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THE EVOLUTION OF BINARY SYSTEM FROM PREDYNASTIC EGYPT TO LEIBNIZ ERA

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Special issue dedicated to Professor Esfandiar Eslami
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Abstract. Egyptians of the predynastic era had a good decimal number system for counting and addition. Although, up to some times, they had problem in counting beyond a million, by the dawn of their history, Narmer, the founder of the first Egyptian dynasty had accountants that could record 400,000 cows and 1,422,000 goats of a war booty. Except for some ambiguities in the case of Mayan number system, specialists in the history of mathematics can guess that how the counting system of the various civilizations evolved into one of the number systems in base 10, 20, 60, etc. There is a puzzle in the mixture of the Egyptian decimal and binary number systems which we are going to discuss and present a justification for it. The novelty of the present paper is the study of the evolution of the binary number system from the predynastic Egypt down to the Leibniz era who, by the benefit of Khwarazmi's "Indian Arithmetics," completed this evolution by representing integers in $0-1$ forms and performing the hybrid decimal/binary Egyptian arithmetic operations purely inside the $0-1$ system. The second author is pleased to dedicate his share of this paper to Esfandiar Eslami showing his love and appreciation for decades of his friendship and collaboration (since 1967) and, of course, the young coauthor joins the joy of this dedication to her former professor.

Keywords: Egyptian fractions, Binary systems, Binary logic, Zero-one.
2020 MSC: Primary 01A11, 01A16, 01A50, 11-03, 11A63.

1. Introduction

The paper is more than a dedication; we have redirected our paper to fit the mathematical topic which are favoured by Professor Eslami. Before choosing the course, we searched for his general interests and found the following three important books authored or coauthored by him:

(A) (With J.J. Buckley) An Introduction to Fuzzy logic and Fuzzy Set Theory; Springer-Verlag, Berlin, Heidelberg GmbH, 2002.
(B) (With J.J. Buckley and T. Feuring) Fuzzy Mathematics in Economics and Engineering; Springer-Verlag, Berlin, Heidelberg GmbH, 2002.

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(C) Fuzzy Logic and its applications; Sh.B. University of Kerman, 2013/2014 (in Persian).

His earlier researches (including the Ph.D. program 1977-1981) were originally in classical, non-classical, and algebraic logic gradually switching to Fuzzy logic and its applications to computer sciences. It looked somehow disappointing! We wanted to dedicate a paper related to Esfandiar's interest and did not see that courage to go beyond the classic 0-1 systems. So, to be able to join his fans, we concretely asked him if he is still fond of binary logic, and he reasonably answered back the following:

"First, a logic is a discipline in which we communicate, express ideas, and evaluate arguments. We may call it the science of reasoning. One part of a logic is its semantics that evaluates the statements. A logic in which every statement is either false (0) or true (1) together with special axioms and rules of inference is called the *classical two valued logic*. If at least one of its postulates is failed, it is called a *non-classical logic*. There are many non-classical logics. One of the well-known non-classical logics is *fuzzy logic* whose semantics is different from our classical or mathematical logic which is usually used in mathematics or in an area where every thing is completely clear. Fuzzy logic has two versions: broad sense and narrow sense. Semantics of Fuzzy logic in broad sense is linguistic that is the statements are evaluated such as: true, very true, very very true, more or less true, ..., false, not true not false, ... We use these values usually in our daily life. Semantics of Fuzzy logic in narrow sense is usually the unit interval [0, 1]; 0 for completely false and 1 for completely true. This logic, some times called *mathematical fuzzy logic*, is an extension of classical two-valued logic. Fuzzy logic is used in areas such as engineering fields where there are some vague and uncertain concepts. It is used also in fields such as sociology, psychology, medicine, law, philosophy,..., in which most notions are not well-defined but are vague or uncertain. In fact, in every field of study we may use a suitable logic and I some times say that every human being has his/her own logic. If we understand each other it seems that we have common logical facts. Otherwise, we can not even talk to each other."

So, we got enough motivation to write the present paper regarding the roots of Boolean algebra and binary number system which formed a basis for the modern binary logic. In Section 2, we show that how a simple distribution of bread among the workers and farmers on the banks of Nile river sparked the binary fractions and how they were developed into binary integers and enriched the Egyptian multiplication and division. With the fall of Egyptian dynasties, their fractions were transfered to Europe labeled as "Greek made" and the Egyptian binary number system lacking the essential place value property, waited for a millennium to be revived and completed, of course independently, by Leibniz and others. (See Section 4). The intermediate Section 3 studies the historical events related to the binary systems between the fall of Pharaohs and the Leibniz era.

We conclude this introductory section with a quick review of the history of fractions. At the dawn of the history, civilizations needed natural numbers for counting and measuring of time, length, area, volume, weight, etc. Before astronomers to realize that a solar year was at lest 365.25 days, in most middle eastern countries, it was a 360−day year consisting of twelve 30-day months. This made it easy for astronomers to find $\frac{m}{n}$ of any number k of years for all natural numbers m and k and any divisor n of 360. (See [17].) For example, the fraction $\frac{1}{120}$ of 250 years can be easily computed as $250 \times \frac{360}{120} = 750$ days = 2 years and 1 month. This could also mean that if 250 loaves of bread had to be distributed among 120 workers on an Egyptian construction site, each worker must get 2 loaves and one-twelfth of a loaf. Certainly, this is not a fast and practical way of distributing bread among the hungry workers; in the next section we learn how Egyptians would make the distribution of bread easy even in the case the number of the loaves was quite arbitrary. Around the same time Chinese had a 364−day solar year consisting of thirteen 28−day months each with exactly 4 weeks. We do not know if ever Chinese used fractions $\frac{1}{13}$, $\frac{1}{26}$, etc. in their calculations, but one may conjecture similar fraction in Mayan Calendars. (See [17].)

2. Egyptian Binary number system

Egypt seems to be the only civilization that used two different number systems in prehistoric times. One for counting and addition, another for fractions. The first one was decimal and the second one was binary. It seems that they first developed the decimal system for their day-to-day counting and later used the binary fractions $\{1/2, 1/4, 1/8, 1/16, 1/32, 1/64\}$ for sharing symmetric goods like bread loaves. The latter fractions were used by the farmers who needed to divide a (regular round) loaf of bread into equal sectors through a series of halving and re-halving. The halving would stop at 1/64 as such a slice of bread was too slim to be halved further. Some authors believe that the ancient Egyptians "used binary systems for multiplication of two numbers, a procedure today known as the peasant multiplication" [15]. The main goal of the present paper is to show that the Egyptian "binary fractions" preceded its so-called "peasant multiplication".

The binary number system entered Egyptian multiplication and division with no approximate time to be known. In this section we put different pieces of the excavated documents to reconstruct a history for the appearance of the Egyptian binary system. However, the (non-positional) decimal number system and the corresponding digits for the numbers $1, 10, 100, 1000, \cdots$ were in use by predynastic Egyptians. See Fig.1 showing the number $1,333,330$; the photo lacks a stick $|$ to contain all the symbols needed for writing the numbers $1 - 9,999,999$. The symbols function like various coins; for example 203 can be written in many ways as follows:

$$\rho\rho||| = |||\rho\rho = |\rho||\rho = |\rho|\rho| = \cdots,$$

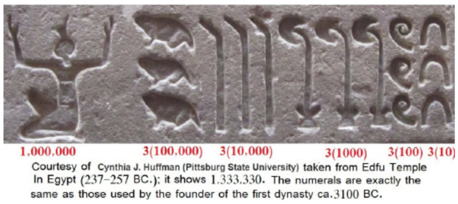

Figure 1. Edfu Temple

where $\rho = 100$ and $| = 1$.

Elementary additions and multiplications were like counting the coins and exchanging the smaller denominations by the larger ones as far as possible. For subtraction, one only needed enough small denominations to replace the large ones by smaller ones if needed. For example,

$$25 + 18 = \cap\cap\,|||||\,\cap\,|||||||| = \cap\cap\cap(||||||||||)||| = \cap\cap\cap\cap\,||| = 43,$$

where $\cap = 10$. For $25 - 18$, one has practically to exchange 1 dime for 10 pennies to get

$$25 - 18 = \cap\cap\,||||| - (\cap||||||||) = \cap||||||||||||||| - (\cap||||||||) = ||||||| = 7.$$

(The scribe's mint worked very simple; All he needed was a pen and a washable tablet to draw as many coins as necessary and to cross or wipe out any ones not needed.)

A monumental wall painting exists showing the founder of the first dynasty (ca.3100 B.C.) boasting on the large numbers of his soldiers, captives, booties, etc. One of these numbers shows 1,422,000 goats which we believe its accurate value is between 1,421,000 and 1,423,000. This shows that 1,000,000 was an accessible ordinary number for the Egyptians of 5000 years ago. However, the word for million was "Heh" which meant "chaos" or "infinity" and was the name of a God. They also used it to show infinity, a number bigger than any number that's ever been written. Comparing with the word "Three" derived from the Indo-European "Throng" meaning "too many" [9], or the equivalent of "Four" in the Africans' language reported in George Gamow's "One, Two, Three, Infinity", reveals that the prehistoric Egyptians were very advanced in counting and writing large quantities.

Thus, at the dawn of the history, the Egyptians could easily count over a million, and add, subtract, double, or triple very large numbers. With the symbols they had for 1, 10, 100, etc., they could easily multiply any number by 10. This is in fact done in some papyri but not frequently. However, there is no Egyptian algorithm for the product of two arbitrary decimal numbers.

The Egyptian golden age came a millennium after the foundation of the first dynasty. The following so far excavated important documents containing almost all of the Egyptian mathematical heritage appeared during this period.

These documents are listed below with their usual abbreviations:

AWT	Akhmim Wooden Tablets, ca.1990 B.C.
EMLR	Egyptian Mathematical Leather Roll, from 2000 to 1850 B.C.
MMP	Moscow Mathematical Papyrus, ca.1850 B.C.
AMP	Ahmes Mathematical Papyrus, originally dated ca.1850 B.C.

The less important (BMP) Berlin Mathematical Papyrus (21st century B.C.), (LMP) Luhan Mathematical papyri (19th century B.C.), and (RP) Reisner Papyrus (18th century B.C.) contain partial mathematical results found in the above mentioned four documents. All these documents are written during the Egyptian golden age of science, economy and social/political prosperity [9], [16] beginning by the 11^{th} dynasty ca.$(2150-1991)$ B.C. and ending with the 12^{th} dynasty ca.$(1991-1783)$ B.C. Amenemhat I, the founder of the 12^{th} dynasty (ca.1991 B.C.) was the minister/co-regent of the last king of the 11^{th} dynasty who (most probably) overtook the power peacefully after it became clear that there was no eligible ruler from the 11^{th} dynasty to continue. Therefore, the AWT could be claimed to have been written by his order. Also, the most important pharaoh from the 12^{th} dynasty and one of the most just rulers of the world was Amenemhat III who first ruled Egypt as a co-regent of his father since 1861 or later and as a king from 1841 until 1797 B.C. Therefore, it is most probable that the four most important Egyptian mathematical documents were written during the time of either Amenemhat I or his descendant Amenemhat III. In fact, Ahmes clearly asserts that he is copying a document which was written at the time of the king of the upper and lower Egypt, Ni-Maat-Re. The latter name was the religious name of Amenemhat III and meant "belonging to the truth of the Sun". Another name for AMP is RMP (Rhind Mathematical Papyrus) named after the Scottish antiquarian Alexander Henry Rhind; the papyrus sold to him in 1858 A.D. was probably found during illegal excavations.

The notion of the decimal number system was known to the Egyptians of the prehistoric times; but, its evolution to the place value number system was not completed until the ninth century A.D., half a millennium after the Roman sack of Egypt (30 B.C.). The development of the Egyptian binary system was not continued by Greeks and Romans who inherited and expanded their science and technology. No one cared to acknowledge the Egyptian share or to investigate how their binary system came into being. In fact, people of the Middle Ages thought that Egyptian fractions were invented by Greeks. Here, we investigate this development and try to share its credit between Egyptians and the Englishman Thomas Harriot (1560-1621), the Spaniard Juan Caramuel de Lobkowitz (1606-1682) and, most importantly, the German Gottfried Wilhelm Leibniz (1646-1716) the latter three living in a scientific era equipped with a full positional decimal number system.

A careful and logical study of the Egyptian mathematical documents reveals that the prehistoric people living along the Nile river had a prime need in distributing bread among the workers and farmers. All these documents show that bread was an item deeply mixed with their daily life and economy. All their mathematical documents, even in the time of their later dynasties, used "bread" as a unit and, in some problems that the unit was not specified at the beginning, their answers ended up with bread as the unit.

According to historians, "Egypt had no cash economy until the coming of the Persians in 525 B.C. ... Egypt operated on a barter system up until the Persian invasion of 525 B.C. and the economy was based on agriculture. ... Laborours were often paid in bread and beer, the staples of the Egyptian diet." [12]. Or "Bread is called in most Arab countries **khobz**. But in Egypt, it is called **Eish**, meaning living (see Fig.2). The word connotes the salience of bread – which was once the method of payment to workers who built the pyramids in Ancient Egypt- in the lives of Egyptians" [2]

FIGURE 2. Eish

A loaf of bread, as a unit, had no multiples. To divide a limited amount of bread among the workers, needed no advanced mathematics; the employer would hand out one loaf of bread at a time to each worker and repeat for the second, third,$\cdots$ rounds until a remainder (strictly smaller than the number of workers) is left. (If the amount of bread was large enough, the employer could save time by giving 2 or more breads at a time to each worker for the first few rounds.) To continue the division was where the submultiples of the unit were really needed. A loaf, as a unit, had six submultiples **half** ($= 2^{-1}$), **quarter** ($= 2^{-2}$), **one eighth** ($= 2^{-3}$), **one sixteenth** ($= 2^{-4}$), **one thirty-second** ($= 2^{-5}$) and **one sixty-fourth** ($= 2^{-6}$). (See Fig. 3.) Since the circular bread and its sectors had (theoretically) axes of symmetry, folding was the easiest way to divide a loaf of bread into its submultiples. In fact, the Egyptian symbol for "half" was $\supset$ which reminds of something being bent or folded. The tiny slice 2^{-6} was so narrow that could not be halved with hand. There was no prejudice against the fraction 2^{-6}; if the remaining slices of size 2^{-6} were less than the number of the workers, the boss could leave them to be consumed by the passersby including birds, or even workers themselves.

Example 2.1. *Imagine a farm owner of five or six thousand years ago on the bank of the Nile river wants to divide* 22 *loaves of bread among* 6 *labourers. The naive owner of the farm starts giving* 1 (*if smart enough,* 2 *or, if smarter,* 3) *loaves at a time to each farmer and repeats until each labourer gets* 3 *loaves. That is the first stage of division is*

$$22 = 6 + 6 + 6 + 4 = 3 \times 6 + 4; \text{ (note that } 0 \leq 4 < 6).$$

It remains to find $4 \div 6$ *to its* 6^{th} *submultiple. Today, a modern educated farm owner would continue the division to two decimals to get* $22 \div 6 = 3.66$. *Thus, each worker gets* 3 *whole loaves,* 6 *tenths, and* 6 *hundredths. Being almost impossible for the farm owner of the antiquity, he/she would switch to binary fractions and enjoy the ease of halving* 4 *loaves to yield* 8 *halves. Each worker gets an extra* 1 *half and* 2 *halves remains. Further halving yields* 4 *quarters coming short of* 2 *and, hence, halving is continued to yield* 8 *eighths. Now, each farmer gets* 1 *with* 2 *eights remained. By twice re-halving of the remaining slices,* 8 *thirty-seconds are obtained and each farmer gets one slice. Halving the remaining* 2 *thirty-seconds, yields* 4 *sixty-fourths which can not be halved further and the division comes to an end, practically.* (*He may leave the remaining* 4 *sixty-fourths to be shared brotherly.*) *In total, the amount of bread that each farmer gets is the sum of the decimal number* 3 *and the binary fraction*

$$(1) \qquad 4 \div 6 \approx \ '2^{-1} + 4^{-1} + \ '8^{-1} + 16^{-1} + \ '32^{-1} + 64^{-1},$$

where the notation $'\gamma$ *means the summand* γ *appears in the quotient.* (*Egyptians would write the right-hand side of* (1) *as a column vector and the quotient is the sum of those entries* γ *which are marked as* $'\gamma$.) *This Egyptian binary expansion* (1) *is equivalent with the modern representation* $(0.101010)_2$ (*in binary number system*).

With the modern terminology, the Egyptian mathematicians have shown that if $a < b$ are positive integers, then

$$(2) \quad \begin{aligned} a \div b \ &\geq \ x_{-1}2^{-1} + x_{-2}2^{-2} + x_{-3}2^{-3} + x_{-4}2^{-4} + x_{-5}2^{-5} + x_{-6}2^{-6} \\ &= \ (0.x_{-1}x_{-2}\cdots x_{-6})_2 \ > \ (a \div b) - 2^{-6}, \\ &\quad \text{with } \ x_{-i} \in \{0,1\}; \ \ 1 \leq i \leq 6. \end{aligned}$$

In particular, if 63 loaves of bread were supposed to be distributed among 64 workers, each person would get half a loaf in the first round and 62 halves would remain. In the next round, each worker gets 1 quarter and again 61 quarters are left and so on. Thus, at the end, each worker gets one loaf of bread minus a slice of size 2^{-6}; i.e.,

$$(3) \qquad 63 \div 64 = (0.111111)_2 = (1 - 0.000001)_2 \approx 1.$$

Hence, each worker gets approximately the whole bread shown in Fig. 3.

The predynastic simple numerical approximation (3) got a mystical fate by the postdynastic priests and politicians. The founder of the first dynasty is credited for the unification of the two upper (southern) and lower (northern) Egypt. The bitter and sad memories of the unification wars were not forgotten for several years or centuries. Politicians and priests made stories that these fierce battles were between the various deities. In one of these wars, an upper Egyptian deity called Seth (a god of deserts, storms, disorder, violence, and foreigners) tore apart an eye of Hur (Persian Khur or Hur, Greek Horus, and the lower Egyptian god of kingship, healing, protection, sun and sky) into six pieces and threw them among the thick bushes on the banks of the Nile river. Anyway, the kind supreme god helped Hur to find them and recover his eye; so, they were named the Hur's eye fractions and priests would write them on armbands to be sold for body safety and perfection. As a result, it was falsely believed that the following theorem was true.

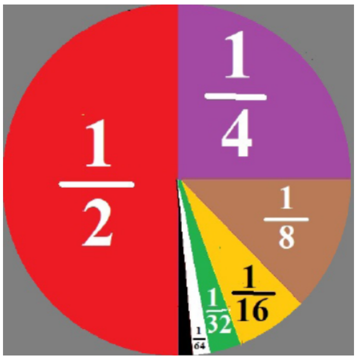

FIGURE 3. [$1 \approx 0.111111$ in base 2]

Theorem 2.2. (False) *The sum of the Hur's eye fractions is exactly* 1*; i.e.,*

$$2^{-1} + 2^{-2} + 2^{-3} + 2^{-4} + 2^{-5} + 2^{-6} = 1.$$

Centuries later a smart young priest, enjoying the cool weather of the temple and using pen and paper playing with the (holy) Hur's eye fractions, noticed that multiplying the sum of the six quantities by 64 yields 63. He hurriedly reported the flaw to the high priests but as in the case of Pythagoreans who were caught surprised by the irrationality of $\sqrt{2}$, the priests tried in vain to suppress the news [9]. As was expected the news spread and the common denominator became a new excitement for the mathematics lovers. An important question was, for example, what happens if in (1) the division $4 \div 6$ is replaced by $256 \div 6$ in which the 4 (full) loaves of bread are replaced with $4 \times 64 = 256$ slices of size 2^{-6}; in fact, Section 5 of AWT contains a much more difficult example in which $32431 \div 926$ is replaced by $2075166 \div 926$, where $2075166 = 32431 \times 64$. (See also Section 2 of [16].) For the moment, we lower the computational level of our examples to fit the knowledge of the mathematicians of the first few dynasties (few centuries before writing AWT). Getting back to (1), we observe that $256 = 42 \times 6 + 4$, by which each worker gets 42 slices of size 2^{-6} and 4 slices are left for the passersby! In comparison, the share of each worker in (1) was a subcollection $\mathcal{A}$ of the collection $\{2^{-1}, 2^{-2}, \cdots, 2^{-6}\}$, while, by the new method, each worker gets β copies of the submultiple 2^{-6}, where $\beta = 2^6 \sum \mathcal{A}$; i.e.,

$$\begin{aligned} \beta &= 42 = 2^6('2^{-1} + 2^{-2} + '2^{-3} + 2^{-4} + '2^{-5} + 2^{-6}) \\ &= '2^5 + 2^4 + '2^3 + 2^2 + '2^1 + 1 = (101010)_2. \end{aligned} \tag{4}$$

In particular, if $32 \leq a < b = 64$ in (2), then $x_{-1} = 1$, the approximation is exact, and

$$a = 2^5 + x_{-2}2^4 + x_{-3}2^3 + x_{-4}2^2 + x_{-5}2^1 + x_{-6}2^0. \tag{5}$$

This proves that Egyptians got the binary expansion of any positive integer $2^5 \leq a < 2^6$. On the other hand, the flaw in Theorem 2.2 as well as the criticism of the pharmacists about the inadequacy of Hur's eye fractions destroyed their sanctity and gave enough courage to mathematicians to ignore the temple's opposition and expand the definition of fractions to any quantity of the form $1/n$. In particular, if $2^n \leq a < 2^{n+1}$, the expansion (5) is generalized as

$$a = 2^n + a_{n-1}2^{n-1} + a_{n-2}2^{n-2} + \cdots + a_0 2^0, \tag{6}$$

where $a_j \in \{0, 1\}$ for $j = 0, 1, 2, \cdots, n-1$.

Now, the simple multiplication of $2^n \times b$ can be extended to any product $a \times b$ for any pair of integers a, b as is shown in the following example.

Example 2.3. *To find 11×15, we write the powers 2^i in the first row of Table 1 and search for the admissible ones. First find n such that $2^n \leq 11 < 2^{n+1}$. Since $8 \leq 11 < 16$, it follows from (6) that $n = 3$ and $11 = 8 + 2^m + \cdots$ or, equivalently, $2^m \leq 11 - 8 = 3 < 2^{m+1}$. That is $m = 1$. So far, $11 = 2^3 + 2^1 + 2^k + \cdots$ or, equivalently, $2^k \leq 11 - 8 - 2 = 1 < 2^{k+1}$. Again, it follows that $k = 0$ and $11 = 2^3 + 2^1 + 2^0$. In modern sense $11_{10} = (1011)_2$. Multiplying these summands into 15 as shown in the second row and adding them up yields $11 \times 15 = 15 + 30 + 120 = 165$. Recall that Egyptians had no name or symbol for our modern 0 and it was convenient for them to work with the transpose of our Table 1. Their multiplication table had two columns. In the forst column, they would write the powers 2^n until they reach $2^3 = 8 \leq 11 < 2^4 = 16$. They would then mark $'8$ as the highest power of 2 that is admissible. Next, they would go back to find the first power of 2 whose sum with 8 does not exceed 11; that is 2 yielding $8 + 2 = 10 < 11$. So, they would mark $'2$ as admissible. Finally, adding 10 to $1 = 2^0$ yields $11 = 2^3 + 2^1 + 2^0$ which yields the final admissible power of 2. Therefore, in this example $4 = 2^2$ is not marked as an admissible power of 2. In modern terminology, $11_{10} = 1011_2$.*

Similarly, we can divide integers by using binary representations.

Example 2.4. (Division with remainder) *To find the quotient x and the remainder r of the division $250 \div 9$, we observe that $250 = 9x + r$ with $0 \leq r < 9$.*

TABLE 1. 11×15

2^i	$'1$	$'2$	4	$'8$	16
$2^i \times 15$	15	30	60	120	-

Expand x as in the first row of Table 2 and find $2^i \times 9$ as in the second row. Since x is unknown we do not know where to stop. Instead, we know in the second row that $2^i \times 9$ cannot exceed 250. Therefore, we first calculate the second row to reach $144 = 2^4 \times 9 < 2^5 \times 9 = 288$. Thus, $'144$ is marked as such. Now, $144 + 72 = 216 \leq 250$ and so $'72$. Next, $216 + 36 = 252 > 250$ and thus 36 is rejected. Now, $216 + 18 = 234 < 250$ and $234 + 9 = 243 < 250$ mark $'18$ and $'9$ as admissible. Hence, $x = \ '16 + \ '8 + 4 + \ '2 + \ '1 = 27$ and $r = 250 - 243 = 7$. (In modern terms $27 = (11011)_2$.)

TABLE 2. $250 = 9x + r$

2^i	1	2	4	8	16	32
$2^i \times 9$	$'9$	$'18$	36	$'72$	$'144$	288

The last two examples as well as the arguments given in the conclusion of this section justify our claim that, in spite of no place value number system, Egyptians achieved a high level of mathematics in dealing with binary number system; a system which was re-discovered by Leibniz almost three millennia later in Europe. The next example shows further how they would continue the divisions to obtain sharper (fractional) quotients with ignorable (fractional) remainders.

Example 2.5. (Division without remainder) *Sharpen the division $250 \div 9$ by using Egyptian allowable binary fractions.*

Predynastic Metod: *In a farm we have to distribute 250 loaves of bread among 9 labourers. In the first round we give one bread to each family and repeat the rounds for 27 times. (Since the quantity of bread is large enough, we may save time by handing out 5 loaves at a time and continue until we fill the quantity has shrunk enough and switch to the safe "1 loaf at a time".) There remains 7 loaves and continue by halving and re-halving according to Example 2.1. Thus, each labouror receives:*

$$27 + \frac{1}{2} + \frac{1}{4} + \frac{1}{64} \text{ loaves of bread.}$$

Akhmim Method: *Multiply the number of loaves by 64 to find the total number 16000 of (possible) slices of size 2^{-6}. (See Table 3.) Now, we follow Table 4 to find $16000 \div 9$ with error $< 2^{-6}$.*

Since $9216 < 16000 < 18432$, since $9216 + 4608 = 13824 < 16000$, since $13824 + 2304 = 16128 > 16000$, since $13824 + 1152 = 14976 <$

Table 3. $64 \times 250 = 16000$

2^i	1	2	4	8	16	32	$'64$
$2^i \times 250$	250	500	1000	2000	4000	8000	16000

Table 4. $16000 \div 9$

Row 1	2^i	1	2	2^2	2^3	2^4	2^5
Row 2	$2^i \times 9$	$'9$	18	$'36$	72	$'144$	$'288$
Row 1	*Cont'd*	2^6	2^7	2^8	2^9	2^{10}	2^{11}
Row 2	*Cont'd*	$'576$	$'1152$	2304	$'4608$	$'9216$	18432

16000*, since* $14976+576 = 15552 < 16000$*, since* $15552+288 = 15840 < 16000$*, since* $15840+144 = 15984 < 16000$*, since* $15984+72 = 16056 > 16000$*, since* $15984 + 36 = 16020 > 1600$*, since* $15984 + 18 = 16002 > 16000$*, and since* $15984+9 = 15993 < 16000$*, it follows that only* $'9216$, $'4608$, $'1152$, $'576$, $'288$, $'144$*, and* $'9$ *are marked admissible and, hence, the remainder is* $16000 - 15993 = 7$.

Therefore, $16000 \div 9 = 2^{10} + 2^9 + 2^7 + 2^6 + 2^5 + 2^4 + 2^2 + 1 + (7 \div 9)$ *copies of the submultiple of size* 2^{-6} *or, equivalently,* $250 \div 9 = 2^4 + 2^3 + 2 + 1 + 2^{-1} + 2^{-2} + 2^{-4} + 2^{-6} + (7 \div 9) \times 2^{-6} = 27 + 2^{-1} + 2^{-2} + 2^{-4} + 2^{-6} + (7 \div 9) \times 2^{-6}$ *loaves of bread. Thus,* $250 \div 9 \approx (11011.110101)_2$.

Simplified Akhmim Method: *By division with remainder,* $250 \div 9 = 27 + (7 \div 9)$. (*See Table* 2.) *It remains to apply Akhmim method to* $7 \div 9$. *First observe that* $7 \times 64 (= 1 \times 64 + 2 \times 64 + 4 \times 64 = 64 + 128 + 256) = 448$ *and, hence, we continue by the following Table* 5 *to find* $448 \div 9$ *without remainder.*

Table 5. $448 \div 9$

2^i	1	2	2^2	2^3	2^4	2^5	2^6
$2^i \times 9$	$'9$	18	36	72	$'144$	$'288$	576

Since $288 < 448 < 578$*, since* $288 + 144 = 432 < 448$*, since* $432 + 72 = 504 > 448$*, since* $432 + 36 = 468 > 448$*, since* $432 + 18 = 458 > 448$*, and since* $432 + 9 = 441 < 448$*, we have marked* $'288$, $'144$ *and* $'18$ *as admissible, and have rejected the rest. Therefore,* $7 \times 64 = 9(2^5 + 2^4 + 2^0) + 7$ *and, hence,* $7 \div 9 = 2^{-1} + 2^{-2} + 2^{-6} + (7 \div 9) 2^{-6} \approx 2^{-1} + 2^{-2} + 2^{-6}$*, with an error* $< 2^{-6}$.

We now conclude this section by reviewing the various evolutionary stages of the binary number system that was shaped in the hands of Egyptians from predynastic to postdynastic times. Recall that Egyptians of those days had no knowledge of place value number system and had no perception of zero as a number or of 0 as a digit. Therefore, our modern formulas should be interpreted in the language of the ancient Egyptians.

Stage (i) Binary fractions: The naive prehistoric farmers on the banks of the Nile river used bread as monetary to pay workers. If the number of the (regular round) loaves of bread was greater than or equal to the number of workers, they would simply give one bread to each worker and repeat the round until the number of the loaves reduces to a number strictly less than the number of workers. So far all the numerals were communicated orally or pictorially in the (no-place valued) decimal number system. For the remainder they would use the submultiples $2^{-1}, 2^{-2}, \cdots, 2^{-6}$ and ignore the smaller ones. Thus besides an integral decimal number of the loaves, each worker would also get 1 or 0 slice of size 2^{-n}, $n = 1, 2, \cdots, 6$. That is, Egyptians of predynastic times would approximate any proper fraction $a \div b$ (with $a < b$) as a $(0, 1)$-sequence of 6 terms after the binary point. (See Examples 2.1, 2.4, 2.5.) As far as the distributor was concerned the remaining slices (of size 2^{-6}) were ignored. However, if the boss in Example 2.1, say, decides to give 1 or more of the remaining slices of size 2^{-6} to some of the workers as bonus, then each modified share would be represented by a unique binary fraction; for example if one gets two extra slices, one's share would be $0.1011(= 0.101010 + 0.000001 + 0.00001)$.

Stage (ii) Binary integers: In predynastic times, Egypt was divided into two Southern and Northern countries with capitals almost one thousand kilometers apart which had different deities and religious beliefs. However, thanks to the Nile waterway they shared the same language and mathematics. In particular, the binary fractions were developed more or less at the same time in the two lands. The founder of the first dynasty is credited for his unification of the two Egyptian lands after a long civil war. However, the war between deities never ended and persisted almost to the last dynasty. Akhmim tablets were written during the Egyptian golden age of science and economy and shows that the northern deities had an upper hand. The sun God Hur monopolized the binary fractions as the six parts of his eye; Assuming their sum is equal to 1 made a barrier on the further development of the binary fractions. On the other hand, pharmacist needed sharper approximations for fractions of their quantities, pushed for smaller fractions and removed the lower barrier 2^{-6}. Mathematicians helped them by generalizing the Hur's eye fractions to any unary fraction of type $1/n$ for all integers $n > 1$. Even, except for the beloved fraction **"two-thirds"** and the misfortune fraction **"three-fourths"** which had their own stories, any other non unary fraction was not regarded as a quantity unless it was written as the sum of distinct unary fractions. For example $2 \div 5$ was an "action" whose result was any "quantity" of the form

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{30}, \quad \text{or } \frac{1}{15}+\frac{1}{3}, \quad \text{or } \frac{1}{15}+\frac{1}{12}+\frac{1}{4}, \quad \text{etc.}$$

Of course, for a long time the (damned) fraction "one third" had no place in Egyptian arithmetic and could only be viewed as the combination

$$\frac{1}{2}(\frac{2}{3});$$

the one on the left being a holy Hur's eye fraction, and the one on the right being the fraction of the solar year in which the generous Nile was peaceful. Later that the unary fractions $1/n$ were permitted to the Egyptian arithmetic, it could be also represented as the sum of distinct unary fractions. Although, the latest mathematical documents had no discrimination among the unary fractions, other fractions, such as $2/(2n+1)$ were not allowed in and took a lot of energy to be interpreted as quantities.

This could bring a halt to the development of the binary number system but, anyway, a different stage of evolution shaped up.

Stage (iii) Binary multiplication/division: The discovery of "common denominator" of the Hur's eye fractions by elite mathematicians (not naive farmers) resulted in the discovery of the binary expansion of any positive integer $a < 64$, see (5) and then, by demystifying the Hur's eye fractions, of any positive integer, see (6). This helped Egyptians to extend the multiplication $2^n \times a$ to the multiplication of any two positive integers (see Table 1). Finally, by a reverse operation, they discovered methods for division with remainder (see Table 2) and without remainder (see Tables 3-4).

Remarks Not every progress in mathematics was helpful in the further development of the Egyptian binary number system. We already mentioned that the introduction of unary fractions reduced the importance of the Hur's eye fractions as well as other binary ones. Also, the ease of multiplication by 10, 100, 1000, etc. was another setback to the use of binary number system. For example, to do 117×66, Egyptians found it easier to use less binary integers in the following way:

$$\begin{aligned} 117 \times 66 &= 66(100) + 17 \times 66 \\ &= 6(1000) + 6(100) + (1 + 2^4)(66) \\ &= 6(1000) + 6(100)) + 66 + 1056 = 7722. \end{aligned}$$

Recall that the Egyptian symbols for $1, 10, 100, 1000, \cdots$ were like various coins and addition was like counting the coins and exchanging the smaller denominations by the larger ones as far as possible.

3. Zero-one systems before Leibniz era

What we regard as the modern binary expansion of a number was not possible to be done by ancient Egyptians; they had no knowledge of place value number system and were not ready to digest the notion of a number called zero and symbolized as 0. Therefore, there was a limitation on their representation of binary numbers. For example, the modern binary expansion of 11 is as 1011 which meant $1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$. However, Egyptians would write these numbers as a column vector and mark those entries which must participate in the sum as shown in the first row of Table 1. (See below.)

$$('1,\ '2,\ 4,\ '8,\ 16)^T.$$

Midway to Leibniz zero-one number system, we have few other mathematical or non mathematical concepts which, like Egyptian binary numbers, can be regarded as pre-zero-one systems. One of these is the principle of non-contradiction in logic which like Euclid's parallel postulate, Zorn's lemma, axiom of choice, etc. is accepted as an axiom. In modern binary logic, as mentioned in the introduction, we assign a value 1 (resp. 0) to any so-called statement P if it is true (resp. false). The principle of non-contradiction asserts that the truth value of a statement P is 1 if and only if the truth value of its negation $\neg P$ is 0. Therefore, to prove a theorem, one can assume the contrary of the conclusion and show that the hypothesis is false. Yet, the mathematicians are displeased to use the proof by contradiction and would avoid it if possible. Mahani, whose name honours the present journal as well as its sponsor "Mahani Mathematical Research Centre," was probably the only moslem mathematician who, following the Greek mathematicians, was reluctant to use such proofs. We have the title of a lost paper by Mahani which indicates such a reluctance:

Mahani, Abu-Abdollah Mohammad-ebn Isa; ca.850 A.D. *On twenty six theorems of Euclid's Elements which can be proved by "non-contradiction" method.* ([4], p. 433)

Recall that in Aristotle's logic, every statement P is either true or false; moreover the Aristotle's principle of "non-contradiction" asserts that if a statement P is true, then its negation $\neg P$ is false. Although Mahani was using the binary logic and Khwarazmi had introduced the notion of 0 to the Islamic world almost fifty years before him, it was too early for Mahani and others to use the $0-1$ language in their logical statements.

Another mathematical concept related to the binary number system is the dyadic number system which is taken by some as a synonym. By fixing our definition of dyadic number system, we make it clear that the two concepts are different in the present paper.

Definition 3.1. Let x be a positive integer.

(i): Dyadic expansion. By the dyadic expansion of x, we mean a series of the form

$$x = \sum_{0 \leq j \leq k} x_j 2^j,$$

for some nonnegative integer k and some $x_j \in \{1, 2\}; (0 \leq j \leq k)$.

(ii): Binary expansion. By the binary expansion of x, we mean a series of the form

$$x = 2^k + \sum_{0 \leq j \leq k-1} x_j 2^j,$$

for some nonnegative integer k and some $x_j \in \{0, 1\}; (0 \leq j \leq k-1)$.

We leave it to the reader to show that every positive integer x has a unique dyadic as well as a unique binary expansion; moreover, the definitions can be extended to any real number x. Note that, ancient Egyptians knew without proof that the binary expansion existed and was unique. Some ethnographers of the nineteenth century who visited the Australian aboriginals reported that the natives' counting system was binary.(See [8] and [6].) If that was the case, then we have to credit them as pioneers of the binary number system alongside with the ancient Egyptians. Here, we briefly show that the report is misleading and their counting is a kind of pseudo-dyadic system which is neither binary nor dyadic.

To prove our claim, we examined all the examples reproduced in [8] and noticed that when all of them are translated into English, they follow the pattern of the fourth column of the Table 6 below, where the symbols "1" and "2" stand for the pronunciation of the aboriginal words "one" and "two", respectively. (See [19] for the variety of languages.) It seems that they had no way for writing their numbers or even they might had no alphabet as well. The word "ras" for 7 and any other number beyond 7 indicates that the visited tribes could not count beyond 6 and would regard all other numbers as infinity. (See a similar phenomenon reported in [3] regarding the numbers beyond 3.) Generally, one may formulate the Australian aboriginal pronunciation of a decimal integer n as Tally(k,"2") if $n = 2k$ and as Tally(k,"2")"1" if $n = 2k + 1$, where Tally(k, †) = † † ⋯ † (k times).

Another prehistoric related concept is the classical Chinese book "I Ching" which was a tool for "kind of" fortune-telling written in antiquity to help people, especially kings and other authorities, to make better decisions! The standard I Ching had 64 chapters each advising the king what to do or not to do about the matter for which I Ching was consulted. Each chapter is labeled by an ordered sequence of length 6 made by two characters a and b. To tell the fortune, the fortune-teller would toss a coin six times and record the outcomes head= a or tail= b in the order of throwing to get a sequence, say, "*bbabaa*" corresponding to a chapter in I Ching. The consultee then interprets the chapter for the customer. The order of the chapters differ from one edition to another; obviously, one of the good orders is something as follows: *aaaaaa*, *aaaaab*,

Table 6. Aboriginal Counting

n	Binary	Dyadic	Aboriginal
1	1	1	"1"
2	10	2	"2"
3	11	11	"2" "1"
4	100	12	"2" "2"
5	101	21	"2" "2" "1"
6	110	22	"2" "2" "2"
7	111	111	ras
8	1000	112	ras

aaaaba, *aaaabb*, *aaabaa* ..., *bbbbbb* which is equivalent with the natural order of the integers $0 = (000000)_2$, $1 = (00001)_2$, $2 = (000010)_2$, $3 = (000011)_2$, $4 = (000100)_2$,..., $63 = (111111)_2$.

Chinese artists have symbolized a as a broken line and b as a solid one and have created various forms of I Ching. One of these was sent to Leibniz which will be discussed in the next section. A clean and more informative I Ching can be seen in [11] on which we have added Arabic numerals as demonstrated in Fig. 4. It has several rings made of characters. The outer ring consists of 365 small circles representing the number of the days in a standard solar year. The second ring consists of 64 hexagrams beginning from 0 at the lowest point of the ring and counts anticlockwise upward to 31; next to it is 63 and counting continues downward to 32 (adjacent to 0). The Arabic numerals $0 - 32$ on the right and $32 - 63$ on the left are written by us. On the third ring there are thirteen $(4 \times 7)-$arrays representing a 364-day year divided into 13 equal 28-day months. Note that each solar year contains at least 12 and at most 13 full moons. Although not of interest to us regarding the study of the binary number system, it may, however, throw some light on the historical background of I Ching and the generality of the bad omen of number 13 [17]. Each of the fourth and the fifth rings represent the 8 trigrams *aaa*, *aab*, *aba*, *abb*, *baa*, *bab*, *bba* and *bbb*, which we have labeled by Arabic numerals $0, 1, 2, 3, 4, 5, 6, 7$. In fact, the second ring is the Cartesian product of the fourth and the fifth rings; that is, each hexagram, say *bbabaa*, is made of two trigrams (bba, baa) each having a certain meaning to the interpreter. Finally, in the center of the plate, we see two pentagons each interpreting the four main elements shaping up the universe plus a supplementary one.

Although we do not claim that Chinese would start counting the chapters of I Ching from 0, yet, we can credit them for getting so close to the idea of the binary number system. But two things should be cleared: (i) Chinese had no intention of numbering the chapters of the I Ching, and (ii) Leibniz was really working with the binary number system and got excited to see a resemblance of his findings with the hexagrams in I Ching.

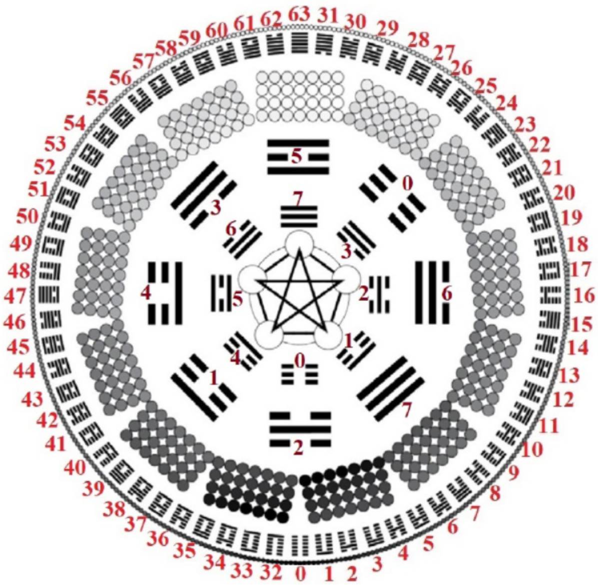

FIGURE 4. A Chinese plate

4. **The Leibniz Era**

Leibniz (1646-1716) did great works in mathematics. But he was so eager for social and religious reform that he could not resist mixing his valuable mathematical achievements with things which may be regarded as superstitions today. In fact, mathematics for him was a secondary goal. As is mentioned in [13], Leibniz became interested in Chinese culture as early as his age of twenties. He has been referred to as the last universalist or "universal genius" with interests and contributions in all areas of European knowledge ([11], [14]). However, he did not receive much attention from his contemporary scholars. In 1714, Leibniz discussed some of his ideas with the well-known L'Hopital and others but felt they paid no more attention to it than if he had told them about a dream of him([11]; pp.523-4). Even in the Leibniz-Newton controversy over who had invented calculus, most mathematicians of the non-German-speaking countries in Europe sided with Newton. To be fair, it must be mentioned that, the non-German L'Hopital acknowledged in his 1696 book about the calculus from a Leibnizian point of view that Newton's published work of the 1680s is as "nearly all about this calculus" but expressed preference for the convenience of Leibniz's notation [5]. After one or two centuries of the death of Leibniz, most judgments credited both Newton and Leibniz as the independent inventors of calculus [7]. But, papers appear off and on that claim Leibniz not only plagiarized Newton on calculus, he also plagiarized Englishman Thomas

Harriot (1560-1621) and the Spaniard Juan Caramuel de Lobkowitz (1606-1682) on binary systems [1] [10].

Harriot as is claimed in [18], did not only work in base 2, but also in number systems with bases 3, 4, 5 etc. As a mathematician, physicist and astronomer, Harriot saw no practical application in his works but believed that useless knowledge may someday be useful theory. Harriot passed away 25 years before Leibniz to be born.

The talented mathematician and astronomer, Caramuel was more interested in theology than mathematics. He dealt with number systems of various bases relating them to different natural phenomena or philosophies; base 2 in relation to doubling or halving the strings in musical tools ...; base 3 in relation to Christian Trinity ...; base 4 in relation to the four winds ...; base 5 in relation with the four main universal elements supplemented by "quintaes sentia" ... and so on. Finally, when he reaches base 60, he does not forget to relate it to degree, minute and second used by the astronomers. In spite of his mathematical interests such as divisibility of natural numbers, dice games, number lottery, and geometric constructions, one is surprised that Caramuel with the description of these number systems does not connect any further mathematical considerations or provide practical tips [10]. Caramuel was 15 years old when Harriot died and he passed away when Leibniz was 36 years old. Anyway, there is no information whether Caramuel knew of the works done by Harriot or Leibniz.

It seems that Leibniz was the first person in the history that corresponded the I Ching symbols with his binary representations of the Arabic numbers $0, 1, 2, \cdots, 63$. A copy of the I Ching less complicated than the one shown in Fig. 4 was sent to him from China by the French Jesuit Joachim Bouvet. The Leibniz's copy is not clear and can be searched in Google; it consists of the single ring of the 64 characters as is seen in Fig. 4 with nothing in the middle except for an 8×8 array of the same characters. Leibniz has marked the characters by the Arabic numerals $0, 1, 2, \cdots, 63$. All the writings are faded. Bouet was working in China between the years of 1697 and 1707 with a group believing the Chinese Fu Xi whom the I Ching was revealed to, was not Chinese but was rather the original Lawgiver of all mankind [13]. Among other things, Leibniz was using base 2 to invent a calculator. After matching his newly invented $0 - 1$ numbers with the hexagrams in the I Ching, Leibniz did not hesitate to claim that "it has been up to him, a European, to restore the lost meaning of I Ching". Leibniz tried to prove (in vain) how the Chinese had lost the "intended meaning" [13].

Leibniz claimed the operations in base 2 are so easy that one shall never have to guess or apply trial and error. He hoped that the binary system would aid him in the creation of the "characteristica universalis", constructing a universal formal language for expressing mathematics, science, and other concepts. The discovery of the hexagrams and their relation to his binary number system gave him encouragement in this area. Simon Marquis de Laplace believed that

Leibniz saw in his binary arithmetic the image of Creation. For him the numeral 1 represented God and the numeral 0 represented nothing and that is how the whole universe was created by God alone. In letters to Rudolph August, Duke of Brunswick, Leibniz expressed that his system of numbers were a suitable analogy to God's omnipotence, just as unity and zero express all numbers in his system of numeration. He asked the Duke of Brunswick to issue a silver medal commemorating this discovery with the following inscriptions: "The model of creation discovered by Gottfried Wilhelm Leibniz ...."

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References

[1] Ares, J., Lara, J., Lizcano, D., and Martínez, M. A. (2018), Who Discovered the Binary System and Arithmetic? Did Leibniz Plagiarize Caramuel? Sci. Eng. Ethics. 1,173-188. doi: 10.1007/ISBN 11948-017-9890-6.

[2] Egypt Today staff. (2021), Bread: types of Egyptians' indispensable food. Egypt Today. CAIRO – 11 November 2021.
https://www.egypttoday.com/Article/1/109827/Bread-types-of-Egyptians-indispensable-food

[3] Gamow, G. (1988) One, Two, Three,... Infinity; Facts and Speculations of Science. Dover Publications INC., New York. ISBN: 0486256642; ISBN13: 9780486256641

[4] Ghorbani. A. (1996), Biography of Mathematicians of Islamic Era. The University Publishing Center, Tehran, Iran. ISBN: 964-01-0265-2 (In Persian)

[5] Guicciardini,N. (2003) Reading the Principia: The Debate on Newton's Mathematical Methods for Natural Philosophy from 1687 to 1736. Cambridge University Press.

[6] Haddon, A.C. (1890), The ethnography of the western tribes of Torres Straits. Journal of the Anthropological institute of Great Britain and Ireland 19, 297-437.

[7] Hall, A. R. (1980). Philosophers at War: The Quarrel between Newton and Leibniz. Cambridge University Press. p. 356. ISBN 0-521-22732-1.

[8] Harris, J. (1987) Australian Aboriginal and Islander mathematics. Australian Aboriginal Studies 2, 29-37.
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[9] Ifrah, G. (2000). The Universal History of Numbers: From prehistory to the invention of the computer. Translated from the French by David Bellos, E.F. Harding, Sophie Wood and Ian Monk. John Wiley and Sons Inc.

[10] Ineichen, R. (2008). Leibniz, Caramuel, Harriot und das Dualsystem. MDMV 16, 12–15.

[11] Lande, D.R. (2014). Development of the Binary Number System and the Foundations of Computer Science, The Mathematics Enthusiast, 11(3),513-540.
DOI: https://doi.org/10.54870/1551-3440.1315
Available at: https://scholarworks.umt.edu/tme/vol11/iss3/6

[12] Mark, J.J. (2017). Jobs in Ancient Egypt. World History Encyclopedia, Retrieved from https://www.worldhistory.org/article/1073/jobs-in-ancient-egypt/

[13] Mungello, D. E. (1971). Leibniz's Interpretation of Neo-Confucianism. Philosophy East and West, 21(1), 3-22.

[14] Perkins, F. (2010). Leibniz: A Guide for the Perplexed. London, GBR: Continuum International Publishing.

[15] Porubsky, S. Binary system. Retrieved 2023/10/19 from Interactive Information Portal for Algorithmic Mathematics, Institute of Computer Science of the Czech Academy of Sciences, Prague, Czech Republic,

https://www.cs.cas.cz/portal/AlgoMath/NumberTheory/Arithmetics/NumeralSystems/PositionalNumeralSystems/BinarySystem.htm

[16] Radjabalipour, M. (2009). Egyptian fractions. Mathematical Culture and Thought; Iranian Mathematical Society, 28(1), 1-38. (In Persian)

[17] Radjabalipour, M. (2024). A history of fractions. (In preparation.)

[18] Shirley, J.W. (1951). Binary Numeration before Leibniz. American Journal of Physics, 19, 452–454.

[19] Zacher, H.J.(1973). The Main Writings on Dyadik by G.W. Leibniz – A contribution to history of the binary number system. Frankfurt A.M.: Vittorio Klostermann. ISBN: 978-3-465-00998-6

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