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A HYBRID CHELYSHKOV WAVELET-FINITE DIFFERENCES METHOD FOR TIME-FRACTIONAL BLACK-SCHOLES EQUATION

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Abstract. In this paper, a hybrid method for solving time-fractional Black-Scholes equation is introduced for option pricing. The presented method is based on time and space discretization. A second order finite difference formula is used to time discretization and space discretization is done by a spectral method based on Chelyshkov wavelets and an operational process by defining Chelyshkov wavelets operational matrices. Convergence and error analysis for Chelyshkov wavelets approximation and also for the proposed method are discussed. The method is validated and its accuracy, convergency and efficiency are demonstrated through some cases with given accurate solutions. The method is also utilize for pricing various European options conducted by a time-fractional Black-Scholes model.

Keywords: Fractional Black-Scholes Equation, Chelyshkov polynomials, Wavelet, Caputo fractional derivative, Option pricing, Error analysis.
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1. Introduction

Simulation of many phenomena in applied sciences and engineering by fractional order differential equations (FDEs) is more accurate and expressive than their simulation by classical differential equations. Fractional derivatives and integrals, because of their non-locality feature, play a paramount rule for characterization of memory. Because, the classical Black-Scholes equation is based on certain stringent assumptions, which include constant volatility, constant rates of return, and the absence of dividends, taxes, or transaction costs. The emergence of fractional structures in the financial market has led to the development of fractional Black-Scholes (fBS) models. A significant limitation of the classical Black-Scholes approach, as well as some other versions, is the integer order derivative in their differential components. Integer order derivatives only offer local information around a specific point. However, changes in market conditions have resulted in the evolution of some unusual structures

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in financial systems. Models that rely on local derivatives are flawed in many ways due to these changes. To address these shortcomings and provide a more realistic representation of reality, fractional models have been introduced.

The memory and non-locality properties, two fundamental aspects of fractional calculus, have enhanced the Black–Scholes equation with fractional time order, making it a more accurate model for option pricing. A nonlocal operator isn't directly determined by function values, hence the concept of non-locality introduces an additional layer of uncertainty into deterministic physical systems, similar to Heisenberg's uncertainty principle.

Moreover, a fractional derivative's covariant formulation is inherently based on a global concept of non-locality, which is also applicable to space-dependent derivative operators. Unlike an integer order differential operator, which is local, a fractional order differential operator is nonlocal. This nonlocal nature of the fractional order derivative in the model makes both exact and numerical solutions more complex compared to those with the integer order model.

According to modern nonlocal theory, this behavior is a direct result of a memory effect. In a society with memory, relationships are more stable, but the strength of these relationships varies depending on how well the society remembers its past. The more the past is remembered, the less likely change occurs. Conversely, in a network with little memory, relationships are in a constant state of flux.

The interpretation of equations involving derivatives and integrals of non-integer order in relation to time is linked to memory effects. Fractional derivatives have the ability to depict the traits of long memory and nonlocal reliance of numerous irregular processes. The system's memory effect is preserved by the fractional-order derivative, which is non-local and surpasses the integer-order derivative in this aspect.

In this context, various fractional Black–Scholes models have been successively introduced. It's widely recognized that fractional calculus is an optimal tool for illustrating memory effects. In fact, these memory effects encompass trend memory, which can be expressed using fractional derivatives. Therefore, we can use fractional calculus to represent the memory process of the increment in traditional stochastic differential equations.

Models based on fractional derivatives are excellent mathematical instruments for elucidating the dynamics of intricate processes, irregular increases, and trend memory effects, which are demonstrated by a variety of financial instruments.

On the whole, it is possible to solve quite a few fractional differential equations analytically. Indeed, actual solutions are not attainable usually for FDEs. Accordingly, numerical approaches are unavoidable to solve such problems. In order to solving FDEs, a wide range of numerical methods has been proposed [6,9–12,14,15,21]. Research on FDEs has obtain much usefulness by reason of their applications in various distinct areas of engineering and science. A portion of the fields of utilization of FDEs cover viscoplasticity, signal processing, control theory, acoustics, material science and fluid mechanics [6,26]. Also, FDEs are used frequently and properly to represent financial problems [24].

A few years after publishing the Black and Scholes well-known partial differential equation (PDE) for the fair value of European call and put options, many attempts were made to solve the problem numerically. At the present, numerical solution of time-dependent PDEs, and specifically, time fractional differential equations set up one of the important bases of computational finance.

This research has considered the time-fractional Black-Scholes (TFBS) equation of the following equation:

$$
\begin{aligned}
&\frac{\partial^\alpha V(s,\tau)}{\partial \tau^\alpha} + \frac{\sigma^2}{2} s^2 \frac{\partial^2 V(s,\tau)}{\partial s^2} + r\, s \frac{\partial V(s,\tau)}{\partial s} - r\, V(s,\tau) = 0, \\
&V(0,\tau) = V_0(\tau)\,, \quad V(\infty,\tau) = V_\infty(\tau)\,, \quad V(s,T) = V_T(s)\,, \\
&(s,\tau) \in [0,\infty) \times (0,T) \text{ and } 0 < \alpha < 1.
\end{aligned} \tag{1}
$$

Various methods have been presented for producing approximate solution for TFBS equation. For example, in [34], proposed an implicit finite difference method (IFDM) to achieve an approximate solution for a spatial fractional Black-Scholes equation. In [32], proposed an implicit discrete scheme for approximating the solution of a European options TFBS model, which has a convergence order of $O(\Delta t^{2-\alpha} + \Delta x^2)$. An implicit FDM was presented in [28], for the approximate solution of a TFBS equation. It was proven that the method is convergent with order of accuracy one in terms of time variable and with order of accuracy two in terms of space variable. In [3] it was introduced a method to get to numerical solution of the TFBS equation. It should be noted that there are few numerical methods offered in researches for solving the TFBS equation (1). Even though various numerical approaches have been introduced to deal with the given problem, these suggested procedures are of less orders in terms of space variable.

In this article, we suggest an approximating procedure with indispensable analyzes for solving the TFBS model (1) which is of high order in terms of space variable. The numerical solution process in this paper is obtained using the popular method of finite difference, followed by a wavelet approach.

The framework of the article is set up as follows: in Section 2, necessary prerequisites such as the definitions and properties of the used wavelet, a little about fractional order derivative, and an approximation of fractional derivative are given. In section 3, notion of operational matrix of integration and required formulaes are explained. Description of the presented method is declared in section 4. In section 5, convergence of the method is analyzed from two perspective. Eventually, illustrative examples are provided in section 6 to expose the method applicability and exactitude.

2. Preliminaries

2.1. **Chelyshkov Wavelets.** Chelyshkov Wavelets (ChWs), $\psi_{n,m}(x) = \psi(k,n,m,x)$, are defined on the interval $[0,1)$ through [19]:

$$\psi_{n,m}(t) = \begin{cases} \sqrt{2^k(2m+1)}\, P_m(2^k t - n), & \frac{n}{2^k} \le t < \frac{n+1}{2^k}, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where $P_m(t)$ is the Chelyshkov polynomial, which is defined as follows:

$$P_m(t) := \rho_{M,m}(t) = \sum_{j=0}^{M-m} a_{m,j}\, t^{m+j}\ , \quad m = 0,1,\ldots,M, \tag{3}$$

where:

$$a_{m,j} = (-1)^j \binom{M-m}{j}\binom{M+m+j+1}{M-m},$$

and M is a fixed predetermined integer.

With simple calculations can be observe that the Chelyshkov polynomials, with respect to the weight function $w(t) = 1$, are orthogonal on the interval $[0,1]$. i.e. :

$$\int_0^1 P_n(t)\, P_m(t)\, dt = \frac{\delta_{mn}}{m+n+1},$$

where δ_{mn} is the Kronecker delta. Besides,

$$\int_0^1 P_n(t)\, dt = \frac{1}{M+1}\ , \quad n = 0,1,\ldots,M. \tag{4}$$

Fixing M, an integer, according to Eq. (3), it is obvious each element of the set formed by polynomials $P_m(t)\,,\quad m = 0,1,\ldots,M$ is of degree M.

The ChWs $\{\psi_{n,m}(t)\,|\, n = 0,1,\ldots 2^k-1,\ m = 0,1,\ldots M\}$ constitutes an orthonormal basis for $L^2[0,1]$. Therefore, any member $f(t) \in L^2[0,1]$ can be built up in terms of ChWs as:

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\, \psi_{n,m}(t), \tag{5}$$

where $c_{n,m} = \langle f, \psi_{n,m}\rangle = \int_0^1 f(t)\, \psi_{n,m}(t)\, dt$. Truncating the infinite series in Eq. (5) is lead to:

$$f(t) \simeq \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} c_{n,m}\, \psi_{n,m}(t) = \mathbf{C}^T \mathbf{\Psi}(t), \tag{6}$$

where $\mathbf{C}$ and $\mathbf{\Psi}(t)$ are $\hat{m} = 2^k(M+1)$-vectors, are of the form:

$$\begin{aligned}\mathbf{C}^T &= [c_{0,0}, c_{0,1}, \ldots, c_{0,M}, c_{1,0}, \ldots, c_{1,M}, \ldots, c_{2^k-1,0} \cdots, c_{2^k-1,M}]\\ &= [c_1, c_2, \ldots, c_{\hat{m}}], \qquad (7)\end{aligned}$$

$$\mathbf{\Psi}(t)^T = [\psi_{0,0}(t), \ldots, \psi_{0,M}(t), \psi_{1,0}(t), \ldots, \psi_{1,M}(t), \ldots, \psi_{2^k-1,0}(t), \ldots, \psi_{2^k-1,M}(t)] \qquad (8)$$

$$= [\psi_1(t), \psi_2(t), \ldots, \psi_{\hat{m}}(t)]. \qquad (9)$$

2.2. **Modified Riemann-Lioville fractional derivative.** The modified Riemann-Lioville fractional derivative is defined as [25]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(x,\tau) - u(x,0)}{(t-\tau)^\alpha} \, d\tau \ , \qquad 0 < \alpha < 1 \ , \qquad (10)$$

Also, the Caputo fractional derivative is defined by [6]:

$$_C D_{0,t}^\alpha \, u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\frac{\partial u(x,s)}{\partial s}}{(t-s)^\alpha} \, ds \ , \quad 0 < \alpha < 1.$$

The Caputo approach's primary benefit is its similarity to integer order differential equations when it comes to initial and boundary conditions. This similarity allows for the same interpretation, making it a popular choice in practical applications. The Caputo derivative proves to be extremely effective in tackling real-world problems. It enables the inclusion of standard initial and boundary conditions in problem-solving, and also maintains that the derivative of a constant is zero. To see the advantages of the Caputo derivative over other fractional derivatives in detail, see the following references [7, 27, 29].

It can be shown that the modified Riemann-Lioville fractional derivative is equivalent to the Caputo fractional derivative as follows:

$$\begin{aligned}\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(x,\tau) - u(x,0)}{(t-\tau)^\alpha} \, d\tau\\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\tau)}{\partial \tau} (t-\tau)^{-\alpha} \, d\tau\\ &= {}_C D_{0,t}^\alpha \, u(x,t) \ , \qquad (11)\end{aligned}$$

After setting uniform grids $\{t_k\}_{k=0}^{N_t}$ with $t_k = k\,h$, $h = \frac{T}{N_t}$, utilizing the forward difference formula $\frac{f(t_{k+1}) - f(t_k)}{h}$ and approximating the derivative of order one on each interval $[t_k, t_{k+1}]$ bring in:

$$\left[{}_C D_{0,t}^\alpha f(t)\right]_{t=t_j} = \sum_{k=0}^{j-1} b_{j-k-1} \left(f(t_{k+1}) - f(t_k)\right) + E_h^j \ , \quad 0 < \alpha < 1 \ , \qquad (12)$$

where $j = 1, 2, \ldots, N_t$, and

$$b_k = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \left((k+1)^{1-\alpha} - k^{1-\alpha}\right) , \quad k = 0, 1, \ldots, j-1 .$$

In equation (12) E_h^j is the truncation error and $E_h^j \leq c_f\, h^{2-\alpha}$ and c_f is a constant where its value merely depends on f [21, 26].

3. ChWs Operational matrix of integration

At this point, we extract a matrix that it makes easy to approximate the integral of functions. This matrix is called the operational matrix of integration for ChWs which will be used in the sequel.

By Eq. (3), for $m = 0, 1, \ldots, M$ and $q = 0, 1, \ldots$, it's easy to calculate that:

$$\int_0^x P_m(t)\, dt = \sum_{j=0}^{M-m} \frac{a_{m,j}}{m+j+1}\, x^{m+j+1}, \tag{13}$$

$$\int_0^1 x^q P_m(x)\, dx = \sum_{j=0}^{M-m} \frac{a_{m,j}}{q+m+j+1}. \tag{14}$$

Lemma 1. The elements of operational matrix of integration to Chelyshkov polynomials are in the following form:

$$\bar{p}_{i,j} = \frac{\langle \int_0^x P_j(t)\, dt, P_i(x)\rangle}{\langle P_i(x)\, dt, P_i(x)\rangle} = \sum_{l=0}^{M-j} \sum_{r=0}^{M-i} \frac{(2i+1)\, a_{j,l}\, a_{i,r}}{(j+l+1)(j+l+i+r+2)}.$$

Proof. According to Eq. (13), we can get:

$$\int_0^1 \left(\int_0^x P_j(t)\, dt\right) P_i(x)\, dx = \int_0^1 \left(\sum_{l=0}^{M-j} \frac{a_{j,l}}{j+l+1} x^{j+l+1}\right) P_i(x)\, dx$$
$$= \sum_{l=0}^{M-j} \frac{a_{j,l}}{j+l+1} \int_0^1 x^{j+l+1} P_i(x)\, dx$$

and by Eq. (14)

$$= \sum_{j=0}^{M-j} \frac{a_{j,l}}{j+l+1} \sum_{r=0}^{M-i} \frac{a_{i,r}}{j+l+1+i+r+1}$$
$$= \sum_{l=0}^{M-j} \sum_{r=0}^{M-i} \frac{a_{j,l}\, a_{i,r}}{(j+l+1)(j+l+i+r+2)}. \tag{15}$$

Thus if $\mathbf{\Phi}(x) = [P_0(x), \ldots, P_M(x)]^T$ is the vector of Chelyshkov polynomials, then Chelyshkov polynomials operational matrix of integration , $\bar{\mathbf{P}}$, with

$\int_0^x \mathbf{\Phi}(t)\,dt \simeq \bar{\mathbf{P}}\,\mathbf{\Phi}(x)$ will be:
(16)

$$\bar{\mathbf{P}} = [\bar{p}_{ij}]_{(M+1)\times(M+1)}\ , \qquad \bar{p}_{i,j} = \sum_{l=0}^{M-j}\sum_{r=0}^{M-i} \frac{(2i+1)\,a_{j,l}\,a_{i,r}}{(j+l+1)(j+l+i+r+2)}.$$

□

Now, using the mentioned topics, we will try to get ChWs operational matrix of integration. If $\mathbf{\Psi}(x)$ stands for the vector of ChWs, then the integral of it can be approximated by itself and written as:

$$\int_0^x \mathbf{\Psi}(t)\,dt \simeq \mathbf{P}\,\mathbf{\Psi}(x). \tag{17}$$

In the above formula, $\mathbf{P}$ is named the ChWs operational matrix of integration which is an $\hat{m}\times\hat{m}$ matrix that is derived by the following process.

Similarly to above topics, by fixing the predetermined integers M, k and real $\frac{n}{2^k} < x < 1$, we start by:

$$\begin{aligned} \int_0^x \psi_{nm}(t)\,dt &= \int_0^x \sqrt{2^k(2m+1)}\,P_m(2^k t - n)\,dt \\ &= \sqrt{2^k(2m+1)}\int_{\frac{n}{2^k}}^x P_m(2^k t - n)\,dt. \end{aligned} \tag{18}$$

Substituting $w = 2^k t - n$ and $x = \frac{n+\zeta}{2^k}$ where $0 < \zeta < 1$, in Eq. (18) and using Eq. (13) we have:

$$\begin{aligned} \int_0^x \psi_{nm}(t)\,dt &= \sqrt{\frac{2m+1}{2^k}}\int_0^\eta P_m(w)\,dw \\ &= \begin{cases} 0 & x < \frac{n}{2^k} \\ \sqrt{\frac{2m+1}{2^k}}\sum_{j=0}^{M-m}\frac{a_{m,j}}{m+j+1}(2^k x - n)^{m+j+1} & \frac{n}{2^k} \le x \le \frac{n+1}{2^k} \\ \sqrt{\frac{2m+1}{2^k}}\frac{1}{M+1} & x > \frac{n+1}{2^k} \end{cases} \end{aligned} \tag{19}$$

So,

$$
\begin{aligned}
\int_0^1 \int_0^x \psi_{nm}(t)\,dt\,dx &= 0 + \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} \sqrt{\frac{2m+1}{2^k}} \sum_{j=0}^{M-m} \frac{a_{m,j}}{m+j+1} (2^k x - n)^{m+j+1}\,dx \\
&\quad + \int_{\frac{n+1}{2^k}}^{1} \sqrt{\frac{2m+1}{2^k}} \frac{1}{M+1}\,dx \\
&= \sqrt{\frac{2m+1}{2^{3k}}} \sum_{j=0}^{M-m} \frac{a_{m,j}}{(m+j+1)(m+j+2)} \\
&\quad + (1 - \frac{n+1}{2^k}) \sqrt{\frac{2m+1}{2^k}} \frac{1}{M+1} \\
&= \sqrt{\frac{2m+1}{2^{3k}}} \left(\sum_{j=0}^{M-m} \frac{a_{m,j}}{(m+j+1)(m+j+2)} + \frac{2^k - n - 1}{M+1} \right). \qquad (20)
\end{aligned}
$$

As we know, elements of the operational matrix of integration for ChWs are $\int_0^1 (\int_0^x \psi_{nm}(t)\,dt)\,\psi_{qi}(x)\,dx$. To reduce the computation time and complexity we try to drive a formula to that which is as simple as possible. Is is easy to find that this integral is zero for $n > q$, hence we consider two cases, i.e. $n < q$ and $n = q$ separately. First, if $n < q$, by (4) and (19),

$$
\begin{aligned}
\int_0^1 \left(\int_0^x \psi_{nm}(t)\,dt \right) \psi_{qi}(x)\,dx &= \sqrt{\frac{2m+1}{2^k}} \frac{1}{M+1} \int_{\frac{q}{2^k}}^{\frac{q+1}{2^k}} \psi_{qi}(x)\,dx \\
&= \sqrt{\frac{2m+1}{2^k}} \frac{1}{M+1} \sqrt{\frac{2i+1}{2^k}} \frac{1}{M+1} \\
&= \frac{\sqrt{(2m+1)(2i+1)}}{2^k (M+1)^2}. \qquad (21)
\end{aligned}
$$

Now, we investigate the case $n = q$, so:

$$\int_0^1 \left(\int_0^x \psi_{nm}(t)\, dt \right) \psi_{ni}(x)\, dx$$
$$= \int_0^1 \sqrt{\frac{2m+1}{2^k}} \left(\sum_{j=0}^{M-m} \frac{a_{m,j}}{m+j+1} (2^k x - n)^{m+j+1} \right) \psi_{ni}(x)\, dx$$
$$= \int_0^1 \sqrt{\frac{2m+1}{2^k}} \left(\sum_{j=0}^{M-m} \frac{a_{m,j}}{m+j+1} (2^k x - n)^{m+j+1} \right) \sqrt{2^k(2i+1)} P_i(2^k x - n)\, dx$$
$$= \sqrt{2m+1}\sqrt{2i+1} \sum_{j=0}^{M-m} \frac{a_{m,j}}{m+j+1} \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} (2^k x - n)^{m+j+1} P_i(2^k x - n)\, dx$$
$$= \frac{\sqrt{(2m+1)(2i+1)}}{2^k} \sum_{j=0}^{M-m} \frac{a_{m,j}}{m+j+1} \int_0^1 u^{m+j+1} P_i(u)\, du$$

(22)
$$= \frac{\sqrt{(2m+1)(2i+1)}}{2^k} \sum_{j=0}^{M-m} \sum_{r=0}^{M-i} \frac{a_{m,j}\, a_{i,r}}{(m+j+1)(m+j+i+r+2)}.$$

It is obvious that for $x < \frac{n}{2^k}$, we have $\int_0^x \psi_{nm}(t)\, dt = 0$. Thus, the operational matrix of integration for ChWs has an upper triangular block matrix as below:

(23)
$$\mathbf{P} = \begin{bmatrix} \mathfrak{D} & \mathfrak{N} & \mathfrak{N} & \cdots & \mathfrak{N} & \mathfrak{N} \\ 0 & \mathfrak{D} & \mathfrak{N} & \cdots & \mathfrak{N} & \mathfrak{N} \\ 0 & 0 & \mathfrak{D} & \cdots & \mathfrak{N} & \mathfrak{N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathfrak{D} & \mathfrak{N} \\ 0 & 0 & 0 & \cdots & 0 & \mathfrak{D} \end{bmatrix},$$

where $\mathfrak{D}$ and $\mathfrak{N}$ are $(M+1) \times (M+1)$ matrices whose elements are:

$$\mathfrak{D} = [d_{ij}]_{(M+1)\times(M+1)},$$

(24)
$$d_{ij} = \frac{\sqrt{(2i+1)(2j+1)}}{2^k} \sum_{l=0}^{M-j} \sum_{r=0}^{M-i} \frac{a_{j,l}\, a_{i,r}}{(j+l+1)(j+l+i+r+2)},$$

and

(25)
$$\mathfrak{N} = [\eta_{ij}]_{(M+1)\times(M+1)}, \qquad \eta_{ij} = \frac{\sqrt{(2j+1)(2i+1)}}{2^k (M+1)^2}.$$

In other words,

(26)
$$\mathbf{P} = I_{2^k} \otimes \mathfrak{D} + \mathbf{U}_k \otimes \mathfrak{N},$$

where I_{2^k} is the identity matrix of order 2^k and $\mathbf{U}_k$ is a strictly upper triangular matrix whose up-diagonal elements are all 1 and the others are 0, i.e.:

$$\mathbf{U}_k = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{2^k \times 2^k},$$

and,

$$\mathbf{P}^2 = \begin{bmatrix} \mathfrak{D}^2 & \bar{\mathfrak{N}} & \bar{\mathfrak{N}} + \mathfrak{N}^2 & \bar{\mathfrak{N}} + 2\mathfrak{N}^2 & \dots & \bar{\mathfrak{N}} + (2^k-2)\mathfrak{N}^2 \\ 0 & \mathfrak{D}^2 & \bar{\mathfrak{N}} & \bar{\mathfrak{N}} + \mathfrak{N}^2 & \dots & \bar{\mathfrak{N}} + (2^k-3)\mathfrak{N}^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \bar{\mathfrak{N}} \\ 0 & 0 & 0 & 0 & \dots & \mathfrak{D}^2 \end{bmatrix},$$

where $\bar{\mathfrak{N}} := \mathfrak{D}\mathfrak{N} + \mathfrak{N}\mathfrak{D}$. If we define $2^k \times 2^k$ matrix $\bar{U}_k$ as:

$$\bar{U}_k = \begin{bmatrix} 0 & 0 & 1 & 2 & \dots & 2^k-2 \\ 0 & 0 & 0 & 1 & \dots & 2^k-3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{2^k \times 2^k},$$

then, we will have $\bar{U}_k = \mathbf{U}_k \times \mathbf{U}_k = \mathbf{U}_k^2$

$$\begin{aligned} \mathbf{P}^2 &= I_{2^k} \otimes \mathfrak{D}^2 + \mathbf{U}_k \otimes \bar{\mathfrak{N}} + \bar{U}_k \otimes \mathfrak{N}^2, \\ &= I_{2^k} \otimes \mathfrak{D}^2 + \mathbf{U}_k \otimes \bar{\mathfrak{N}} + \mathbf{U}_k^2 \otimes \mathfrak{N}^2 \end{aligned}$$

Consequently, to obtain the operational matrix if integration for ChWs it is sufficient to calculate $\mathfrak{D}$ and $\mathfrak{N}$ in (23). In order to determine $\mathfrak{D}$, it is enough to consider $n = q$, say, $n = q = 0$ in (22). Also, to obtain $\mathfrak{N}$ we can set $n = 0$ and $i = 1$ in (21).

For example if $k = 3$ and $M = 3$, we have:

$$\mathfrak{D} = \frac{1}{2^8 \cdot 3 \cdot 5 \cdot 7} \begin{bmatrix} 105 & 273\sqrt{3} & 189\sqrt{5} & 213\sqrt{7} \\ -63\sqrt{3} & 315 & 245\sqrt{15} & 205\sqrt{21} \\ 21\sqrt{5} & -35\sqrt{15} & 525 & 225\sqrt{35} \\ -3\sqrt{7} & 5\sqrt{21} & -15\sqrt{35} & 735 \end{bmatrix},$$

and:

$$\mathfrak{N} = \frac{1}{2^7} \begin{bmatrix} 1 & \sqrt{3} & \sqrt{5} & \sqrt{7} \\ \sqrt{3} & 3 & \sqrt{15} & \sqrt{21} \\ \sqrt{5} & \sqrt{15} & 5 & \sqrt{35} \\ \sqrt{7} & \sqrt{21} & \sqrt{35} & 7 \end{bmatrix}.$$

3.1. **Solving an ODE with boundary conditions by ChWs.** We start this section by trying to find an approximate solution for the following second order boundary value problem using ChWs.

$$\begin{aligned} \alpha\, y''(x) + \beta\, y'(x) + \gamma\, y(x) = r(x) \ , \qquad 0 \leq x \leq 1 \ , \\ y(0) = y_0 \ , y(1) = y_1, \end{aligned} \tag{27}$$

where $\alpha, \beta, \gamma, y_0$ and y_1 are given constants.

According Eq. (6), we begin by considering an approximation for $y''(x)$ in terms of ChWs, i.e.

$$y''(x) \simeq \mathbf{C}^T\, \mathbf{\Psi}(x) \tag{28}$$

where the vector $\mathbf{C}$ is unknown at the moment. Then we have:

$$y'(x) \simeq \mathbf{C}^T\, \mathbf{P}\, \mathbf{\Psi}(x) + y'(0) \tag{29}$$

and thereby:

$$y(x) \simeq \mathbf{C}^T\, \mathbf{P}^2\, \mathbf{\Psi}(x) + y'(0)\, x + y_0 \ , \tag{30}$$

where $\mathbf{P}$ is operational matrix of integratopn for ChWs. Note that $y'(0)$ in Eqs (29) and (30) is not given and must be determine. By imposing the boundary condition $y(1) = y_1$ in Eq. (30), we have:

$$y_1 \simeq \mathbf{C}^T\, \mathbf{P}^2\, \mathbf{\Psi}(1) + y'(0) + y_0 \ ,$$

and this gives us:

$$y'(0) = y_1 - y_0 - \mathbf{C}^T \mathbf{P}^2 \mathbf{\Psi}(1). \tag{31}$$

Substituting $y'(0)$ in Eqs. (29) and (30) by (31), respectively, we get:

$$\begin{aligned} y'(x) &= \mathbf{C}^T \mathbf{P} \mathbf{\Psi}(x) - \mathbf{C}^T \mathbf{P}^2 \mathbf{\Psi}(1) + y_1 - y_0 \ , \\ y(x) &= \mathbf{C}^T \mathbf{P}^2 \mathbf{\Psi}(x) - \mathbf{C}^T \mathbf{P}^2 \mathbf{\Psi}(1)\, x + (y_1 - y_0)x + y_0. \end{aligned}$$

For simplicity and ease of calculations we define $y_{10} := y_1 - y_0$, $\mathbf{F} := \mathbf{P}^2 \mathbf{\Psi}(1)$. According to (20) and definition of operational matrix of integration, (17), $\mathbf{F}$ can be approximated by:

$$\mathbf{F} = \mathbf{P}^2 \mathbf{\Psi}(1) \simeq \bar{\mathbf{F}}, \tag{32}$$

where $\bar{\mathbf{F}}$ is an $\hat{m} = 2^k(M+1)$-vector whose elements are calculated by (20).

Then, calculate expansion of functions $1, x$ and $r(x)$ in terms of ChWs, and obtain

$$1 := \mathbf{E}^T \mathbf{\Psi}(x) \ , \tag{33}$$

$$x := \mathbf{B}^T \mathbf{\Psi}(x) \ , \text{equal holds for } n \geq 1, \tag{34}$$

$$r(x) \simeq \mathbf{R}^T \mathbf{\Psi}(x), \tag{35}$$

and substitute these together with Eqs. (28) – (30) in Eq. (27) and simplifying, we will have:

$$\mathbf{C}^T \left(\alpha\mathbf{I} + \beta\mathbf{P} + \gamma\mathbf{P}^2 - \mathbf{F}(\beta\mathbf{E}^T + \gamma\mathbf{B}^T)\right) \mathbf{\Psi}(x)$$
$$= \left(\mathbf{R}^T - y_{10}(\beta\mathbf{E}^T + \gamma\mathbf{B}^T) - \gamma y_1\mathbf{E}^T\right) \mathbf{\Psi}(x) \ ,$$

or equivalently:

$$\left(\alpha\mathbf{I} + \beta\mathbf{P}^T + \gamma(\mathbf{P}^2)^T - (\beta\mathbf{E} + \gamma\mathbf{B})\mathbf{F}^T\right) \mathbf{C} = \left(\mathbf{R} - y_{10}(\beta\mathbf{E} + \gamma\mathbf{B}) - \gamma y_1\mathbf{E}\right). \tag{36}$$

Solving this system of linear equations gives $\mathbf{C}$ and leads us to the approximate solution of Eq. (27) as:

$$y(x) \simeq \mathbf{C}^T\mathbf{P}^2\mathbf{\Psi}(x) + \left(y_{10} - \mathbf{C}^T\mathbf{F}\right) x + y_0. \tag{37}$$

It should be noted that it is easy to see that the dominant sentence in the implementation of this method is the calculation of the coefficients of the right-hand function in terms of the given wavelet basis, i.e. vector $\mathbf{R}$ in the notation of subsection 3.1. For a better understanding of the discussion, consider the following example

Example 1. Consider the following ODE with boundary conditions:

$$y'' - 6y' - 2y = -2e^{3x}\sin(2x), \qquad y(0) = 0, \quad y(1) = -e^3.$$

Exact solution for the ODE is:

$$y(x) = \frac{1}{15}e^{3x}\left(2\sin(2x) - (15 + 2\sin(2))\frac{\sinh(\sqrt{11}x)}{\sinh(\sqrt{11})}\right).$$

The ODE has been solved both by the proposed method and by using NDSolve command in WOLFRAM MATHEMATICA, and the results obtained are given below for comparison. Using the proposed method, for different values of M and k, the L_∞-error and the CPU time are shown in the table 1. Using NDSolve command, the resulted error is 5.2×10^{-7}. As can be seen, by

TABLE 1. L_∞ Error and CPU time for ODE in example 1

M	k	L_∞ Err.	CPU time
3	4	2.0×10^{-4}	0.29
	5	1.4×10^{-5}	0.73
	6	8.2×10^{-7}	1.88
4	4	4.2×10^{-6}	0.46
	5	8.1×10^{-8}	0.96
	6	6.1×10^{-8}	2.57
5	4	6.6×10^{-8}	0.54
	5	4.0×10^{-8}	1.28
	6	3.5×10^{-8}	3.30

increasing M and k in the proposed method, the error can be reduced as much as desired.

4. Description of the method

4.1. **Problem Statement.** In this paper we deal with the subsequent TFBS equation:

$$\frac{\partial^\alpha V(s,\tau)}{\partial \tau^\alpha} + \frac{\sigma^2}{2} s^2 \frac{\partial^2 V(s,\tau)}{\partial s^2} + r\, s \frac{\partial V(s,\tau)}{\partial s} - r\, V(s,\tau) = 0, \tag{38}$$

accompanying the subsequent boundary and terminal conditions:

$$V(0,\tau) = V_0(\tau)\,, \quad V(\infty,\tau) = V_\infty(\tau)\,, \quad V(s,T) = V_T(s)\,, \tag{39}$$

where $(s,\tau) \in [0,\infty) \times (0,T)$ and $0 < \alpha < 1$.

In Eq. (38), s is the stock (underlying asset) price, τ represents the ongoing moment (time), V is the value of a European option price, T is the exercise time or maturity date of contract, r is the risk-free interest rate and σ is the dispersion (volatility) of the revenue from the underlying asset. The fractional derivative in the equation is the modified Reimann-Liouville derivative defined in (10).

Changing variables $s = e^x$ and $\tau = T - t$ reconstructs Eq. (38) to the following equation with constant coefficients:

$$\begin{aligned} -\frac{\partial^\alpha V(e^x, T-t)}{\partial t^\alpha} + \frac{\sigma^2}{2}\frac{\partial^2 V(e^x, T-t)}{\partial x^2} + (r - \frac{\sigma^2}{2})\frac{\partial V(e^x, T-t)}{\partial x} - \\ rV(e^x, T-t) = 0 \ , \quad (x,t) \in \mathbb{R} \times (0,T) \ , \end{aligned} \tag{40}$$

together with terminal and initial conditions

$$\begin{aligned} &V(e^{-\infty}, T-t) = V_0(T-t) \ , \quad V(e^{\infty}, T-t) = V_\infty(T-t) \ , \\ &V(e^x, T) = V_T(e^x). \end{aligned} \tag{41}$$

By defining $u(x,t) = V(e^x, T-t)$, Eq. (40) leads to:

$$\begin{aligned} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\sigma^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2} + (r - \frac{\sigma^2}{2})\frac{\partial u(x,t)}{\partial x} - r\, u(x,t) \ , \\ (x,t) \in \mathbb{R} \times (0,T) \ , \end{aligned} \tag{42}$$

together with the pursuing terminal and initial conditions

$$u(-\infty, t) = \underline{u}(t) \ , \quad u(\infty, t) = \overline{u}(t) \ , \quad u(x,0) = u_0(x). \tag{43}$$

To get a solving to the equation (42) with conditions (43) practically, we curtail the region $\mathbb{R} \times (0,T)$ to $(B_x, E_x) \times (0,T)$, where B_x and E_x are lower and upper bounds for x, respectively. Therefore, the equation converts to:

$$\begin{aligned} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = C_2\frac{\partial^2 u(x,t)}{\partial x^2} + C_1\frac{\partial u(x,t)}{\partial x} - C_0 u(x,t) + f(x,t) \ , \\ (x,t) \in (B_x, E_x) \times (0,T) \ , \end{aligned} \tag{44}$$

where $C_2 = \frac{\sigma^2}{2}$, $C_1 = r - \frac{\sigma^2}{2}$, $C_0 = r$ and the mentioned two boundary conditions and one initial condition are rewritten as:

$$u(B_x,t) = u_B(t)\ ,\quad u(E_x,t) = u_E(t)\ ,\quad u(x,0) = u_0(x). \tag{45}$$

The debate over the existence and uniqueness of the solution to the equation (44) is cited in a multitude of references, including [2,5,20,22,30,33].

Firstly, we set a uniform partition on time domain by choosing a positive integer N_t and defining $h = \frac{T}{N_t}$ and $t_k = k\,h\ , k = 0, 1, \ldots, N_t$. As we know from (11) the modified fractional Reimann-Liouville derivative is equivalent to the Caputo fractional derivative, and we can approximate Caputo fractional derivative at $t = t_k\ ,\ k = 1, 2, \ldots, N_t$ by (12) as follows:

$$\frac{\partial^\alpha u(x,t_k)}{\partial t^\alpha} \simeq \frac{1}{h^\alpha}\frac{1}{\Gamma(2-\alpha)}\sum_{l=0}^{k-1}\delta_l(u(x,t_{k-l}) - u(x,t_{k-l-1})), \tag{46}$$

where $\delta_l = (l+1)^{1-\alpha} - l^{1-\alpha}\ ,\quad l = 0, 1, \ldots, k-1$.

Hence, Eq. (44) at $t = t_k$, converts to:

$$\begin{aligned}&\frac{1}{h^\alpha}\frac{1}{\Gamma(2-\alpha)}\sum_{l=0}^{k-1}\delta_l(u(x,t_{k-l}) - u(x,t_{k-l-1}))\\ &\simeq C_2\frac{\partial^2 u(x,t_k)}{\partial x^2} + C_1\frac{\partial u(x,t_k)}{\partial x} - C_0 u(x,t_k) + f(x,t_k)\ ,\end{aligned} \tag{47}$$

Also, terminal and initial conditions at $t = t_k$ take the form:

$$u(x,0) = u_0(x)\ ,\quad u(B_x,t_k) = u_B(t_k)\ ,\quad u(E_x,t_k) = u_E(t_k).$$

Defining $u_k(x) := u(x,t_k)$ and $f_k(x) := f(x,t_k)$, the Eq. (47) becomes as the following:

$$\begin{aligned}&\frac{1}{h^\alpha}\frac{1}{\Gamma(2-\alpha)}\sum_{l=0}^{k-1}\delta_l(u_{k-l}(x) - u_{k-l-1}(x))\\ &= C_2\frac{\partial^2 u_k(x)}{\partial x^2} + C_1\frac{\partial u_k(x)}{\partial x} - C_0 u_k(x) + f_k(x),\end{aligned} \tag{48}$$

with:

$$u(x,0) = u_0(x)\ ,\quad u(B_x,t_k) =: u_{B,k}\ ,\quad u(E_x,t_k) =: u_{E,k}. \tag{49}$$

Setting $\theta = \frac{1}{h^\alpha}\frac{1}{\Gamma(2-\alpha)}$, Eq. (48) leads to:

$$\begin{aligned}&\theta\delta_0 u_k(x) + \theta\sum_{l=1}^{k-1}(\delta_l - \delta_{l-1})u_{k-l}(x) - \theta\delta_{k-1}u_0(x)\\ &= C_2\frac{\partial^2 u_k(x)}{\partial x^2} + C_1\frac{\partial u_k(x)}{\partial x} - C_0 u_k(x) + f_k(x),\end{aligned} \tag{50}$$

and thereby:

$$\begin{aligned} C_2\frac{\partial^2 u_k(x)}{\partial x^2} + C_1\frac{\partial u_k(x)}{\partial x} - (\theta\delta_0 + C_0)u_k(x) = \\ \theta\sum_{l=0}^{k-1}(\delta_l - \delta_{l-1})u_{k-l}(x) - \theta\delta_{k-1}u_0(x) - f_k(x). \end{aligned} \tag{51}$$

with boundary conditions (49). Thus, according to Eq. (51), we have a linear ODE with constant coefficients and boundary conditions which can be numerically solved using the method previously described in subsection 3.1.

4.2. **Notes about existence of the solution.** According to the method mentioned in subsection 3.1 and considering Eq. (51), if we define:

$$\begin{aligned} &\alpha = C_2, \quad \beta = C_1, \quad \gamma = -(C_0 + \theta\delta_0), \\ &A = \alpha\mathbf{I} + \beta\mathbf{P}^T + \gamma(\mathbf{P}^2)^T - (\beta\mathbf{E} + \gamma\mathbf{B})\mathbf{F}^T \\ &\quad = \alpha\mathbf{I} + \beta\mathbf{P}^T + \gamma(\mathbf{P}^2)^T - (\beta\mathbf{E}\mathbf{\Psi}^T(1) + \gamma\mathbf{B}\mathbf{\Psi}^T(1))(\mathbf{P}^2)^T \\ &u_j(x) \simeq X_j^T\mathbf{\Psi}(x), \end{aligned}$$

As observed, to solve Eq. (38) numerically we reached an equation of the form:

$$A\,X_n = -\theta\,\delta_{n-1}\,X_0 - \theta\sum_{j=1}^{n-1}\bar{\delta}_{n-j}X_j\ , \quad n = 1, 2, \ldots, N_t, \tag{52}$$

where $\theta = \frac{1}{h^\alpha}\frac{1}{\Gamma(2-\alpha)}$, $\delta_i = (i+1)^{1-\alpha} - i^{1-\alpha}$ and $\bar{\delta}_i = \delta_i - \delta_{i-1}$. Or equivalently,

$$\sum_{j=1}^{n-1}\bar{\delta}_{n-j}X_j + \tilde{A}\,X_n = \delta_{n-1}\,X_0\ , \quad n = 1, 2, \ldots, N_t, \tag{53}$$

where $\tilde{A} = -\frac{1}{\theta}A$ and X_0 is a given vector obtained from initial condition.

One of the important advantages of using this method is that the matrix of coefficients, $\tilde{A}$, is constant in all steps and only the vector on the righthand side changes, so to solve the system of linear equations obtained, there is no need to calculate the inverse of this matrix in every step and its inverse can be Calculated only once.

Setting $n = 1, 2, \ldots, N_t$ gives us

$$\begin{aligned} &\tilde{A}X_1 = \delta_0 X_0, \\ &\bar{\delta}_0 X_1 + \tilde{A}X_2 = \delta_1 X_0, \\ &\vdots \\ &\sum_{j=1}^{N_t-1}\bar{\delta}_{N_t-j}X_j + \tilde{A}X_{N_t} = \delta_{N_t-1}X_0, \end{aligned}$$

If we write the above system in matrix form, we have:

$$\begin{bmatrix} \tilde{A} & 0 & 0 & \cdots & 0 & 0 \\ \bar{\delta}_0 I & \tilde{A} & 0 & \cdots & 0 & 0 \\ \bar{\delta}_1 I & \bar{\delta}_0 I & \tilde{A} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \bar{\delta}_{N_t-3} I & \bar{\delta}_{N_t-4} I & \bar{\delta}_{N_t-5} I & \cdots & \tilde{A} & 0 \\ \bar{\delta}_{N_t-2} I & \bar{\delta}_{N_t-3} I & \bar{\delta}_{N_t-4} I & \cdots & \bar{\delta}_0 I & \tilde{A} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_{N_t-1} \\ X_{N_t} \end{bmatrix} = \begin{bmatrix} \delta_0 X_0 \\ \delta_1 X_0 \\ \delta_2 X_0 \\ \vdots \\ \delta_{N_t-2} X_0 \\ \delta_{N_t-1} X_0 \end{bmatrix},$$

or:

$$\left(I \otimes \tilde{A} + \tilde{\Delta} \otimes I\right) \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N_t} \end{bmatrix} = \begin{bmatrix} \delta_0 X_0 \\ \delta_1 X_0 \\ \vdots \\ \delta_{N_t-1} X_0 \end{bmatrix},$$

where $\otimes$ is the Kronecker product, I is the identity matrix of appropriate order and:

$$\tilde{\Delta} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \bar{\delta}_0 & 0 & 0 & \cdots & 0 \\ \bar{\delta}_1 & \bar{\delta}_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{\delta}_{N_t-2} & \bar{\delta}_{N_t-3} & \bar{\delta}_{N_t-4} & \cdots & 0 \end{bmatrix}.$$

If we define matrices $X := [X_1\ X_2\ \dots\ X_{N_t}]$ and $\bar{C} := [\delta_0 X_0\ \delta_1 X_0\ \dots\ \delta_{N_t-1} X_0]$, then by theorem 3 we have the Sylvester equation, that is,

$$\tilde{A}X + X\tilde{\Delta}^T = \bar{C}. \tag{54}$$

Definition 2. [13] Let $c_i \in \mathbb{R}^n$ denote the columns of $C \in \mathbb{R}^{n\times m}$ so that $C = [c_1, \dots, c_n]$. Then $\mathbf{Vec}(C)$ is charactrized to be the mn-vector constituted by stacking the columns of C on top of another, i.e. $\mathbf{Vec}(C) = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \in \mathbb{R}^{mn}$.

[13]Using Definition 2 the linear system $AX + XB = C$ can be adjusted in the form

$$[(I_m \otimes A) + (B^T \otimes I_n)]\mathbf{Vec}(X) = \mathbf{Vec}(C).$$

Theorem 3. [13] Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$, $C \in \mathbb{R}^{n\times m}$, then the Sylvester equation $AX + XB = C$ has one and only one solution if and only if the matrices A and $-B$ do not have a common eigenvalue.

It's obvious that the matrix $\bar{\Delta}$ have no eigenvalues other than 0. Hence only condition that guarantees existence and uniqueness of solution to the equation is that all eigenvalues of the matrix $\tilde{A}$ differ from 0 or equivalently $\det \tilde{A} \neq 0$.

From another perspective. Now, we look at the linear system obtained by the mentioned method from a different view. Remember that we begin with a fractional PDE which was converted to an ODE at each $t_k \quad k = 1, 2, \ldots, N_t$, afterwards, each ODE was transformed to a system of linear equation. Beginning with $k = 1$ and solving the linear system, numerically solves the ODE related to its t_k, in turn. In next step, with solution of ODE at former step in hand, ODE at t_{k+1} can be solved and so forth. As we know, the final solution is one correspond to t_{N_t} that determins the option price at $\tau = 0$.

As briefly described, $u_{N_t}(x)$ solves the problem completely. Thus if we can get $u_{N_t}(x)$ without having to solve the middle problems, $u_k(x)$, $k = 1, 2, \ldots, N_t - 1$, i.e. if we can eliminate intermediary problems, we have saved a lot of time and avoided a lot of calculations. This is possible as you can see in following.

From Eq. (52), we have:

$$\tilde{A}X_n = \delta_{n-1}X_0 + \sum_{j=1}^{n-1} \bar{\delta}_{n-j}X_j, \quad n = 1, 2, \ldots, N_t,$$

multiplying this equation by $\tilde{A}^{n-1}$ from left, gives

$$\begin{aligned} \tilde{A}X_1 &= X_0, \qquad (\delta_0 = 1) \\ \tilde{A}^n X_n &= \delta_{n-1}\tilde{A}^{n-1}X_0 + \sum_{j=1}^{n-1} \bar{\delta}_{n-j}\tilde{A}^{n-j-1}(\tilde{A}^j X_j), \quad n = 1, 2, \ldots, N_t, \end{aligned} \tag{55}$$

this means, we can define a polynomial recursively as:

$$\begin{aligned} p_0(x) &= 1 \\ p_n(x) &= \delta_n x^n + \sum_{j=1}^{n} \bar{\delta}_{n-j+1}x^{n-j}p_{j-1}(x), \quad n = 1, 2, \ldots, N_t, \end{aligned} \tag{56}$$

and using this, now, we enable to define

$$\tilde{A}^n X_n = p_{n-1}(\tilde{A})X_0, \quad n = 1, 2, \ldots, N_t, \tag{57}$$

where $p_{n-1}(\tilde{A})$ is the value of $p_{n-1}(x)$ at $\tilde{A}$ with appropriate modifications.

Another advantage of using this method and the above formula is that it is not depend on intermediate solutions, X_is, $i = 1, \ldots, N_t - 1$, and only depends on X_0, hence, this enables us to calculate final solution at once.

To calculate the polynomials $p_n(x)$ we can proceed as follows too. We define $p_n(x) = \sum_{i=0}^{n} \alpha_{n,i}x^i$ then calculate the coefficients $\alpha_{n,i}$ as:

$$\begin{aligned} \alpha_{n,n} &= \delta_n , \qquad n = 0, 1, 2, \ldots \\ \alpha_{n,m} &= \sum_{k=1}^{m+1} \bar{\delta}_k \alpha_{n-k,m-k+1} , \qquad m < n. \end{aligned}$$

5. Notes on Convergence

In this section, we will have some argue about conditions that guarantees convergence of the presented method and introduce definitions, theorems and etc about convergence.

Definition 4. [31] A mapping $T : X \to Y$ is bounded if there exists a constant $c > 0$, independent of $x \in X$, such that $\|Tx\|_Y \leq c\|x\|_X$ for all $x \in X$.

For a linear mapping T, boundedness is equivalent to continuity [31].

Definition 5. [1] Assume V and W be spaces that equipped with a norm, and let operator $\mathcal{K} : V \to W$ be linear. Then $\mathcal{K}$ is compact if the set $\{\mathcal{K}v \mid \|v\|_V \leq 1\}$ has compact closure in W. This is equivalent to saying that for every bounded sequence $\{v_n\} \subset V$, the sequence $\{\mathcal{K}v_n\}$ has a subsequence that is convergent to some point in W. Compact operators are also called completely continuous operators.

Suppose $T : X \to Y$ be a bounded linear operator and X, Y Banach spaces. If T has finite rank, then T is compact [31]. Therefore:
Every operator with a finite dimension image is compact. For example, matrices are compact operators of finite rank as same as functionals [23].

Corollary 6. [4] Every Volterra operator,

$$Tf(x) = \int_0^x k(x, y)f(y)\, dy.$$

where $k(x, y)$ is continuous on $\Delta = \{(x, y) \in [a, b] \times [a, b]\,; a \leq y \leq x \leq b\}$ is a compact operator.

Lemma 7. [1] Suppose V be a Banach space, and assume $\{\mathcal{P}_n\}$ be a collection of bounded projections on V with

$$\mathcal{P}_n u \to u \text{ as } n \to \infty, u \in V.$$

If operator $\mathcal{K} : V \to V$ is compact, then

$$\|\mathcal{K} - \mathcal{P}_n\mathcal{K}\| \to 0 \text{ as } n \to \infty.$$

Theorem 8. The presented method in subsection 3.1 to solve ODE (27) numerically is convergent.

Proof. First, we define operator

$$\mathcal{P}_n : L^2[0, 1] \to W_n := \text{span}\{\psi_0, \ldots, \psi_n\} \subseteq L^2[0, 1],$$

as the orthogonal projection operator that maps every $f \in L^2[0, 1]$ to its approximation in W_n, i.e. $\mathcal{P}_n f(x) = \mathbf{C}^T\mathbf{\Psi}(x)$. Observing that $\mathcal{P}_n$ is a bounded linear operator is uncomplicated. Now, we define operator $\mathcal{K} : L^2[0, 1] \to L^2[0, 1]$ as $\mathcal{K}f = \int_0^x f(t)\, dt$. By corollary 6 $\mathcal{K}$ is compact.

We know that:

$$\mathcal{P}_n y'' = \mathbf{C}^T\mathbf{\Psi}(x) \to y'', \text{ as } n \to \infty,$$

for $y'' \in L^2[0,1]$. Also, verifying that,

$$\mathcal{P}_n \mathcal{K} y'' = \mathcal{P}_n \int_0^x y''(t)\,dt = \mathbf{C}^T \mathbf{P} \mathbf{\Psi}(x),$$

is uncomplicated. Here $\mathbf{P}$ is the operational matrix of integration for $\mathbf{\Psi}(x)$, (ChWs). So, by Lemma 7 we have:

$$\|\mathcal{K} - \mathcal{P}_n \mathcal{K}\| \to 0, \text{ as } n \to \infty.$$

Therefore, $y' \to \mathbf{C}^T P \mathbf{\Psi}(x) + y'_0$.

And

$$\mathcal{K}^2 f = \int_0^x \int_0^t f(u)\,du = \int_0^x (x-t) f(t)\,dt,$$

agian, by corollary 6, in a similar way as above, can deduce that

$$y \to \mathbf{C}^T P^2 \mathbf{\Psi}(x) + y'_0 x + y_0,$$

and this complets the proof. □

Definition 9. [1] Suppose W is a Banach space. A linear operator P on W satisfying the trait $P^2 = P$ is named a projection operator.

If W, in the definition, be a Hilbert space, P be a projection operator, and $W = P(W) \oplus (I-P)(W)$ be an orthogonal direct sum, then we name P an orthogonal projection operator.
It isn't difficult to observe that a projection operator is orthogonal if and only if $\langle Pv, (I-P)w \rangle = 0$, $\forall v, w \in W$ [1].

Theorem 10. [1] Suppose W an inner product space and K is a complete subspace of it. Then the orthogonal projection operator $P_K : W \to W$, best approximation in K, is linear self-adjoint, i.e.,

$$\langle P_K u, v \rangle = \langle u, P_K v \rangle,$$

and

$$\|P_K\| = 1.$$

Theorem 11. [1] Assume W_1 is a closed linear subspace of the Hilbert space W, and $W_1^{\perp}$ be the orthogonal complement of W_1. Let $P : W \to W_1$. Then

(1) If the operator P be a self-adjoint projection, then, it is an orthogonal projection and vice versa.
(2) Every orthogonal projection $P \neq 0$ is continuous and $\|P\| = 1$.
(3) $W = W_1 \oplus W_1^{\perp}$.
(4) The orthogonal projection operator from W onto W_1 is unique.

Lemma 12. [16] Assume H be a Hilbert space, and let P be an orthogonal projection operator on H, i.e., $P \in BL(H)$, that is, P is a bounded linear operator, $P^2 = P$ and $R(P) \perp Z(P)$. Then P is a positive operator.

Theorem 13. [8] Assume $a < b$ be two real numbers and suppose $L_1, L_2, \ldots$ be positive operators all defined on domain D, which contains the restrictions of $1, t, t^2$ to the interval $[a, b]$. For $n = 1, 2, \ldots,$ assume $L_n(1)$ is bounded. Suppose $g \in D$ be continuous in $[a, b]$, with modulus of continuity ω. Then for $n = 1, 2, \ldots,$

$$\|g - L_n(g)\| \leq \|g\| \cdot \|L_n(1) - 1\| + \|L_n(1) + 1\|\omega(\mu_n),$$

where $\mu_n = \|(L_n([t-x]^2))(x)\|^{\frac{1}{2}}$, and $\|\cdot\|$ denote the sup-norm on $[a, b]$. Notably, if $L_n(1) = 1$, then the conclusion becomes

$$\|g - L_n(g)\| \leq 2\omega(\mu_n).$$

Definition 14. [1] Modulus of continuity of function g on $[a, b]$ is represented by $\omega(g; \delta)$ for $\delta \leq b - a$ and defined by:

$$\omega(g; \delta) = \max_{\substack{|t_1 - t_2| < \delta \\ a \leqslant t_1, t_2 \leqslant b}} |g(t_1) - g(t_2)|.$$

Theorem 15. [8] In case that the function g is continuous in the interval $[-1, 1]$, then

$$E_n(g) = \|P_n(g, \cdot) - g\| \leq 6\omega(g; n^{-1}),$$

wherein $P_n(g, t)$ is an algebraic polynomial whose degree do not exceed n, with the least deviation from the function g.

Corollary 16. The presented method in subsection 3.1 is convergent.

Proof. Let $n = 1, 2, \ldots$ and define L_n be the operator that maps the (accurate) solution of the ODE (27), $y(x)$, to the approximate solution, $y_n(x)$, obtained by (37). As, by corollary 12, we can verify easily, the operator is positive and the functions 1, x and x^2 are solutions to the following ODEs respectively:

$$\alpha y'' + \beta y' + \gamma y = \gamma \ , \quad y_0 = y_1 = 1 \tag{58}$$

$$\alpha y'' + \beta y' + \gamma y = \gamma x + \beta \ , \quad y_0 = 0, \ y_1 = 1 \tag{59}$$

$$\alpha y'' + \beta y' + \gamma y = \gamma x^2 + 2\beta x + 2\alpha \ , \quad y_0 = 0, \ y_1 = 1 \tag{60}$$

and if we try to solve the first equation using the presented method, we have $r(x) = \gamma$ and hence $\mathbf{R} = \gamma\mathbf{E}$ where the vector $\mathbf{E}$ introduced in Eq (33). So, by Eq (36)

$$\Big(\alpha I + \beta\mathbf{P}^T + \gamma(\mathbf{P}^T)^2 - (\beta\mathbf{E} + \gamma\mathbf{B})\mathbf{\Psi}^T(1)(\mathbf{P}^T)^2\Big)\mathbf{C} = \gamma E - \gamma E = \mathbf{0} \ ,$$

consequently, by Eqns (33) and (34)

$$\Big(\alpha I + \beta\mathbf{P}^T + \gamma(\mathbf{P}^T)^2 - (\beta\mathbf{E}\mathbf{\Psi}^T(1) + \gamma\mathbf{B}\mathbf{\Psi}^T(1))(\mathbf{P}^T)^2\Big)\mathbf{C} = \mathbf{0}.$$

Therefore, $\mathbf{C} = \mathbf{0}$ and thereby according to Eq (37) we will have $y_n(x) = 1$. This means $L_n(1) = 1$. Consequently, by theorem 13

$$\|y - L_n(y)\| \leq 2\omega(\mu_n) \ ,$$

and since μ_n is continuous on compact set $[0, 1]$, the method is convergent. □

5.1. **Notes to facilitate calculations.** In this subsection, we consider some vectors and matrices that appear in calculations of the method and calculating their elements will considerably reduce calculations time and complexity.

First, lets begin with defining some vectors as follows:

$$\mathcal{E}_k := [\underbrace{0,\dots,0}_{2^k-1 \text{ times}},1]^T ,$$

$$\mathcal{J}_k := [\underbrace{1,\dots,1}_{2^k \text{ times}}]^T ,$$

$$\mathcal{H}_M := \left[(-1)^M,(-1)^{M-1},\dots,(-1)^0\right]^T ,$$

$$\mathcal{M}_k := [0,1,\dots,2^k-1]^T ,$$

$$\mathcal{N}_M := \left[\sqrt{1},\sqrt{3},\dots,\sqrt{2M+1}\right]^T ,$$

$$\mathcal{A}_M := \left[\sum_{j=0}^{M}\frac{a_{0,j}}{j+2},\sum_{j=0}^{M-1}\frac{a_{1,j}}{j+3},\dots,\sum_{j=0}^{0}\frac{a_{M,j}}{M+j+2}\right]^T ,$$

then, calculate $\mathbf{\Psi}(1)=[\psi_{00}(1),\dots,\psi_{nm}(1)]^T$. By definition of ChWs (2) we have:

$$\mathbf{\Psi}(1)=2^{k/2}\Big[\underbrace{0,0,\dots,0,0}_{(2^k-1)(M+1)\text{ times}},(-1)^M\sqrt{1},(-1)^{M-1}\sqrt{3},\dots,(-1)^0\sqrt{2M+1}\Big]^T$$

$$= 2^{k/2}\mathcal{E}_k\otimes(\mathcal{H}_M\circ\mathcal{N}_M) , \tag{61}$$

where $\circ$ denotes the Hadamard (elementwise) multiplication. Now, we try to calculate elements of vector $\mathbf{E}$. Remember that we had $1=\mathbf{E}^T\mathbf{\Psi}(x)$. So $E_i=\int_0^1\psi_i(x)\,dx,\quad i=1,\dots,\hat{m}=2^k(M+1)$. According to Eq (19) $E_i=\frac{\sqrt{2m+1}}{2^{k/2}(M+1)}$ where $m=(i-1)\mod(M+1)$. Therefore, in vector notation, we have:

$$\mathbf{E}=\frac{1}{2^{k/2}(M+1)}(\mathcal{J}_k\otimes\mathcal{N}_M). \tag{62}$$

Also, remember that $x=\mathbf{B}^T\mathbf{\Psi}(x)$, so, $B_i=\int_0^1 x\psi_i(x)\,dx=\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}}x\psi_{nm}(x)\,dx$, $i=1,\dots,\hat{m}$ where $n=(i-1)\div(M+1)$ and $m=(i-1)\mod(M+1)$. After integrating, we deduce $B_i=\frac{\sqrt{2m+1}}{2^{3k/2}(M+1)}\big(n+(M+1)\sum_{j=0}^{M-m}\frac{a_{m,j}}{m+j+2}\big)$, or in vector notation

$$\mathbf{B}=\frac{1}{2^{3k/2}(M+1)}\left(\mathcal{M}_k\otimes\mathcal{N}_M+(M+1)\mathcal{J}_k\otimes(\mathcal{N}_M\circ\mathcal{A}_M)\right). \tag{63}$$

Finally, matrix $\mathfrak{N}$ in Eq (25) can be calculated as:

$$\mathfrak{N}=\frac{1}{2^k(M+1)^2}\mathcal{N}_M\cdot\mathcal{N}_M^T. \tag{64}$$

6. **Illustrative Examples**

In this section, to show the effectiveness of the presented method, we present various examples and compare them with other methods if possible. As we will see, all the results obtained in the examples are confirmed by theoretical results.

Example 2. [17, 18, 26, 32] Consider the following problem with accurate solution $u(x,t) = (x^3 + x^2 + 1)(t + 1)^2$

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = C_2 \frac{\partial^2 u(x,t)}{\partial x^2} + C_1 \frac{\partial u(x,t)}{\partial x} + C_0\, u(x,t) + f(x,t), \qquad (x,t) \in (0,1) \times (0,1),$$

$$u(0,t) = (t+1)^2, \quad u(1,t) = 3(t+1)^2\ ,$$
$$u(x,0) = x^3 + x^2 + 1,$$

where, $C_2 = 1$, $C_0 = -0.5$, $C_1 = -(C_0 + C_2)$ and $f(x,t) = \frac{2t^{1-\alpha}}{\Gamma(1-\alpha)}\big(\frac{t}{(2-\alpha)(1-\alpha)} - \frac{1}{1-\alpha}\big)(x^3 + x^2 + 1) - (t+1)^2\big(C_0 x^3 + (3C_1 + C_0)x^2 + (6C_2 + 2C_1)x + 2C_2 + C_0\big)$. The presented method is applied on the problem with $k = 3$, $M = 3$, $\alpha = 0.1, 0.25, 0.5, 0.75, 0.9$ and $N_t = 20, 40, 60, 80, 100, 120$. The results are showed in table 2. Also in Fig 1, the graphs of the L_∞ of the error for $\alpha = 0.1$ at the $t = t_{N_t} = T$ are displayed. As can be seen and expected, Reducing the value of α and increasing the value of N_t both lead to a reduction in the error value. Of course, theory also confirms this.

Table 2. L_∞ Error and evaluated order of convergence w.r.t. time for different values of α and N_t in Example 2

		N_t						Order of
		20	40	60	80	100	120	convergence
α	0.1	5.16739×10^{-5}	1.49800×10^{-5}	7.22394×10^{-6}	4.29732×10^{-6}	2.86937×10^{-6}	2.06160×10^{-6}	1.81334
	0.25	1.95118×10^{-4}	6.06715×10^{-5}	3.05086×10^{-5}	1.87029×10^{-5}	1.27857×10^{-5}	9.36580×10^{-6}	1.70722
	0.5	8.34043×10^{-4}	2.99211×10^{-4}	1.63911×10^{-4}	1.06865×10^{-4}	7.66629×10^{-5}	5.84296×10^{-5}	1.48965
	0.75	2.84050×10^{-3}	1.19842×10^{-3}	7.22874×10^{-4}	5.04889×10^{-4}	3.82168×10^{-4}	3.04379×10^{-4}	1.24827
	0.9	5.65629×10^{-3}	2.64139×10^{-3}	1.69153×10^{-3}	1.23288×10^{-3}	9.64641×10^{-4}	7.89400×10^{-4}	1.09961

Table 3. L_∞ error with $\alpha = 0.7$ for problem of Example 2

N_t	Method in [17]	Method in [18]	Method in [32]	Presented Method
80	3.79591×10^{-4}	2.9376×10^{-4}	3.5176×10^{-4}	3.72833×10^{-4}
160	1.54434×10^{-4}	1.1949×10^{-4}	1.3065×10^{-4}	1.51684×10^{-4}

Comparing the results with results in the cited references, Table 3, it can be seen that solutions with accuracy of the same order have relatively large N_t and more calculations in the references.

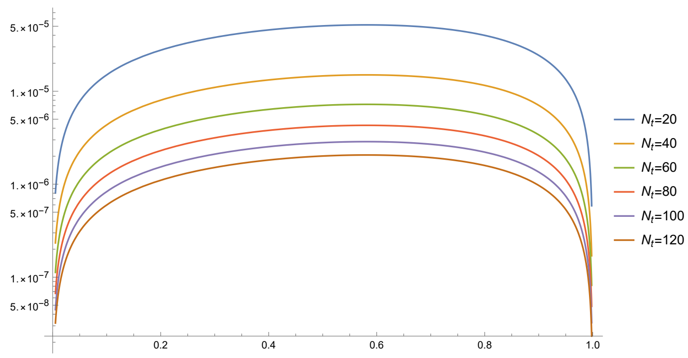

FIGURE 1. Error function for $N_t = 20, 40, 60, 80, 100, 120$ and $\alpha = 0.1$ at $t = T$ for Example 2

Example 3. [17, 18, 26, 32] Consider the following problem with exact solution $u(x,t) = (t+1)^2 x^2 (1-x)$

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = C_2 \frac{\partial^2 u(x,t)}{\partial x^2} + C_1 \frac{\partial u(x,t)}{\partial x} + C_0\, u(x,t) + f(x,t),$$
$$(x,t) \in (0,1) \times (0,1),$$
$$u(0,t) = 0, \quad u(1,t) = 0\ ,$$
$$u(x,0) = x^2(1-x),$$

where, $C_2 = \frac{\sigma^2}{2}$, $C_0 = -r$, $C_1 = r - \frac{\sigma^2}{2}$, $\sigma = 0.25$, $r = 0.05$ and $f(x,t) = 2C_2(1+t)^2(3x-1) + C_1(1+t)^2 x(3x-2) - \frac{2t^{1-\alpha}(t+2-\alpha)x^2(x-1)}{\Gamma(3-\alpha)}$. The presented method is applied on the problem for $\alpha = \frac{7}{10}$, $k = 4$, $M = 3$ and $N_t = 120$. The results are showed in table 4 and Fig 2.

TABLE 4. L_∞ error with $\alpha = 0.7$ for problem of Example 3

N_t	Method in [17]	Method in [18]	Method in [32]	Presented Method
80	3.09290×10^{-4}	2.28581×10^{-4}	2.5638×10^{-4}	2.58647×10^{-4}
160	1.26173×10^{-5}	9.32188×10^{-5}	9.5056×10^{-5}	9.05819×10^{-5}

As can be seen, the presented method produces less error compared to the example 1 mentioned in [32].

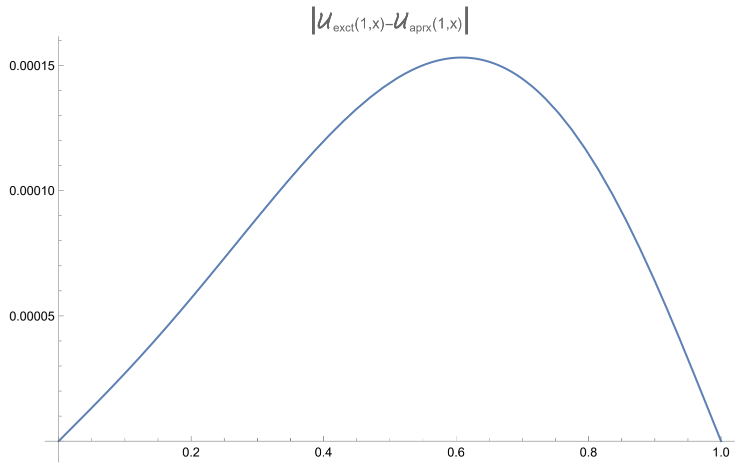

FIGURE 2. Function $\mathcal{U}(x,1)$ with $\alpha = 0.7$ and $N_t = 120$ for example 3

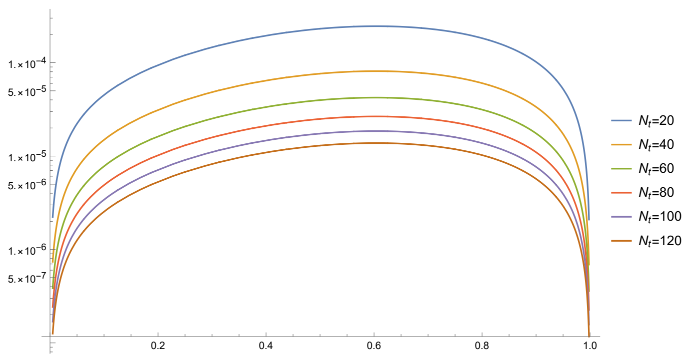

FIGURE 3. Error function for various N_t for Example 3

Example 4. Consider the following problem

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = C_2 \frac{\partial^2 u(x,t)}{\partial x^2} + C_1 \frac{\partial u(x,t)}{\partial x} + C_0\, u(x,t) + f(x,t), \qquad (x,t) \in (0,1) \times (0,1),$$

$$u(0,t) = t^{4\alpha}, \quad u(1,t) = -\frac{t^{4\alpha}}{e^2}\ ,$$

$$u(x,0) = 0,$$

with exact solution $u(x,t) = e^{-2x}\cos(3\pi x)t^{4\alpha}$, where, $C_2 = \frac{\sigma^2}{2}$, $C_0 = -r$, $C_1 = r - \frac{\sigma^2}{2}$, $\sigma = 0.25$, $r = 0.05$ and

$$f(x,t) = \frac{1}{2}e^{-2x}t^{3\alpha}\Bigg(3t^{\alpha}\Big(\pi\left(2r-5\sigma^2\right)\sin(3\pi x) + \left(2r+\left(3\pi^2-2\right)\sigma^2\right)\cos(3\pi x)\Big) + \frac{2\Gamma(4\alpha+1)\cos(3\pi x)}{\Gamma(3\alpha+1)}\Bigg).$$

The presented method is applied on the problem for $\alpha = 0.2,\ 0.7$, $k = 5,6$, $M = 1,2,3$ and $N_t = 1000$. The results are showed in table 5.

TABLE 5. L_∞ of error of example 4 with $k = 5,6$, $M = 1,2,3$ and $\alpha = 0.2, 0.7$

		$\alpha = 0.2$		$\alpha = 0.7$	
		k		k	
		5	6	5	6
M	1	1.74818×10^{-3}	4.61812×10^{-4}	2.04976×10^{-3}	4.42312×10^{-4}
	2	3.82118×10^{-5}	1.19771×10^{-5}	1.37218×10^{-4}	3.25044×10^{-5}
	3	1.15894×10^{-6}	1.07891×10^{-7}	3.10334×10^{-5}	3.02873×10^{-5}

As it can be seen in table 5 and the theory also confirms it, with the increase of M and/or k and the decrease of α, the error of the method decreases.

Example 5. Take into account the following fractional Black-Scholes equation for an European option:

$$\frac{\partial^\alpha \mathcal{U}(s,\tau)}{\partial\tau^\alpha} + \frac{\sigma^2}{2}s^2\frac{\partial^2\mathcal{U}(s,\tau)}{\partial s^2} + (r-D)s\frac{\partial\mathcal{U}(s,\tau)}{\partial s} - r\,\mathcal{U}(s,\tau) = 0,$$
$$(s,\tau) \in (3,15)\times(0,1),$$
$$\mathcal{U}(3,\tau) = 0\ ,\quad \mathcal{U}(15,\tau) = 0,$$
$$\mathcal{U}(s,T) = \max\{S-K,0\}\ ,$$

where, $\sigma = 0.45, r = 0.03, D = 0.01$ and $T = 1, K = 10$. The equation express's a European double barrier knock-out call option and has no exact solution for the given α values. This equation has been solved by the presented method for $\alpha = 0.1, 0.5, 0.9$ and $N_t = 100, 200$, and the results are given in the fig 4.

Example 6. Consider the following time-fractional Black-Scholes equation conducting an European call option:

$$\frac{\partial^\alpha \mathcal{U}(s,\tau)}{\partial\tau^\alpha} + \frac{\sigma^2}{2}s^2\frac{\partial^2\mathcal{U}(s,\tau)}{\partial s^2} + (r-D)s\frac{\partial\mathcal{U}(s,\tau)}{\partial s} - r\,\mathcal{U}(s,\tau) = 0,$$
$$(s,\tau) \in (0.1,100)\times(0,1),$$
$$\mathcal{U}(0.1,\tau) = 0\ ,\quad \mathcal{U}(100,\tau) = 100 - K\exp(-r(1-\tau)),$$
$$\mathcal{U}(s,T) = \max\{S-K,0\}\ ,$$

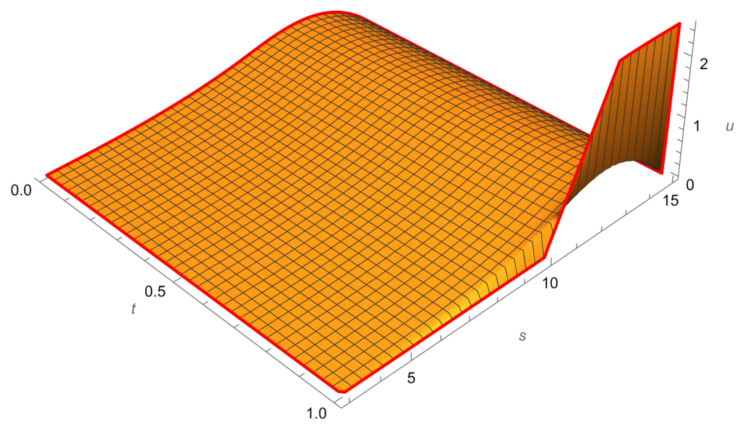

FIGURE 4. Function $\mathcal{U}(s,\tau)$ with $\alpha = 0.1$ and $N_t = 100$

where, $\sigma = 0.25, r = 0.05, D = 0$ and $T = 1, K = 50$.

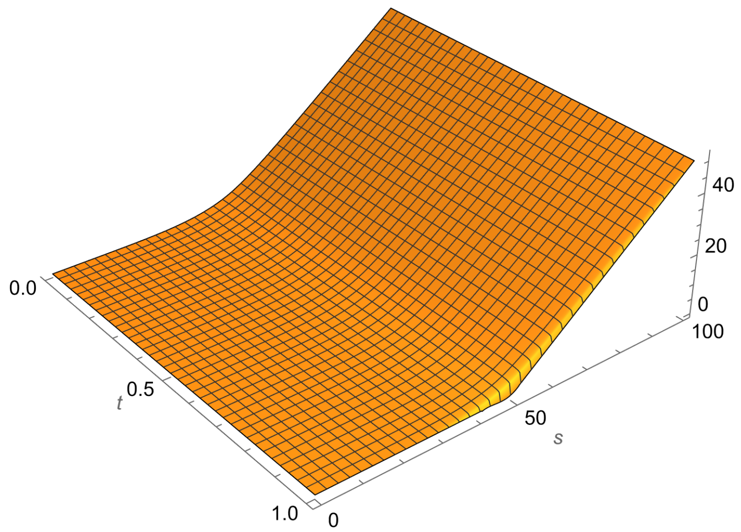

FIGURE 5. Function $\mathcal{U}(s,\tau)$ with $\alpha = 0.1$ and $N_t = 200$

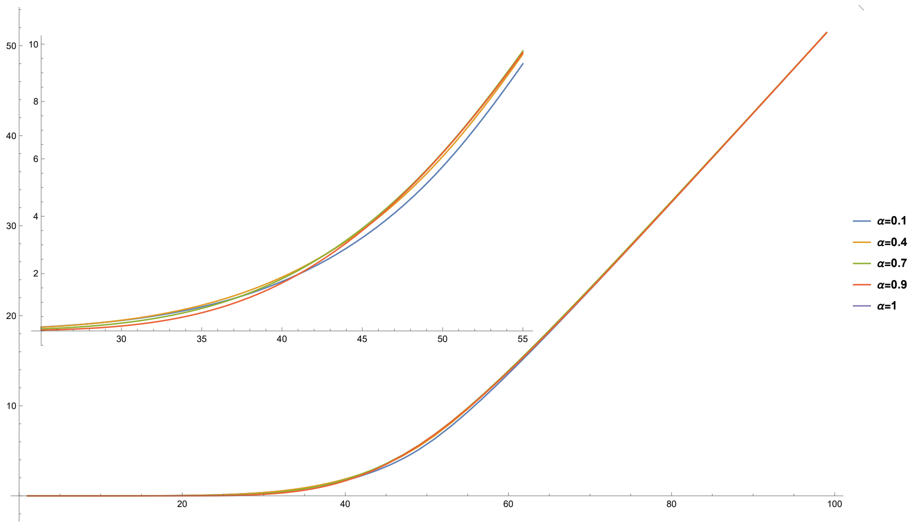

FIGURE 6. function $u(s,t)$ with various αs at $t = t_{N_t}$

The equation has been solved by the presented method for $\alpha = 0.1, 0.4, 0.7, 0.9, 1$ and $N_t = 100, 200$, and the results are shown in the figs 5 and 6.

7. Conclusion

In this paper, a novel method based on Chelyshkov wavelets is utilized to approximate the solution of TFBS equation for pricing standard European call/put options. TFBS model is the general form of classical Black-Scholes model. The method introduced in the paper is a hybrid method that combines a $(2-\beta)$-order finite difference method for discretization in time dimension and an operational method based on Chelyshkov wavelets for discretization in space dimension from a computational point of view.

The proposed method has two main advantages: First, the coefficient matrix of the resulting linear system of equations remains constant during calculation processes and the second, if one need only final solution (soulution of TFBS at initial time $t = 0$) it can be calculated with no need for intermediate steps calculation as explained in (57). In economics, the time-fractional Black-Scholes PDE plays a key role in defining European options in financial activities. The Black-Scholes model is a popular choice for option pricing due to its simplicity and availability from a practical point of view. The convergence of Chelyshkov wavelets approximation has been proved and applied to show that the introduced method is also convergent. The method's applicability and theoretical orders of convergence are demonstrated through numerical experiments. The approximate solution obtained by the presented method has notable contiguity with the accurate solution of the equation. Also, as seen, the calculated numerical solutions are endorsed with the stated theoretical topics. Additionally, the considered method has been applied to price three distinctive European options governed by a time-fractional Black-Scholes model.

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