equation $A^{\gamma+1}x = A^\gamma b$ has a solution $x = A^D(A^\gamma b)$.

Therefore, to compute the Drazin solution of an arbitrary linear systems of equations $Ax = b$, it is enough to compute the solution of the (consistent) linear system $A^{\gamma+1}x = A^\gamma b$. Let x_0 be an arbitrary initial guess. Then $\mathbf{r_0} = A^\gamma b - (A^{\gamma+1}x_0) = A^\gamma(b - Ax_0) = A^\gamma(r_0)$. By (5), we know that the Drazin-inverse solution $A^D b = q(A^{\gamma+1})A^\gamma b$, for some polynomial $q(\cdot)$ of degree at most $m-1$. Then we consider the following Krylov subspace:

$$\mathcal{K}_k(A^{\gamma+1}, \mathbf{r_0}) = \operatorname{span}\{\mathbf{r_0}, A^{\gamma+1}\mathbf{r_0}, A^{2(\gamma+1)}\mathbf{r_0}, \ldots, A^{(k-1)(\gamma+1)}\mathbf{r_0}\}.$$

Note that when $\gamma = 0$, the above Krylov subspace is the same as the Krylov subspace in GMRES method. The IGMRES process works as follows: Let x_0 be an initial guess, the initial residual $r_0 := b - Ax_0$, If $A^\gamma r_0 = 0$, then x_0 is a solution of $A^{\gamma+1}x = A^\gamma b$. Now, assume $\beta = \|A^\gamma r_0\| \neq 0$. Define $v_1 := (A^\gamma r_0)/\beta$. Note that $\|\cdot\|$ denotes the Euclidean norm for vectors and the induced operator norm for matrices. The approximate solutions $x_k = x_0 + \eta$, where $\eta \in \mathcal{K}_k(A^{\gamma+1}, v_1)$ is chosen to minimize $\|b - Ax_k\|$. Arnoldi algorithm constructs an orthonormal basis $\{v_1, \ldots, v_k\}$ for the Krylov space $\mathcal{K}_k(A^{\gamma+1}, v_1)$. Define $V_k := [v_1|\cdots|v_k]$. Then

$$A^{\gamma+1}V_k = V_{k+1}H_{k+1,k}, \tag{7}$$

where $H_{k+1,k} \in M_{k+1,k}(\mathbb{C})$ is an upper Hessenburg matrix. Therefore, the approximate solution x_k is of the form $x_k = x_0 + V_k\xi$, where $\xi \in \mathbb{C}^k$. Then the residual

$$r_k := b - Ax_k = b - A(x_0 + V_k\xi) = r_0 - AV_k\xi.$$

Note that the linear system $Ax = b$ may be inconsistent, so r_k may not goes to zero. But, we know that $A^{\gamma+1}x = A^\gamma b$ has $x = A^D b$ as a solution. Therefore, we should consider $A^\gamma r_k$ instead of r_k. By using (7), $A^\gamma r_k$ can be written as

$$A^\gamma r_k = A^\gamma r_0 - A^{\gamma+1}V_k\xi = A^\gamma r_0 - V_{k+1}H_{k+1,k}\xi.$$

Since $A^\gamma r_0 = V_{k+1}(\beta\mathbf{e_1})$, where $\mathbf{e_1} = (1, 0, \ldots, 0)^T \in \mathbb{R}^k$, we can write

$$A^\gamma r_k = V_{k+1}(\beta\mathbf{e_1} - H_{k+1,k}\xi). \tag{8}$$

By solving the least squares problem $H_{k+1,k}\xi = \beta\mathbf{e_1}$, the vector ξ can be chosen to minimize $\|A^\gamma r_k\|$. Now we state the IGMRES algorithm

Algorithm 1 IGMRES algorithm

Compute comput $r_0 = b - Ax_0$; $v_1 = A^\gamma r_0; \beta := \|v_1\|$
$v_1 := v_1/\beta$
for $j = 1$ to k **do**
  $w_j = A^{\gamma+1}v_j$
  for i = 1 to j **do**
    $h_{ij} = (w_j, v_i)$
    $w_j = w_j - h_{ij}v_i$
  end for
  $h_{j+1,j} = \|w_j\|$
  if $h_{j+1,j} = 0$ **then**
    $k = j$ break
  else
    set $v_{j+1} = w_j/h_{j+1,j}$
  end if
end for
Form the matrix $\hat{H}_k \in \mathbb{R}^{(k+1)\times(k-\gamma)}$
Compute the QR factorization of $\hat{H}_k$: $\hat{H}_k = Q_kR_k; Q_k \in \mathbb{R}^{(k+1)\times(k-\gamma)}$ and $R_k \in \mathbb{R}^{(k-\gamma)\times(k-\gamma)}$ (R_k is upper triangular).
Solve the (upper triangular) system $R_kz_k = \beta(Q_k^*e_1)$,
where $e_1 = [1, 0, \ldots, 0]^t$.
Compute $\|A^\gamma r_k\| = \|\beta e_1 - Q_kR_kz_k\|$ and $x_k = x_0 + V_kz_k$.

This GMRES-like implementation is stable numerically, because the errors are tapered off and do not increase at least for several iterations. In the following, we show that the errors in the IGMRES method are decreasing and the m^{th} error is zero. Then the IGMRES method terminates at most after m iteration, where m is the order of nonsingular part of A as in (3).

Theorem 2.3. *Let A be as in (3) with index γ and let m be the order of E as in (3) and $b \in \mathbb{C}^n$. Then the residual error in the IGMRES method, $A^\gamma r_m = 0$. Therefore, the IGMRES method terminates at most after m iterations.*

Proof. In IGMRES method, we are looking to find the Drazin-inverse solution $A^D b$ of the linear system $Ax = b$ or the exact solution of $A^{\gamma+1}x = A^\gamma b$. By (5), there exists a polynomial $q(x)$ of degree at most $m-1$ such that $A^D = q(A^{\gamma+1})A^\gamma$.

$$\|A^\gamma r_m\| = \|A^\gamma(b - Ax_m)\| = \min_{e_1,\ldots,e_{m-1}}\|A^\gamma(b - A(x_0 + \sum_{i=0}^{m-1} e_i A^{i(\gamma+1)}A^\gamma r_0)\|$$

$$= \min_{e_1,\ldots,e_{m-1}}\|A^\gamma r_0 - A^{\gamma+1}\left(\sum_{i=0}^{m-1} e_i A^{i(\gamma+1)}\right)A^\gamma r_0\|$$

$$\leq \|A^\gamma r_0 - A^{\gamma+1}q(A^{\gamma+1})A^\gamma r_0\| \leq \|(A^\gamma - A^{\gamma+1}A^D)\|\|r_0\| = 0. \quad \square$$

Remark 2.4. Note that the order of convergence for the constructed sequence x_k in the IGMRES method is $\mathcal{O}(k\gamma n^2)$, where n is the size of matrix A, γ is the index of A and k is the number of iterations. This order is the same as DGMRES method.

3. k^{th} ideal IGMRES polynomial

Let A and P be as in (3) and let $P = QR$ be the QR decomposition of P. Then by using this QR decomposition, we obtain

$$A = QR\begin{bmatrix} \mathbf{E} & 0 \\ 0 & \mathbf{N} \end{bmatrix} R^{-1}Q^* = Q\begin{bmatrix} E & * \\ 0 & N \end{bmatrix} Q^*, \tag{9}$$

where E and N are $m \times m$ nonsingular and $n-m \times n-m$ nilpotent matrices, respectively.

Let A be as in (9) with index γ. The IGMRES algorithm described in Section 2 generates vectors x_k, $k = 1, 2, \ldots, m$, of the form $x_k = x_0 + \sum_{i=1}^{k} e_i A^{i(\gamma+1)} r_0$. The k^{th} residual $r_k := b - Ax_k = r_0 - \sum_{i=1}^{k} e_i A^{i(\gamma+1)} r_0$. Then

$$A^\gamma r_k = \left(I_n - \sum_{i=1}^{k} e_i A^{i(\gamma+1)}\right) A^\gamma r_0.$$

.

The coefficients $e_1, \ldots, e_k$ are chosen to minimize $\|A^\gamma r_k\|$.

$$\|A^\gamma r_k\| = \min_{e_1,\ldots,e_k} \|(I_n - \sum_{i=1}^{k} e_i A^{i(\gamma+1)})A^\gamma r_0\|. \tag{10}$$

From (9) and (10),

$$\|A^\gamma r_k\| = \min_{e_1,\ldots,e_k}\left\|Q\begin{bmatrix} I_m - \sum_{i=1}^{k} e_i E^{i(\gamma+1)} & * \\ 0 & I_{n-m} \end{bmatrix} Q^* A^\gamma r_0\right\|. \tag{11}$$

Note that the first m columns of Q form an orthonormal basis for the range of A^γ and the remaining $n-m$ columns of Q form an orthonormal basis for the orthogonal complement of the range of A^γ. Then the last $n-m$ entries of $Q^*A^\gamma r_0$ are zeros and the first m entries of the vector $Q^*A^\gamma r_0$ is denoted by $\hat{r}_0 \in \mathbb{R}^m$, i.e. $Q^*A^\gamma r_0 = (\hat{r}_0, 0_{n-m})^T$ (see [6]).

$$\begin{aligned}\|A^\gamma r_k\| &= \min_{e_1,\dots,e_k} \|(I_m - \sum_{i=1}^{k} e_i E^{i(\gamma+1)})\hat{r}_0\| \\ &\le \min_{e_1,\dots,e_k} \|(I_m - \sum_{i=1}^{k} e_i E^{i(\gamma+1)})\|\|\hat{r}_0\|.\end{aligned}$$

Since $\|\cdot\|$ is unitary invariant norm, $\|\hat{r}_0\| = \|Q^*A^\gamma r_0\| = \|A^\gamma r_0\|$. Therefore,

$$\|A^\gamma r_k\|/\|A^\gamma r_0\| \le \min_{e_1,\dots,e_k} \|I_m - \sum_{i=1}^{k} e_i E^{i(\gamma+1)}\|. \tag{12}$$

Since $E^{\gamma+1}$ is nonsingular, we obtain that the minimum in (12) is obtained.

The bound in (12) is called the k^{th} ideal residual norm bound for IGMRES. Note that this bound is the same as the k^{th} ideal residual norm bounds for GMRES, when the algorithm is applied to a problem with the nonsingular coefficient matrix $E^{(\gamma+1)}$ and initial residual $\hat{r}_0$. If $\min_{e_1,\dots,e_k} \|I_m - \sum_{i=1}^{k} e_i E^{i(\gamma+1)}\| > 0$, then the coefficients $e_1, \dots, e_k$ are uniquely determined [11]. Therefore, there exist $\tilde{e}_1, \dots, \tilde{e}_k$ such that

$$\min_{e_1,\dots,e_k} \|I - \sum_{i=1}^{k} e_i E^{i(\gamma+1)}\| = \|I - \sum_{i=1}^{k} \tilde{e}_i E^{i(\gamma+1)}\|.$$

Therefore, the following upper bound for the relative error in the k^{th} iteration of the IGMRES method is obtained.

$$\frac{\|A^\gamma r_k\|}{\|A^\gamma r_0\|} \le \|I - \sum_{i=1}^{k} \tilde{e}_i E^{i(\gamma+1)}\|. \tag{13}$$

Now, we define the following polynomial as the $k^{th}-$ ideal IGMRES polynomial of A.

$$p(x) := 1 + \tilde{e}_1 x^{\gamma+1} + \tilde{e}_2 x^{2(\gamma+1)} + \cdots + \tilde{e}_k x^{k(\gamma+1)}. \tag{14}$$

Note that the $k^{th}-$ ideal GMRES polynomial of $E^{(\gamma+1)}$ is as follows:

$$q(x) := 1 + \tilde{e}_1 x + \tilde{e}_2 x^2 + \cdots + \tilde{e}_k x^k. \tag{15}$$

3.1. **The roots of the k^{th} ideal IGMRES polynomial.** In this subsection, we compute the roots of k^{th} ideal IGMRES polynomial of A. First, we define the Ritz and harmonic Ritz values of square matrix A.

Definition 3.1. Let A be an $n \times n$ matrix and let $\mathcal{V}$ be a k-dimensional subspace of $\mathbb{C}^n$, and let the columns of $V \in \mathbb{C}^{n\times k}$ form an orthonormal basis for $\mathcal{V}$. The eigenvalues of V^*AV are called the *Ritz values* of A with respect to $\mathcal{V}$. These values are independent of the orthonormal basis V for $\mathcal{V}$. Also, if A is an invertible, then the harmonic Ritz values are the reciprocals of the (ordinary) Ritz values of A^{-1} computed from $\mathcal{V}$.

Now, for singular matrix A, we define the harmonic Ritz values as follws:

Definition 3.2. Let A be an $n \times n$ matrix with $\mathrm{ind}(A) = \gamma$ and let V be a linear subspace of range of A^γ. Then μ is a *harmonic Ritz value* of A with respect to V if μ^{-1} is a Ritz value of A^D with respect to V if $(A^D v - \mu^{-1}v) \perp V$, for some $0 \neq v \in V$.

Note that if A is a nonsingular matrix, then $\gamma = 0$ and $A^D = A^{-1}$. Then the above definition coincide with the definition of harmonic Ritz values for nonsingular matrices. For more details see [1, 5]

Let A be a nonsingular $n\times n$ matrix. Kim-Chuan Toh in his Ph.D. thesis [17, Theorem 5. 11] shows that the roots of the ideal GMRES polynomials of A are the harmonic Ritz values of A. In the following, we extend this theorem for any $n \times n$ matrix A.

Theorem 3.3. *Let A be an $n \times n$ matrix with $\mathrm{ind}(A) = \gamma$. Then the roots of the k^{th} ideal IGMRES polynomial of A are in the set of all $(\gamma + 1)^{th}$ roots of the harmonic Ritz values of A corresponding to the Krylov subspace $\{A^\gamma r_0, A^{\gamma+1}(A^\gamma r_0), \ldots, A^{(k-1)(\gamma+1)}(A^\gamma r_0)\}$, where $r_0 = b - Ax_0$ and x_0 is the initial guess.*

Proof. Let $p(x) := 1 + \tilde{e}_1 x^{\gamma+1} + \tilde{e}_2 x^{2(\gamma+1)} + \cdots + \tilde{e}_k x^{k(\gamma+1)}$ be the $k^{th}-$ ideal IGMRES polynomial of A. By (15), $q(x) = 1 + \tilde{e}_1 x + \tilde{e}_2 x^2 + \cdots + \tilde{e}_k x^k$ is the k^{th}-ideal GMRES polynomial of $E^{\gamma+1}$. Toh in [17, Theorem 5. 11] shows that the roots of $q(x)$ are the harmonic Ritz values of $E^{\gamma+1}$ corresponding to the $k^{th}-$Krylov subspace $\mathrm{Span}\{v_0, E^{\gamma+1}v_0, \ldots, E^{(\gamma+1)(k-1)}v_0\}$, where $v_0 \in \mathcal{R}(A^\gamma)$. Therefore, the roots of $p(x)$ are the $(\gamma + 1)^{th}$ roots of the harmonic Ritz values of $E^{\gamma+1}$ corresponding to the k^{th} Krylov subspace $\{\hat{r}_0, E^{\gamma+1}\hat{r}_0, \ldots, E^{(\gamma+1)(k-1)}\hat{r}_0\}$. □

4. Numerical Results

In the following two examples the residual errors and CPU times in the DGMRES and IGMRES methods are compared.

Example 4.1. *Let $A = \begin{bmatrix} E & 0 \\ 0 & N \end{bmatrix}$, where $E \in \mathbb{C}^{m\times m}$ be a random matrix, $N \in \mathbb{C}^{n-m\times n-m}$ be the Jordan block with index $n - m$ and $b \in \mathbb{C}^n$. In the following figures, we depict the residual errors of DGMRES and IGMRES for different values of n and m. In Figure 1-(a) we consider $n = 100$ and $m = 97$. The CPU times of the IGMRES method is 0.0872 sec. and the CPU times of* *the DGMRES method is 0.0811 sec. In Figure 1-(b) we consider* $n = 20$ *and* $m = 18$. *The CPU times of the IGMRES method is 0.0131 sec. and the CPU times of the DGMRES method is 0.0116 sec.*

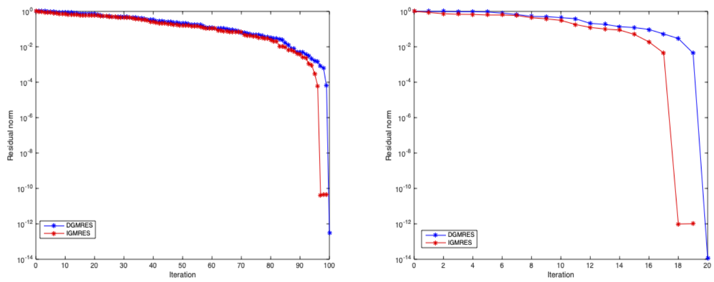

FIGURE 1. (a) (b)

Example 4.2. *Let* $A = \begin{bmatrix} E & 0 \\ 0 & N \end{bmatrix}$, *where E=gallery('smoke',m) from Matlab gallery,* $N \in \mathbb{C}^{n-m \times n-m}$ *be the Jordan block with index* $n - m$ *and* $b \in \mathbb{C}^n$. *In the following figures, we depict the residual errors of DGMRES and IGMRES for different values of* n *and* m. *In Figure 2-(a) we consider* $n = 50$ *and* $m = 47$. *The CPU times of the IGMRES method is 0.0342 sec. and the CPU times of the DGMRES method is 0.0311 sec. In Figure 2-(b) we consider* $n = 30$ *and* $m = 27$. *The CPU times of the IGMRES method is 0.0244 sec. and the CPU times of the DGMRES method is 0.0170 sec.*

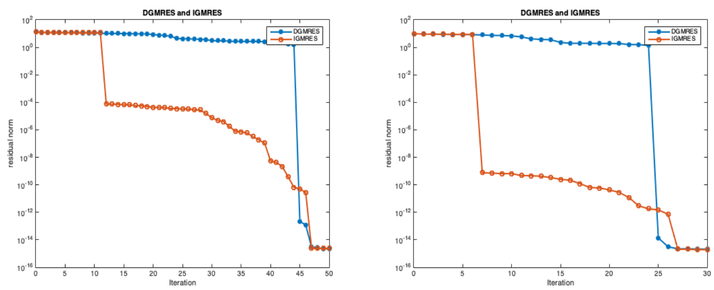

FIGURE 2. (a) (b)

In the following example by using the IGMRES method, the Drazin invese solution of $Ax = b$ is obtained, where A is 4×4 matrix and b is an arbitrary vector in $\mathbb{R}^4$.

Example 4.3. *Let* $A = \begin{bmatrix} E & 0 \\ 0 & N \end{bmatrix}$*, where E=gallery('smoke',2)=* $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ *from Matlab gallery,* $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *be the Jordan block with index* 2 *and* b *be an arbitrary vector in* $\mathbb{R}^4$*. Then by choosing the initial guess* $x_0 = 0$*, the residual* $r_0 = b - Ax_0 = b$ *and* $\mathbf{r_0} = A^2(b)$*. So, the Krylov subspace corresponding to IGMRES method is*

$$\mathcal{K}_2(A^3, \mathbf{r_0}) = span\{\mathbf{r_0}, A^3(\mathbf{r_0})\}.$$

Note that $A^3 = \begin{bmatrix} E^3 & 0 \\ 0 & 0 \end{bmatrix}$*. By using (5),* $E^{-3} = q(E^3)$*, where* $q(x) = \frac{1}{8}x$ *and hence*

$$A^D = q(A^3)A^2 = \begin{bmatrix} \mathbf{E}^{-1} & 0 \\ 0 & 0 \end{bmatrix}, where\ E^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Therefore, the Drazin solution $A^D b \in \mathcal{K}_2(A^3, \mathbf{r_0})$*, which means that the IGMRES method find the solution after two iteration.*

5. Conclusion

A new iterative method IGMRES is introduced to obtain the Drazin inverse solution of the singular system of equations $Ax = b$. In this method we are using a new Krylov subspace. The convergence and the bounds for the relative errors of IGMRES are investigated. The k^{th} ideal IGMRES polynomial as the upper bound for the relative error of the IGMRES and the harmonic Ritz value for singular matrices are introduced and the roots of these polynomials are studied. Although the CPU time in DGMRES method is a little bit less the CPU time in IGMRES method, but the convergence rate of IGMRES method is faster than DGMRES method, see Example 4.2.

References

[1] Carden R. & Embree M. (2012) Ritz value localization for non-Hermitian matrices, SIAM Journal Matrix Analysis and its Applications, 33, 1320-1338. https://doi.org/10.1137/120872693

[2] Campbell, S. L., & Meyer, C. D. (2009). Generalized inverses of linear transformations. Society for industrial and applied Mathematics.

[3] Eiermann, M., Marek, I., & Niethammer, W. (1988). On the solution of singular linear systems of algebraic equations by semiiterative methods. Numerische Mathematik, 53(3), 265-283. https://doi.org/10.1007/BF01404464

[4] Freund, R. W., & Hochbruck, M. (1994). On the use of two QMR algorithms for solving singular systems and applications in Markov chain modeling. Numerical linear algebra with applications, 1(4), 403-420. https://doi.org/10.1002/nla.1680010406

[5] Goossens S. & Roose D. (1999) Ritz and harmonic Ritz values and the convergence of FOM and GMRES, Numerical Linear Algebra and its Applications, 6, 281-293. https://doi.org/10.1002/(SICI)1099-1506(199906)6:4¡281::AID-NLA158¿3.0.CO;2-B

[6] Greenbaum A. & Kyanfar F. & Salemi A. (2018) On the convergence rate of DGMRES, Linear Algebra and its Applications 552, 219-238. https://doi.org/10.1016/j.laa.2018.04.027

[7] Horn R. & Johnson C. (2013) Matrix Analysis, second edition, Cambridge.
[8] Ipsen I.C.F. & Meyer C.D. (1998) The idea behind Krylov methods, Amer. Math. Monthly, 105, 889-899. https://doi.org/10.1080/00029890.1998.12004985
[9] Kyanfar F. (2022) On stagnation of the DGMRES method, Iranian Journal of Numerical Analysis and Optimization, Vol 12. no.3, 533-541. https://doi.org/10.22067/IJNAO.2022.73913.1081
[10] Levine, J., & Hartwig, R. E. (1980). Applications of the Drazin Inverse to the Hill Cryptographic System. Part I. Cryptologia, 4(2), 71-85. https://doi.org/10.1080/0161-118091854906
[11] Liesen J. & Tichy P. (2009) On best approximation of polynomials in matrices in the matrix 2-norm, SIAM Journal Matrix Analysis and its Applications, 31, 853-863. https://doi.org/10.1137/080728299
[12] Nevanlinna, O.(1993) Convergence of iterations for linear equations, Lectures in Mathematics ETH Zürich. Basel: Birkhäuser Verlag.
[13] Saad Y. & Schultz M.H. (1986;), GMRES: A generalized minimal residual algorithm for solving non symmetric linear systems, SIAM J. Sci. Statist. Comput., 7, 856-869. https://doi.org/10.1137/0907058
[14] Safarzadeh M. & Salemi A. (2018) DGMRES and index numerical range of matrices, Journal of Computational and Applied Mathematics, 335, 349-360. https://doi.org/10.1016/j.cam.2017.12.009
[15] Sidi A. (2001) DGMRES: A GMRES-type algorithm for Drazin-inverse solution of singular non symmetric linear systems, Linear Algebra and its Applications, 335, 189-204. https://doi.org/10.1016/S0024-3795(01)00289-0
[16] Simeon, B., Führer, C., & Rentrop, P. (1993). The Drazin inverse in multibody system dynamics. Numerische Mathematik, 64, 521-539. https://doi.org/10.1007/BF01388703
[17] Toh K.C. (1996) Matrix approximation problems and nonsymmetric iterative methods, PhD dissertation, Cornell University, Ithaca, NY.

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