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THE CYCLIC-FIBONACCI HYBRID SEQUENCE IN GROUPS

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Abstract. The aim of this paper is to introduce the cyclic-Fibonacci hybrid sequence and give some properties. By taking into account the cyclic-Fibonacci hybrid sequence modulo m, the method will be given to determine the period lengths of this sequence according to the different m values. In the final part of this paper, we study the cyclic-Fibonacci hybrid sequence in groups and then we calculate the cyclic-Fibonacci hybrid lengths of polyhedral groups $(2, 2, 2)$, $(2, n, 2)$ and $(n, 2, 2)$ as applications of the results produced.

Keywords: Fibonacci Hybrid Sequence, Groups, Period, Presentation.
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1. Introduction

There is a long history of studying sequences of numbers greater than one dimensional, such as, 2-dimensional, 4-dimensional and so on. Complex, dual and hyperbolic numbers are well-known two dimensional number systems. Especially in recent years, a lot of researchers deal with the geometric, algebraic and physical applications of these numbers. In 1998, the authors generalized the 2-dimensional number systems to higher dimensions using a very natural way [10]. In [18], Ozdemir introduced a hybrid number as a generalization of the complex ($\mathbf{i}^2 = -1$), dual ($\epsilon^2 = 0$) and hyperbolic ($\mathbf{h}^2 = 1$) numbers. The set of hybrid numbers denoted by $\mathbb{K}$, is defined as

$$\mathbb{K} = \left\{ u + v\mathbf{i} + w\epsilon + z\mathbf{h} : \ \mathbf{i}^2 = -1, \epsilon^2 = 0, \mathbf{h}^2 = 1, u, v, w, z \in \mathbb{R} \right\}.$$

For any two hybrid numbers $K_1 = u_1 + v_1\mathbf{i} + w_1\epsilon + z_1\mathbf{h}$ and $K_2 = u_2 + v_2\mathbf{i} + w_2\epsilon + z_2\mathbf{h}$, it write

- $K_1 = K_2$ only if $u_1 = u_2$, $v_1 = v_2$, $w_1 = w_2$, $z_1 = z_2$ (equality),
- $K_1 + K_2 = (u_1 + u_2) + (v_1 + v_2)\mathbf{i} + (w_1 + w_2)\epsilon + (z_1 + z_2)\mathbf{h}$ (addition),
- $K_1 - K_2 = (u_1 - u_2) + (v_1 - v_2)\mathbf{i} + (w_1 - w_2)\epsilon + (z_1 - z_2)\mathbf{h}$ (substraction),
- $sK_1 = su_1 + sv_1\mathbf{i} + sw_1\epsilon + sz_1\mathbf{h}$ (the multiplication by scalar),
- $\overline{K_1} = u_1 - v_1\mathbf{i} - w_1\epsilon - z_1\mathbf{h}$ (the conjugate of a hybrid number).

The addition operation in the hybrid numbers is both associative and commutative. The multiplication of hybrid numbers is not commutative, but it has

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the property of associativity. The product table of the basis of hybrid numbers are as Table 1.

TABLE 1. The product table for the basis of $\mathbb{K}$

.	1	$\mathbf{i}$	ϵ	$\mathbf{h}$
1	1	$\mathbf{i}$	ϵ	$\mathbf{h}$
$\mathbf{i}$	i	-1	$1-\mathbf{h}$	$\epsilon+\mathbf{i}$
ϵ	ϵ	$\mathbf{h}+1$	0	$-\epsilon$
$\mathbf{h}$	$\mathbf{h}$	$-\epsilon-\mathbf{i}$	ϵ	1

For $n \geq 2$, the Fibonacci number is defined as $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$. There are lots of astonishing identities belonging to the Fibonacci number in [15]. In recent years, Fibonacci, Lucas and Pell hybrid numbers cover a wide range of interests in modern mathematics as they appear in the comprehensive works of [4, 5, 16, 17, 19–24]. Kızılateş gave the generalizations of the Fibonacci and Lucas hybrid numbers, see [12, 13]. In [23], the Fibonacci hybrid numbers are defined as

$$FH_n = F_n + F_{n+1}\mathbf{i} + F_{n+2}\epsilon + F_{n+3}\mathbf{h}.$$

On the other hand, in [11], Kalman derived several closed-form formulas for the generalized sequence by companion matrix method.

For a finitely generated group $G = \langle A \rangle$, where $A = \{a_1, a_2, \cdots, a_n\}$, the sequence $x_i = a_{i+1}, 0 \leq i \leq n-1$, $x_{n+i} = \prod_{j=1}^{n} x_{i+j-1}$, $i \geq 0$, is called the Fibonacci orbit of G with respect to the generating set A, denoted $F_A(G)$, see [1–3].

Definition 1.1. For $k, l, m > 1$, the polyhedral (triangle) group presented by

$$\left\langle x, y \mid x^k = y^l = z^m = xyz = 1 \right\rangle,$$

or

$$\left\langle x, y \mid x^k = y^l = (xy)^m = 1 \right\rangle.$$

The polyhedral group (k, l, m) is finite if and only if the number

$$\mu = klm \left(\frac{1}{k} + \frac{1}{l} + \frac{1}{m} - 1 \right)$$

is positive, that is, in the cases $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$. Its order is $2klm/\mu$. By thinking in Combinatorial Group theory Tietze transformations, we can obtain that $(l, m, n) \cong (m, n, l) \cong (n, l, m)$. For more information on these groups, see [6, 7].

If a sequence consists only of repetitions of a fixed subsequence after a certain point, it is periodic. The period of the sequence is the number of elements in the shortest repetition subsequence. For instance, the sequence $k, l, m, n, o, l, m, n, o, \ldots,$ is periodic after the first element k and has period 4. As a special case, a sequence is simply periodic with period m if the initial m elements in the sequence form a repeating subsequence. For instance, the sequence $k, l, m, n, o, k, l, m, n, o, k, l, m, n, o, \ldots,$ is simply periodic with period 5. Recently, many authors have studied some special linear recurrence sequences in groups; see for example, [8, 9, 14, 26].

In Section 2, we define the cyclic-Fibonacci hybrid sequence and then we present some properties. In Section 3, we study the cyclic-Fibonacci hybrid sequence modulo m and then we give the relationships among the lengths of periods of the cyclic-Fibonacci hybrid sequences according to the different m values. In Section 4, we introduce the cyclic-Fibonacci hybrid sequence in groups. Finally, we calculate the cyclic-Fibonacci hybrid length in some finite polyhedral groups.

2. The Cyclic-Fibonacci Hybrid Sequence

In this section, we will introduce cyclic-Fibonacci hybrid sequence for $n \geq 2$ any positive integer numbers. Then, we will present the miscellaneous properties of these sequences.

Definition 2.1. The cyclic-Fibonacci hybrid sequence is defined as follows:

$$x_n = \begin{cases} \mathbf{h}x_{n-1} + \epsilon x_{n-2} & \text{if } n \equiv 0 \ (\mathrm{m}od3), \\ \mathbf{i}x_{n-1} + \mathbf{h}x_{n-2} & \text{if } n \equiv 1 \ (\mathrm{m}od3), \\ \epsilon x_{n-1} + \mathbf{i}x_{n-2} & \text{if } n \equiv 2 \ (\mathrm{m}od3) \end{cases}$$

with initial conditions $x_0 = 0$ and $x_1 = 1$.

The first eleven terms of the cyclic-Fibonacci hybrid sequence are as follows:
$x_0 = 0,$
$x_1 = 1,$
$x_2 = \epsilon,$
$x_3 = 2\epsilon,$
$x_4 = 2 + \epsilon - 2\mathbf{h},$
$x_5 = 2 + 4\epsilon - 2\mathbf{h},$
$x_6 = -2 + 8\epsilon + 2\mathbf{h},$
$x_7 = 6 + 6\epsilon - 6\mathbf{h},$
$x_8 = 8 + 14\epsilon - 8\mathbf{h},$
$x_9 = -8 + 26\epsilon + 8\mathbf{h},$
$x_{10} = 18 + 22\epsilon - 18\mathbf{h}.$

We will give the property

$$\begin{cases} x_{3n-1} = \left[(3^{n-1}-1)\mathbf{i} + 5.3^{n-2} - 1\right]\epsilon \\ x_{3n} = \left[(1-3^{n-1})\mathbf{i} + 3^{n} - 1\right]\epsilon \\ x_{3n+1} = \left[2.3^{n-1}\mathbf{i} + 8.3^{n-2} - 2\right]\epsilon \end{cases}$$

where $n \geq 2$. We can write for the cyclic-Fibonacci hybrid sequence

(1) $$G = \begin{bmatrix} 2+\epsilon-2\mathbf{h} & -\mathbf{i}-\epsilon+\mathbf{h} \\ 2\epsilon & -\mathbf{i}-\epsilon \end{bmatrix}.$$

By mathematical induction on n, we find

(2) $$G^{n} = \begin{bmatrix} x_{3n+1} & g_{12}^{n} \\ x_{3n} & g_{22}^{n} \end{bmatrix},$$

where $n \geq 1$, $g_{12}^{n} = (2+\epsilon-2\mathbf{h})g_{12}^{n-1} + (-\mathbf{i}-\epsilon+\mathbf{h})g_{22}^{n-1}$ and $g_{22} = 2\epsilon g_{12}^{n-1} + (-\mathbf{i}-\epsilon)g_{22}^{n-1}$.

Lemma 2.2. *We give the recurrence relation for the cyclic-Fibonacci hybrid sequence as follows:*

$$x_n = 4x_{n-3} - 3x_{n-6},$$

where $n \geq 11$.

Proof. Let us use the principle of mathematical induction on n. For $n = 11$, it is easy to see that

$$\begin{aligned} x_{11} =& 4x_8 - 3x_5 \\ =& 4(8+14\epsilon-8\mathbf{h}) - 3(2+4\epsilon-2\mathbf{h}) \\ =& 26+44\epsilon-26\mathbf{h}. \end{aligned}$$

As the usual next step of inductions, let us assume that it is true for all positive integers $k \leq n$. In other words, $x_k = 4x_{k-3} - 3x_{k-6}$.

Finally, we need to show that it is true for $k+1$. There are three conditions. Firstly, if $k+1 \equiv 0\ (\mathrm{mod} 3)$, we can write from Definition 2.1

$$\begin{aligned} x_{k+1} =& \mathbf{h}x_k + \epsilon x_{k-1} \\ =& \mathbf{h}(4x_{k-3} - 3x_{k-6}) + \epsilon(4x_{k-4} - 3x_{k-7}) \\ =& 4x_{k-2} - 3x_{k-5}. \end{aligned}$$

Secondly, if $k+1 \equiv 1\ (\mathrm{mod} 3)$, we can obtain again from Definition 2.1

$$\begin{aligned} x_{k+1} =& \mathbf{i}x_k + \mathbf{h}x_{k-1} \\ =& \mathbf{i}(4x_{k-3} - 3x_{k-6}) + \mathbf{h}(4x_{k-4} - 3x_{k-7}) \\ =& 4x_{k-2} - 3x_{k-5}. \end{aligned}$$

Eventually, if $k+1 \equiv 2\ (\mathrm{mod} 3)$, the result can be obtained with similar operations. Hence the proof is complete. □

In the following Theorem, we develop the generating function for the cyclic-Fibonacci hybrid sequence.

Theorem 2.3. *The generating function of the sequence $\{x_n\}$ is*

$$\sum_{n=0}^{\infty} x_n t^n = \frac{t+\epsilon t^2+2\epsilon t^3+(-2+\epsilon-2\mathbf{h})t^4+(2-2\mathbf{h})t^5+(-2+2\mathbf{h})t^6+(1+2\epsilon+2\mathbf{h})t^7+\epsilon t^8+\epsilon t^{10}}{1-4t^3+3t^6}.$$

Proof. Assume that $f(t)$ is the generating function of $\{x_n\}$. Then we have

$$f\left(t\right)=\sum_{n=0}^{\infty} x_n t^n$$

From Lemma 2.2, we obtain

$$\begin{aligned} f\left(t\right) &= \sum_{n=0}^{10} x_n t^n + \sum_{n=11}^{\infty}\left(4x_{n-3}-3x_{n-6}\right)t^n \\ &= \sum_{n=0}^{10} x_n t^n + 4\left(f(t)-\sum_{n=0}^{7} x_n t^n\right)t^3 - 3\left(f(t)-\sum_{n=0}^{4} x_n t^n\right)t^6 . \end{aligned}$$

Now rearrangement of the equation implies that

$$f(t)=\frac{x_1 t+x_2 t^2+x_3 t^3+\left(x_4-4x_1\right)t^4+\left(x_5-4x_2\right)t^5+\left(x_6-4x_3\right)t^6+\sum_{n=7}^{10}\left(x_n-4x_{n-3}+3x_{n-6}\right)t^n}{1-4t^3+3t^6},$$

which is equal to the $\sum_{n=0}^{\infty} x_n t^n$ in Theorem. □

3. The Cyclic-Fibonacci Hybrid Sequence Modulo m

In this section, we study the cyclic-Fibonacci hybrid sequence modulo m. Then, we obtain the length of the period of the cyclic-Fibonacci hybrid sequence for modulo m.

Let f_n denote the nth member of the Fibonacci sequences $f_0 = a$, $f_1 = b$, $f_{n+1} = f_n + f_{n-1}$ $(n \geq 1)$.

Theorem 3.1. *(Wall [25]) $f_n\,(\mathrm{mod}\,m)$ forms a simple periodic sequence. That is, the sequence is periodic and repeats by returning to its starting values.*

The length of the period of the ordinary Fibonacci sequence $\{F_n\}$ modulo m was denoted by $k(m)$.

If we reduce the cyclic-Fibonacci hybrid sequence of modulo m, taking the smallest nonnegative residues, then we get the following recurrence sequences:

$$\{x_n\left(m\right)\}=\{x_1\left(m\right),\ x_2\left(m\right),\ \ldots,\ x_u\left(m\right),\ \ldots\}$$

where $x_u\left(m\right)$ is used to mean the uth element of the cyclic-Fibonacci hybrid sequence when reading modulo m. We note here that the recurrence relations in the sequences $\{x_n\left(m\right)\}$ and $\{x_n\}$ are the same.

Theorem 3.2. *The sequence* $\{x_n(m)\}$ *is periodic and the length of its period is divisible by* 3.

Proof. Consider the set

$$\begin{aligned} H &= \{(H_1, H_2) \mid H_j\text{'s are hybrid numbers } u_j + v_j\mathbf{i} + w_j\epsilon + z_j\mathbf{h} \text{ where} \\ &\quad u_j, v_j,\ w_j \text{ and } z_j \text{ are integers such that } 0 \le u_j, v_j,\ w_j,\ z_j \le m-1 \text{ and } j \in \{1,2\}\}. \end{aligned}$$

Suppose that the cardinality of the set H is denoted by the notation $|H|$. Since the set H is finite, there are $|H|$ distinct 2-tuples of the cyclic-Fibonacci hybrid sequence $\{x_n\}$ modulo m. Thus, it is clear that at least one of these 2-tuples appears twice in the sequence $\{x_n(m)\}$. Let $x_\alpha(m) \equiv x_\beta(m)$ and

$x_{\alpha+1}(m) \equiv x_{\beta+1}(m)$. If $\beta - \alpha \equiv 0\,(\text{mod}3)$, then we get $x_{\alpha+2}(m) \equiv x_{\beta+2}(m)$, $x_{\alpha+3}(m) \equiv x_{\beta+3}(m)$, ... So, it is easy to see that the subsequence following this 2 -tuple repeats; that is, the sequence $\{x_n(m)\}$ is a periodic sequence and the length of its period must be divisible by 3. □

We next denote the length of the period of the sequence $\{x_n(m)\}$ by $h_{x_n}(m)$. Consider the matrices

$$A_1 = \begin{bmatrix} \epsilon & \mathbf{i} \\ 1 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} \mathbf{h} & \epsilon \\ 1 & 0 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} \mathbf{i} & \mathbf{h} \\ 1 & 0 \end{bmatrix}.$$

Suppose that $G = A_3A_2A_1$. Using the above, we define the following matrix:

$$M^n = \begin{cases} G^{\frac{n}{3}} & \text{if } n \equiv 0\ (mod3), \\ A_1 G^{\frac{n-1}{3}} & \text{if } n \equiv 1\ (mod3), \\ A_2A_1 G^{\frac{n-2}{3}} & \text{if } n \equiv 2\ (mod3). \end{cases}$$

Then we get

$$M^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}.$$

Therefore, we immediately deduce that $h_{x_n}(m)$ is the smallest positive integer β such that $M^\beta \equiv I(\text{mod } m)$.

4. The Cyclic-Fibonacci Hybrid Sequence in Groups

In this subsection, we extend the concept to groups and then we examine the periods the cyclic-Fibonacci hybrid sequences in finite groups. Additionally, for a better understanding of the idea, we calculate the lengths of the periods of the cyclic-Fibonacci hybrid sequences in the polyhedral groups $(2,2,2)$, $(2,n,2)$ and $(n,2,2)$ with respect to the generating pair (x,y).

Let G be a 2-generator group and let

$$X = \{(x_1, x_2) \in G \times G \mid \langle\{x_1, x_2\}\rangle = G\}.$$

We call (x_1, x_2) a generating pair for G.

Definition 4.1. Let G be a 2-generator group. For the generating pair (x, y), we define the cyclic-Fibonacci hybrid orbit as follows:

$$a_n = \begin{cases} (a_{n-2})^{\epsilon}(a_{n-1})^{\mathbf{h}} & \text{if } n \equiv 0 \ (mod3), \\ (a_{n-2})^{\mathbf{h}}(a_{n-1})^{\mathbf{i}} & \text{if } n \equiv 1 \ (mod3), \\ (a_{n-2})^{\mathbf{i}}(a_{n-1})^{\epsilon} & \text{if } n \equiv 2 \ (mod3) \end{cases}$$

for $n \geq 2$, with initial conditions $a_0 = x$ and $a_1 = y$, where the following conditions hold for every $x, y \in G$:

(i). Let $q = a + b\mathbf{i} + c\epsilon + d\mathbf{h}$ such that a, b, c and d are integers and let e be the identity of G, then
 - $x^q = x^{a(\mathrm{mod}|x|)+b(\mathrm{mod}|x|)\mathbf{i}+c(\mathrm{mod}|x|)\epsilon+d(\mathrm{mod}|x|)\mathbf{h}} = x^{a(\mathrm{mod}|x|)}x^{b(\mathrm{mod}|x|)\mathbf{i}}x^{c(\mathrm{mod}|x|)\epsilon}x^{d(\mathrm{mod}|x|)\mathbf{h}}$.
 - $(x^u)^a = (x^a)^u$, where $u \in \{\mathbf{i}, \epsilon, \mathbf{h}\}$ and a is an integer.
 - $e^q = e$ and $x^{0+0\mathbf{i}+0\epsilon+0\mathbf{h}} = e$.

(ii). Let $q_1 = a_1 + b_1\mathbf{i} + c_1\epsilon + d_1\mathbf{h}$ and $q_2 = a_2 + b_2\mathbf{i} + c_2\epsilon + d_2\mathbf{h}$ such that $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$ are integers, then $(x^{q_1}x^{q_2})^{-1} = x^{-q_2}x^{-q_1}$.

(iii). If $xy \neq yx$, then $x^u y^u \neq y^u x^u$ for $u \in \{\mathbf{i}, \epsilon, \mathbf{h}\}$.

(iv). $(xy)^u = y^u x^u$ for $u \in \{\mathbf{i}, \epsilon, \mathbf{h}\}$.

(v). $(x^{u_1}y^{u_2})^{u_3} = x^{u_3u_1}y^{u_3u_2}$ for $u_1, u_2, u_3 \in \{\mathbf{i}, \epsilon, \mathbf{h}\}$.

(vi). For $u_1, u_2 \in \{\mathbf{i}, \epsilon, \mathbf{h}\}$ such that $u_1 \neq u_2$, $x^{u_1}y^{u_2} = y^{u_2}x^{u_1}$, $xy^{u_1} = y^{u_1}x$, $x^{u_1}y = yx^{u_1}$.

Let the notation $FH_{(x,y)}(G)$ denote the cyclic-Fibonacci hybrid orbit of the group G for the generating pair (x, y). From the definition of the orbit $FH_{(x,y)}(G)$ it is clear that the length of the period of this sequence in a finite group depends on the chosen generating pair.

Theorem 4.2. *Let G be a finite 2-generator group. The cyclic-Fibonacci hybrid orbit of the sequence a_n is periodic and the length of its period is divisible by 3.*

Proof. We take into account the sequence a_n the cyclic-Fibonacci hybrid orbit of the group G. Consider the set

$$S = \Big\{(s_1)^{a_1(\mathrm{mod}|s_1|)+b_1(\mathrm{mod}|s_1|)\mathbf{i}+c_1(\mathrm{mod}|s_1|)\epsilon+d_1(\mathrm{mod}|s_1|)\mathbf{h}}, \\ (s_2)^{a_2(\mathrm{mod}|s_2|)+b_2(\mathrm{mod}|s_2|)\mathbf{i}+c_2(\mathrm{mod}|s_2|)\epsilon+d_2(\mathrm{mod}|s_2|)\mathbf{h}} : \\ s_1, s_2 \in G \text{ and } a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{Z}\Big\}.$$

Since the group G is finite, S is a finite set. Then for any $u \geq 0$, there exists $v > u$ such that $a_u^2 = a_v^2$ and $a_{u+1}^2 = a_{v+1}^2$. If $v - u \equiv 0\,(\mathrm{mod}3)$, then we get $a_{u+2}^2 = a_{v+2}^2$, $a_{u+3}^2 = a_{v+3}^2$, ... Because of the reduplicating, for all generating pairs, the sequence $FH_{(x,y)}(G)$ is periodic and the length of its period must be divisible by 3. □

We next denote the length of the period of the orbit $CFH_{(x,y)}(G)$ by $LCFH_{(x,y)}(G)$.

We shall now address the lengths of the periods of the orbits $CFH_{(x,y)}\left((2,2,2)\right)$, $CFH_{(x,y)}\left((2,n,2)\right)$ and $CFH_{(x,y)}\left((n,2,2)\right)$. Now we compute the polyhedral groups $(2,2,2)$, $(2,n,2)$ and $(n,2,2)$ for the generating pair (x,y).

Theorem 4.3. *The group defined by the presentation* $\left\langle x,y \mid x^2=y^2=1,\ (xy)^2=1\right\rangle$ *has the cyclic-Fibonacci hybrid length* $LCFH_{x,y}((2,2,2))=6$.

Proof. By a simple calculation, we obtain the cyclic-Fibonacci hybrid orbit of the polyhedral $(2,2,2)$ as shown:

$$\begin{aligned} x_0 &= x,\ x_1=y,\ x_2=y^{\epsilon}x^{\mathbf{i}},\ x_3=x^{-\mathbf{i}-\epsilon},\ x_4=y^{\epsilon}x^{-\mathbf{i}-\epsilon+\mathbf{h}},\\ x_5 &= x^{-1-\epsilon},\ x_6=x,\ x_7=x^{\mathbf{i}-\epsilon-\mathbf{h}},\ x_8=x^{1+\mathbf{i}+\epsilon+\mathbf{h}},\ x_9=x^{-\mathbf{i}+\epsilon},\\ x_{10} &= x^{1-\mathbf{i}},\ x_{11}=x^{-1-\epsilon},\ x_{12}=x,\ x_{13}=x^{\mathbf{i}-\epsilon-\mathbf{h}},\ x_{14}=x^{1+\mathbf{i}+\epsilon+\mathbf{h}},\\ x_{15} &= x^{-i+\epsilon},\ x_{16}=x^{1-i},\ x_{17}=x^{-1-\epsilon},\ x_{18}=x,\ldots. \end{aligned}$$

Since $x_5=x_{11}=x_{17}=x^{-1-\epsilon}$ and $x_6=x_{12}=x_{18}=x$, we get $LCFH_{x,y}((2,2,2))=6$. □

Theorem 4.4. *For* $n>2$,

$$LCFH_{x,y}((2,n,2))=lcm\left(6,h_{x_n}(n)\right).$$

Proof. We prove this by direct calculations. We first note that in the group defined by $\left\langle x,y \mid x^2=y^n=1,\ (xy)^2=1\right\rangle$ We have the sequence

$$\begin{aligned} & x,\ y,\ y^{\epsilon}x^{\mathbf{i}},\ y^{2\epsilon}x^{\mathbf{i}-\epsilon},\ y^{2+\epsilon-2\mathbf{h}}x^{-\mathbf{i}-\epsilon+\mathbf{h}},\ y^{2+4\epsilon-2\mathbf{h}}x^{-1-\epsilon},\\ & y^{-2+8\epsilon+2\mathbf{h}}x,\ y^{6+6\epsilon-6\mathbf{h}}x^{\mathbf{i}+\epsilon+\mathbf{h}},\ y^{8+14\epsilon-8\mathbf{h}}x^{1+\mathbf{i}+\epsilon+\mathbf{h}},\\ & y^{-8+26\epsilon+8\mathbf{h}}x^{\mathbf{i}+\epsilon},y^{18+22\epsilon-18\mathbf{h}}x^{1+\mathbf{i}},\ y^{26+44\epsilon-26\mathbf{h}}x^{1+\epsilon},\\ & y^{-26+80\epsilon+26\mathbf{h}}x,\ y^{54+70\epsilon-54\mathbf{h}}x^{\mathbf{i}+\epsilon+\mathbf{h}},y^{80+134\epsilon-80\mathbf{h}}x^{1+\mathbf{i}+\epsilon+\mathbf{h}},\\ & y^{-80+242\epsilon+80\mathbf{h}}x^{\mathbf{i}+\epsilon},y^{162+214\epsilon-162\mathbf{h}}x^{1+\mathbf{i}},\ y^{242+404\epsilon-242\mathbf{h}}x^{1+\epsilon},\\ & y^{-242+728\epsilon+242\mathbf{h}}x,\ y^{486+646\epsilon-486\mathbf{h}}x^{\mathbf{i}+\epsilon+\mathbf{h}},y^{728+1214\epsilon-728\mathbf{h}}x^{1+\mathbf{i}+\epsilon+\mathbf{h}},\\ & y^{-728+2186\epsilon+728\mathbf{h}}x^{\mathbf{i}+\epsilon},y^{1458+1942\epsilon-1458\mathbf{h}}x^{1+\mathbf{i}},\ y^{2186+3644\epsilon-2186\mathbf{h}}x^{1+\epsilon},\ \ldots. \end{aligned}$$

So we get the sequence with initial conditions $a_0=x,\ a_1=y,\ a_2=y^{\epsilon}x^{\mathbf{i}},\ a_3=y^{2\epsilon}x^{\mathbf{i}+\epsilon}$ and $a_4=y^{2+\epsilon-2\mathbf{h}}x^{\mathbf{i}+\epsilon+\mathbf{h}}$ as follows:

$$\begin{aligned} & a_5=y^{2+4\epsilon-2\mathbf{h}}x^{1+\epsilon},\ a_6=y^{-2+8\epsilon+2\mathbf{h}}x,\ a_7=y^{6+6\epsilon-6\mathbf{h}}x^{\mathbf{i}+\epsilon+\mathbf{h}},\ a_8=y^{8+14\epsilon-8\mathbf{h}}x^{1+\mathbf{i}+\epsilon+\mathbf{h}},\cdots,\\ & a_{11}=y^{26+44\epsilon-26\mathbf{h}}x^{1+\epsilon},\ a_{12}=y^{-26+80\epsilon+26\mathbf{h}}x,\ a_{13}=y^{54+70\epsilon-54\mathbf{h}}x^{\mathbf{i}+\epsilon+\mathbf{h}},\cdots,\\ & a_{17}=y^{242+404\epsilon-242\mathbf{h}}x^{1+\epsilon},\ a_{18}=y^{-242+728\epsilon+242\mathbf{h}}x,\ a_{19}=y^{486+646\epsilon-486\mathbf{h}}x^{\mathbf{i}+\epsilon+\mathbf{h}},\cdots,\\ & \cdots\\ & a_{6n+5}=y^{x_{6n+5}}x^{1+\epsilon},\qquad a_{6n+6}=y^{x_{6n+6}}x,\qquad a_{6n+7}=y^{x_{6n+7}}x^{\mathbf{i}+\epsilon+\mathbf{h}},\\ & a_{6n+8}=y^{x_{6n+8}}x^{1+\mathbf{i}+\epsilon+\mathbf{h}},\ a_{6n+9}=y^{x_{6n+9}}x^{\mathbf{i}+\epsilon},\quad a_{6n+10}=y^{x_{6n+10}}x^{1+\mathbf{i}}. \end{aligned}$$

Since the order of the element y is n and the period of the sequence $\{x_n(n)\}$ is $h_{x_n}(n)$, we obtain the period of the sequence $\{a_n\}$ as $\mathrm{lcm}\left(6,h_{x_n}(n)\right)$. □

Consider the sequence
$c_0=1$,
$c_1=0$,

$c_2 = \mathbf{i}$,
$c_3 = -\mathbf{i} - \epsilon$,
$c_4 = -\mathbf{i} - \epsilon + \mathbf{h}$,
$c_5 = -1 - \epsilon$,
$c_6 = -1 - 2\epsilon - 2\mathbf{h}$,
$c_7 = -2 - 3\mathbf{i} - 3\epsilon + \mathbf{h}$,
$c_8 = -5 - 3\mathbf{i} - 5\epsilon - \mathbf{h}$,
$c_9 = -4 + 3\mathbf{i} - 5\epsilon - 8\mathbf{h}$,
$c_{10} = -9 - 9\mathbf{i} - 10\epsilon$.
$\vdots$
$c_n = 4c_{n-3} - 3c_{n-6}$, where $n \geq 11$.
It is easy to prove that the sequence $\{c_n\}$ for modulo t is periodic. Reducing the sequence $\{c_n\}$ by a modulo t, then we get the repeating sequence, denoted by

$$\{c_n(t)\} = \{c_0(t), c_1(t), \ldots c_u(t), \ldots\}.$$

We denote the lengths of the period of the sequence $\{c_n(t)\}$ by $h_{c_n}(t)$. We take into consideration the generating matrix

$$A = \begin{bmatrix} 0 & 0 & 4 & 0 & 0 & -3 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

By direct calculations it is easy to see that the sequence $\{c_n\}$ conforms to the following pattern:

$$A \begin{bmatrix} -9 - 9\mathbf{i} - 10\epsilon \\ -4 + 3\mathbf{i} - 5\epsilon - 8\mathbf{h} \\ -5 - 3\mathbf{i} - 5\epsilon - \mathbf{h} \\ -2 - 3\mathbf{i} - 3\epsilon + \mathbf{h} \\ -1 - 2\epsilon - 2\mathbf{h} \\ -1 - \epsilon \end{bmatrix} = \begin{bmatrix} c_{11} \\ c_{10} \\ c_9 \\ c_8 \\ c_7 \\ c_6 \end{bmatrix}, \ A^2 \begin{bmatrix} -9 - 9\mathbf{i} - 10\epsilon \\ -4 + 3\mathbf{i} - 5\epsilon - 8\mathbf{h} \\ -5 - 3\mathbf{i} - 5\epsilon - \mathbf{h} \\ -2 - 3\mathbf{i} - 3\epsilon + \mathbf{h} \\ -1 - 2\epsilon - 2\mathbf{h} \\ -1 - \epsilon \end{bmatrix} = \begin{bmatrix} c_{17} \\ c_{16} \\ c_{15} \\ c_{14} \\ c_{13} \\ c_{12} \end{bmatrix}, \ldots.$$

Using the above, we define the following matrices:

$$A^n \begin{bmatrix} -9 - 9\mathbf{i} - 10\epsilon \\ -4 + 3\mathbf{i} - 5\epsilon - 8\mathbf{h} \\ -5 - 3\mathbf{i} - 5\epsilon - \mathbf{h} \\ -2 - 3\mathbf{i} - 3\epsilon + \mathbf{h} \\ -1 - 2\epsilon - 2\mathbf{h} \\ -1 - \epsilon \end{bmatrix} = \begin{bmatrix} c_{n+10} \\ c_{n+9} \\ c_{n+8} \\ c_{n+7} \\ c_{n+6} \\ c_{n+5} \end{bmatrix},$$

where $n \geq 0$. From these equations we immediately deduce:
$h_{c_n}(t)$ is the smallest positive integer β such that $A^\beta \equiv I (\mathrm{mod}\ t)$.

Now we give the lengths of the period of the sequence $LCFH_{x,y}((n,2,2))$ by the aid of the above useful results.

Theorem 4.5. *For $n>2$, the cyclic-Fibonacci hybrid length of the polyhedral group $(n,2,2)$ is $h_{c_n}(n)$.*

Proof. The polyhedral group $(n,2,2)$ is defined by the presentation $\left\langle x,y \mid x^n=y^2=1,\ (xy)^2=1\right\rangle$, then the cyclic-Fibonacci hybrid orbit of $(n,2,2)$ is as follows:

$$\begin{aligned}
a_0 &= x,\ a_1=y,\ a_2=y^{\epsilon}x^{\mathbf{i}},\ a_3=x^{-\mathbf{i}-\epsilon},\ a_4=y^{\epsilon}x^{-\mathbf{i}-\epsilon+\mathbf{h}},\\
a_5 &= x^{-1-\epsilon}, \qquad a_6=x^{-1-2\epsilon-2\mathbf{h}},\\
a_7 &= x^{-2-3\mathbf{i}-3\epsilon+\mathbf{h}}, \qquad a_8=x^{-5-3\mathbf{i}-5\epsilon-\mathbf{h}},\\
a_9 &= x^{-4+3\mathbf{i}-5\epsilon-8\mathbf{h}}, \qquad a_{10}=x^{-9-9\mathbf{i}-10\epsilon},\\
a_{11} &= x^{-17-12\mathbf{i}-17\epsilon-4\mathbf{h}}, \qquad a_{12}=x^{-13+12\mathbf{i}-14\epsilon-26\mathbf{h}},\\
a_{13} &= x^{-30-27\mathbf{i}-31\epsilon-3\mathbf{h}}, \qquad a_{14}=x^{-53-39\mathbf{i}-53\epsilon-13\mathbf{h}},\\
a_{15} &= x^{-40+39\mathbf{i}-41\epsilon-80\mathbf{h}}, \qquad a_{16}=x^{-93-81\mathbf{i}-94\epsilon-12\mathbf{h}},\\
a_{17} &= x^{-161-120\mathbf{i}-161\epsilon-40\mathbf{h}}, \qquad a_{18}=x^{-121+120\mathbf{i}-122\epsilon-242\mathbf{h}},\\
a_{19} &= x^{-282-243\mathbf{i}-283\epsilon-39\mathbf{h}}, \qquad a_{20}=x^{-485-363\mathbf{i}-485\epsilon-121\mathbf{h}},\\
a_{21} &= x^{-364+363\mathbf{i}-365\epsilon-728\mathbf{h}}, \qquad a_{22}=x^{-849-729\mathbf{i}-850\epsilon-120\mathbf{h}},\\
a_{23} &= x^{-1457-1092\mathbf{i}-1457\epsilon-364\mathbf{h}}, \qquad a_{24}=x^{-1093+1092\mathbf{i}-1094\epsilon-2186\mathbf{h}},\\
a_{25} &= x^{-2550-2187\mathbf{i}-2551\epsilon-363\mathbf{h}},\\
& \dots\\
a_{3n-1} &= x^{1-2.3^{n-2}-\frac{1}{2}(3^{n-2}-1)\mathbf{i}+(1-2.3^{n-2})\epsilon-\frac{1}{2}(3^{n-2}-1)\mathbf{h}}\\
a_{3n} &= x^{\frac{1}{2}-\frac{3^{n-1}}{2}+2.3^{n-2}\mathbf{i}+(\frac{1}{2}+\frac{3^{n-1}}{2})\epsilon-(1-3^{n-1})\mathbf{h}}\\
a_{3n+1} &= x^{\frac{3}{2}-\frac{7}{2}.3^{n-2}-3^{n-1}\mathbf{i}+(\frac{1}{2}+\frac{7}{2}3^{n-1}3^{n-3})\epsilon+(\frac{3}{2}-\frac{3^{n-2}}{2})\mathbf{h}}.
\end{aligned}$$

By direct calculation it is easy to see that the sequence $CFH_{x,y}((n,2,2))$ conforms to the following pattern:

$$\begin{aligned}
a_0 &= x,\ a_1=y,\ a_2=y^{\epsilon}x^{\mathbf{i}}, a_3=x^{-\mathbf{i}-\epsilon}, a_4=y^{\epsilon}x^{-\mathbf{i}-\epsilon+\mathbf{h}},\\
a_5 &= x^{c_5},\ a_6=x^{c_6},\ a_7=x^{c_7}, a_8=x^{c_8},\ a_9=x^{c_9},\\
a_{10} &= x^{c_{10}}, a_{11}=x^{c_{11}}, a_{12}=x^{c_{12}}, \dots.
\end{aligned}$$

Since the sequence $\{c_n\}$ appears as the power of x and the order of x is n, the period of the sequence $\{c_n(n)\}$ with the cyclic-Fibonacci hybrid length of group $(n,2,2)$ are the same. So we have the conclusion. □

5. Conclusion

In this paper, we define the cyclic-Fibonacci hybrid sequence by considering hybrid numbers and Fibonacci recurrence. Firstly, we study the number theoretic properties of the sequence defined. Further, we determine the periods of the cyclic-Fibonacci hybrid sequence when reading modulo m by the matrix method. Finally, we extend the sequence stated to groups. Then we describe the cyclic-Fibonacci hybrid orbit of a 2-generator group and investigate it in non-abelian groups. Additionally, we obtain the lengths of the periods of certain classes of finite polyhedral groups as applications of the results produced.

As mentioned above, $(2, n, 2) \cong (n, 2, 2)$ and they are different presentations of the dihedral group D_{2n}. Our main purpose here is to show that the lengths of the periods of the cyclic-Fibonacci hybrid orbits of two isomorphic groups may be different.

6. Data Availability Statement

Our manuscript has no associate data.

7. Declaration of interest

The authors declare no competing interests.

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