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SOME PROPERTIES OF FINITE GENERALIZED-GROUPS

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Abstract. In this article, we discuss the concept of completely simple-semigroups, which serves as a natural extension of the group structures. These semigroups, also known as generalized-groups, provide an interesting generalization beyond the realm of the groups. Many scientists have investigated various applications of generalized-groups. Notably, this algebraic structure has connections to the unified gauge theory. In this article, we investigate the structures and properties of generalized-groups, providing examples and results within this fascinating subject. Specially, we show that the generalized Lagrange Theorem may not be true for generalized-groups.

Keywords: Completely simple semigroups, Groups, Generalized-groups, Algebraic structure.
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1. Introduction

In 1998, M. R. Molaei [6] introduced the generalized-groups as an extension of the traditional group structure. A generalized-group is a set $G \neq \emptyset$ equipped with a binary operation called multiplication, satisfying the following rules for each x, y, and z in G:

- $x(yz) = (xy)z$; (Associativity)
- For each x in G, there is exactly one corresponding element $e(x)$ in G such that $xe(x) = e(x)x = x$; (Identity element(s))
- For each x in G, there is a corresponding element x' in G such that $xx' = x'x = e(x)$. (Inverse element)

Remarkably, J. Araujo and J. Konieczny [5] established the equivalence between the generalized-groups and the completely simple semigroups. Specifically, consider G be a semigroup where for each element x in G, we had $G \cdot x \cdot G = G$, and consider α and β are idempotent elements in G such that $\alpha \cdot \beta = \beta \cdot \alpha$, then G qualifies as a completely simple semigroup. In this article, we collectively refer to these structures as generalized-groups. Many scientists, including Professors V. V. Vagner [10], M. R. Molaei [7], M. R. Ahmadi Zand and S. Rostami [3],

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P. G. Romeo and K. K. Sneha [8], A. A. A. Agbola [2], have explored various applications of generalized-groups. It should be noted that this algebraic structure is related to the unified gauge theory.

2. **Generalized Groups**

In this section, we explore some properties of the generalized-groups.

Definition 2.1. [9] Suppose $G \neq \emptyset$ be a set, and assume that "$\star$" denotes a binary operation over the set G. We introduce the following terms:
1. If G is a Groupoid and if for each g and h in G, the equations $g \star x = h$ and $y \star g = h$ have solutions in G, then couple $(G, \star)$ is called a quasi-group.
2. If $(G, \star)$ is a groupoid and for each g, h, and k in G, we have $(g \star h) \star k = g \star (h \star k)$, then $(G, \star)$ is classified as a semigroup.

Definition 2.2. [6] A semigroup $(G, \star)$ which satisfies the following conditions, is called a generalized-group:

- For each element g in G, there is a unique element $e(g)$ in G such that $e(g) \star g = g \star e(g) = g$.
- For each element g in G, there is an element g^{-1} in G such that $g^{-1} \star g = g \star g^{-1} = e(g)$.

Example 2.3. *Every group is a generalized-group. It is well-known that every group naturally falls into the category of generalized-groups. Specifically, consider a group G. We can define the set of elements $\{e(g) : g \in G\}$ to be equal to the singleton set $\{e\}$. This simple observation highlights the inherent connection between groups and generalized-groups.*

Example 2.4. [5] *Let G be a group and e be the identity element of G. Additionally, let $\Gamma \neq \emptyset$ and $I \neq \emptyset$ be sets. Consider the $\Gamma \times I$ matrix $P = (g_{\gamma i})$ over the group G. Now, for elements i, j in I, and γ, μ in Γ, as well as k, h in G, we define "$\star$" the binary operation on the set $I \times G \times \Gamma$ as follows:*

$$(i, k, \gamma) \star (j, h, \mu) := (i, k g_{\gamma j} h, \mu)$$

Observations:

- *The identity element of (i, k, γ) the resulting structure is given by:*

$$e((i, k, \gamma)) = (i, g_{\gamma i}^{-1}, \gamma)$$

- *The inverse of (i, k, γ) is:*

$$(i, k, \gamma)^{-1} = (i, g_{\gamma i}^{-1} k^{-1} g_{\gamma i}^{-1}, \gamma)$$

Hence, the structure $(I \times G \times \Gamma, \star)$ forms a generalized-group. Moreover, we have the union:

$$I \times G \times \Gamma := \bigcup_{(i,\ \gamma) \in I \times \Gamma} \{i\} \times G \times \{\gamma\}$$

Each group $\{i\} \times G \times \{\gamma\}$ isomorph to the original group G. Hence, $I \times G \times \Gamma$ is the union disjoint $Card(I \times \Gamma)$ isomorphic groups to G.

Definition 2.5. [6] Consider a generalized-group $(G, \star)$. If for all $g,\ h$ in G: $e(g \star h) = e(g) \star e(h)$, then $(G, \star)$ is called as a normal generalized-group.

Example 2.6. *In general, based on the concepts introduced in Example 2.4, the structure $(I \times G \times \Gamma, \star)$ is not a normal generalized-group. We have:*

- *The identity element of the product $(i, k, \gamma) \star (j, h, \mu)$ is:*

$$e((i, k, \gamma) \star (j, h, \mu)) = e((i, kg_{\gamma j}h, \mu)) = (i, g_{\mu i}^{-1}, \mu)$$

- *The product of individual identity elements is:*

$$e((i, k, \gamma)) \star e((j, h, \mu)) = (i, g_{\gamma i}^{-1}, \gamma) \star (j, g_{\mu j}^{-1}, \mu) = (i, g_{\gamma i}^{-1} g_{\gamma j} g_{\mu j}^{-1}, \mu)$$

Interestingly, it can be demonstrated that $(I \times G \times \Gamma, \star)$ is a normal generalized-group if and only if there exist functions $\theta : \ I \longrightarrow G$ and $\sigma : \ \Gamma \longrightarrow G$ such that:

$$g_{\gamma i} = \sigma(\gamma)\theta(i) \quad \text{for all} \quad \gamma \in \Gamma,\ i \in I.$$

Example 2.7. *Consider a field F and H to be the set defined as:*

$$H = \left\{ \begin{bmatrix} 0 & 0 \\ x & y \end{bmatrix} \mid 0 \neq y, x \in F \right\}$$

We claim that H forms a normal generalized-group under ordinary matrix multiplication. In fact,

For any element $(\begin{bmatrix} 0 & 0 \\ x & y \end{bmatrix})$ in H, we have: $e\left(\begin{bmatrix} 0 & 0 \\ x & y \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ xy^{-1} & 1 \end{bmatrix}$, and $\left(\begin{bmatrix} 0 & 0 \\ x & y \end{bmatrix}\right)^{-1} = \begin{bmatrix} 0 & 0 \\ x^2y^{-1} & y^{-1} \end{bmatrix}$. Then, if $(\begin{bmatrix} 0 & 0 \\ x & y \end{bmatrix})$ and $(\begin{bmatrix} 0 & 0 \\ z & t \end{bmatrix})$ be in H. Their product is:

$$\begin{bmatrix} 0 & 0 \\ xy^{-1} & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ zt^{-1} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ zt^{-1} & 1 \end{bmatrix}.$$

Hence, $e\left(\begin{bmatrix} 0 & 0 \\ x & y \end{bmatrix}\right) e\left(\begin{bmatrix} 0 & 0 \\ z & t \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ xy^{-1} & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ zt^{-1} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ zt^{-1} & 1 \end{bmatrix}$. Hence, H with the ordinary matrix multiplication, is a normal generalized-group.

Definition 2.8. [6] A generalized-group $(G, \star)$ is called an Abelian generalized-group if $(g \star h = h \star g)$ for all $g,\ h \in G$.

It can be shown that, if G is an Abelian generalized-group, then the cardinal number of the set $\{e(g) | g \in G\}$ is one, so G is an Abelian group. Then all Abelian generalized-groups are Abelian groups.

Some parts of the following Theorem can be found in [1] and [4]. They are mentioned in the proof.

Theorem 2.9. *Assume that $(G, \star)$ be a generalized-group and g, h be two arbitrary elements in G. Then, we have:*

1. *$e(e(g)) = e(g)$,i.e., implying that $e(g)$ is unique.*
2. *$e(g)$ is an idempotent element.*
3. *g^{-1} is a unique element, and $(g^{-1})^{-1} = g$.*
4. *If $(G, \star)$ be a normal generalized-group where the elements $e(g)$ and h^{-1} commute together, then $(g \star h)^{-1} = h^{-1} \star g^{-1}$.*
5. *For each integer number n, $e(g)^n = e(g^n) = e(g)$.*
6. *For each $x \in G$, the set $G(x) = \{y \in G : e(x) = e(y)\}$ forms a group, and we have, $G = \bigcup_{x \in G} G(x)$, making G a union of disjoint groups $G(x)$.*
7. *If the cardinal number of G is finite, then there is a positive integer number k such that $g^k = e(g)$.*

Proof. The proof of certain aspects of this theorem can be found in the references. Now, let's break down the proof into different parts:
Proof of parts (1), (2), (3), and (4) can be found in reference [1].
Proof of Part (5): We consider the following cases for n:
Case 1: $n = 0$, let's consider $g^0 = e(g)$. Then, we have, $e(g^0) = e(e(g)) = e(g)$.
Case 2: $n > 0$, we observe that, $g^n \star e(g) = g^{n-1} \star g \star e(g) = g^{n-1} \star g = g^n = g \star g^{n-1} = e(g) \star g \star g^{n-1} = e(g) \star g^n$.
Case 3: $n < 0$, since $-n > 0$, referring to Case 2, we deduce that, $e(g^n) = e((g^n)^{-1}) = e(g^{-n}) = e(g)$.
Proof of the first part of (6) and (7) can be found in reference [4].
Proof the second part of (6), let $g \in G$. Since g is in $G(g)$, we have $G \subseteq \bigcup_{x \in G} G(x) \subseteq G$. Therefore, $G = \bigcup_{x \in G} G(x)$. □

Theorem 2.10. *Suppose that $(G, \star)$ be a generalized-group, and let $g \in G$. If the cardinal number of G is finite, then for $g \in G$ the cardinality of the subgroup $G(g)$ divides the cardinality of G.*

Proof. By Theorem 1 in [5], there is a finite group H and finite non-empty sets I and Γ and a $\Gamma \times I$ matrix P with entries $g_{\gamma i} \in H$, such that G is isomorphic to the generalized-group $I \times H \times \Gamma$. Now, let's focus on an arbitrary member g of G. This corresponds to an element (j, h, μ) in the product set $I \times H \times \Gamma$. We can define a subgroup $G(g)$ within G. This subgroup is isomorphic to the group $(I \times H \times \Gamma)_{(j,h,\mu)} = j \times H \times \mu$. Therefore, we have the following cardinality relationships:

$$Card\ G(g) = Card\ \{j\} \times H \times \{\mu\} = Card\ H$$

i.e., $Card\ G = Card\ I \times H \times \Gamma = Card\ I \times Card\ H \times Card\ \Gamma$. Therefore, the cardinality of $G(g)$ divides the cardinality of G. □

Theorem 2.11. *Let $(G, \star)$ be a generalized-group, and the cardinal number of G be finite, and let $G = \bigcup_{g \in A} G(g)$, which, $A \neq$ is a subset of G such that for* *all $x,\ y \in A$ and $x \neq y$, it implies $G(x) \neq G(y)$. Then, the cardinal number of A divides the cardinal number of G.*

Proof. According to the symbols introduced in Theorem 2.10, we have the equality: $Card\ A = Card\ I \times Card\ \Gamma$. Therefore, we can express the cardinality of G as follows, $Card\ G = Card\ (I \times H \times \Gamma) = Card\ I \times Card\ H \times Card\ \Gamma = Card\ A \times Card\ H$. In other words, the cardinal number of A divides the cardinal number of G. □

Definition 2.12. Consider a generalized-group denoted as G with the binary operation $\star$. Let S be a non-empty subset of G such that $(S, \star)$ is also a generalized-group, where, $\star$ is the same binary operation on G, limited on $S \times S$. We recalled that S is a generalized-subgroup of G and is denoted as $S \leq G$.

Theorem 2.13. *The intersection of an arbitrary family of generalized-subgroups of a generalized-group G is generalized-subgroup of G.*

Proof. Let $\{S_i\}_{i \in I}$ be an arbitrary family of generalized-subgroups of a generalized-group G, we have:

$$\begin{aligned} x \in \bigcap_{i \in I} S_i \Longrightarrow &\quad x \in S_i,\ \forall i \in I \Longrightarrow x \in G,\ \exists!\ e(x) \in G \Longrightarrow e(x) \in S_i,\ \forall i \in I \\ \Longrightarrow &\quad e(x) \in \bigcap_{i \in I} S_i. \\ x \in \bigcap_{i \in I} S_i \Longrightarrow &\quad x \in S_i,\ \forall i \in I \Longrightarrow x^{-1} \in S_i,\ \forall i \in I \Longrightarrow x^{-1} \in \bigcap_{i \in I} S_i \\ x, y \in \bigcap_{i \in I} S_i \Longrightarrow &\quad x, y \in S_i,\ \forall i \in I \Longrightarrow x \star y \in S_i,\ \forall i \in I \Longrightarrow x \star y \in \bigcap_{i \in I} S_i \end{aligned}$$

Therefore, $\bigcap_{i \in I} S_i$ is generalized-subgroup of G. □

Remark 2.14. The union of some generalized-subgroups of a generalized-group maybe not a generalized-subgroup.

Theorem 2.15. *Consider a generalized-group denoted as G, and let $S \leq G$. Assume that $x \in S$. Then, the subgroup $S(x)$ is a subgroup of $G(x)$. In the special case where G is finite, we have:*

$$Card\ S(x)\ \text{ divides } Card\ G(x) \text{ and therefore } Card\ S(x)\ \text{ divides }\ Card\ G.$$

Proof. Given that S is a subset of G, we have:

$$S(x) = \{y \in S : e(y) = e(x)\} \subseteq \{y \in G : e(y) = e(x)\} = G(x).$$

We observe that both $S(x)$ and $G(x)$ are groups, and $S(x)$ is a subgroup of $G(x)$. If G is a finite generalized-group, then $G(x)$ is also finite. By using the Lagrange Theorem, we conclude that the cardinal number of $S(x)$ divides the cardinal number of $G(x)$. Furthermore, based on Theorem 2.10, the cardinal number of $G(x)$ divides the cardinal number of G. Therefore, the cardinal number of $S(x)$ must also divide the cardinal number of G. □

Theorem 2.16. *Suppose G is a generalized-group, and S is a generalized-subgroup of G (denoted as $S \leq G$). Then, there exist sets B and A such that, $B \subseteq A \subseteq G$ and we have:*

$$S = \bigcup_{x \in B} S(x) \quad \text{and} \quad G = \bigcup_{x \in A} G(x).$$

Moreover, for all y and z in A (where $y \neq z$), we have $G_y \cap G_z = \emptyset$. In the special case where G is finite, we find that, Card S devides Card G if and only if Card B divides Card $A \times [G(x) : S(x)]$, which, $[G(x) : S(x)]$ is the index of subgroup $S(x)$ of the group $G(x)$.

Proof. Assuming that B be a subset of S, such that, $S = \bigcup_{x \in B} S(x)$, and for each $y, z \in B$ where $(y \neq z)$, it implies $S_y \cap S_z = \emptyset$. Since $S(x) \leq G(x)$ for all $x \in B$, we can extend B to the set A such that: $G = \bigcup_{x \in A} G(x)$, and for each $y, z \in A$ where $(y \neq z)$, it implies $G_y \cap G_z = \emptyset$. Now, considering a fixed element $x \in B$, we have: $Card\ G = Card\ A \times\ Card\ G(x)$ and $Card\ S = Card\ B \times Card\ S(x)$. By Theorem 2.15, we know that $Card\ S(x)$ divides $Card\ G(x)$. Therefore, $Card\ S$ divides $Card\ G$ if and only if $Card\ B \times Card\ S(x)$ divides $Card\ A \times Card\ G(x)$. This is equivalent to $Card\ B$ dividing $Card\ A \times \frac{Card\ G(x)}{Card\ S(x)}$ if and only if $Card\ B$ dividing $Card\ A \times [G(x) : S(x)]$. □

Corollary 2.17. *Assume that G is a finite generalized-group, and H is a generalized-subgroup of G. It is conceivable that the generalized Lagrange Theorem does not hold for H and G, meaning that the cardinality of H may not evenly divides the cardinality of G.*

Proof. Referring to Theorem 2.16, we recognize that it is essential for the number of elements in S to divide the number of elements in G if the number of elements in B divides the number of elements in A. Consequently, we can deliberately select sets A and B such that the cardinality of B does not divide the cardinality of A. □

3. Conclusion

In this article, we discussed the concept of generalized-groups, which serves as an extension of the group structures. We investigated the structures and properties of generalized-groups, providing examples and results within this fascinating subject. Finally, we show that the generalized Lagrange Theorem may not be true for generalized-groups.

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