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# EXACT ANALYTICAL SOLUTION OF TEMPERED FRACTIONAL HEAT-LIKE (DIFFUSION) EQUATIONS BY THE MODIFIED VARIATIONAL ITERATION METHOD

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Abstract. This paper introduces a modified version of the Variational Iteration Method, incorporating $\mathbb{P}$-transformation. We propose a novel semi-analytical technique named the modified variational iteration method for addressing fractional differential equations featuring tempered Liouville-Caputo derivatives. The modified variational iteration method emerges as a highly efficient and powerful mathematical tool, offering exact or approximate solutions for a diverse range of real-world problems in engineering and the natural sciences, specifically those expressed through differential equations. To assess its effectiveness and accuracy, we scrutinize the modified variational iteration method by applying it to three problems related to the heat-like multidimensional diffusion equation with a fractional time derivative in a tempered Liouville-Caputo form.

*Keywords*: Tempered fractional derivative, Mittag-Leffler function, fractional diffusion equation.
*2020 MSC*: Primary 34K37, 35R11.

## 1. Introduction

Fractional calculus, originating from Leibniz's conceptualization in 1695, has undergone a transformative journey, evolving from a theoretical mathematical construct into an indispensable tool with diverse practical applications [36]. Initially confined to pure mathematics for about three centuries, the field witnessed a pivotal shift with the emergence of fractional partial differential equations. Researchers recognized the profound implications of these equations, observing their broad applicability across disciplines such as physics, chemistry, ecology, biology, and engineering [5, 20, 22, 39, 43, 44].

The intrinsic capacity of fractional calculus to adeptly model systems characterized by memory, long-range dependence, and anomalous diffusion marked a paradigm shift. This enabled the field to address intricate phenomena that conventional calculus struggled to encapsulate. Presently, fractional calculus

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stands as an invaluable tool, facilitating nuanced comprehension and modeling of complex systems within various scientific and engineering domains. The historical narrative underscores the dynamic synergy between theoretical exploration and practical efficacy, exemplifying the enduring pertinence and adaptability of mathematical concepts in meeting the challenges posed by real-world complexities [4, 9, 12, 26, 27, 29, 30, 47].

The utilization of derivatives with non-integer orders has proven exceptionally effective in describing complex physical phenomena, notably in Rheology, damping laws, and diffusion processes. In Rheology, fractional calculus enhances the modeling of viscoelastic materials, providing a more nuanced understanding of how these substances respond to stress and strain [15]. Similarly, in the realm of damping laws, fractional calculus excels in capturing the intricacies of materials with non-local or memory-dependent damping characteristics, offering heightened precision in dynamic response modeling. Additionally, fractional calculus proves invaluable in describing anomalous diffusion processes, where traditional models may falter in accounting for long-range dependence or memory effects [2, 5].

These applications have sparked a notable surge in interest, prompting extensive studies across scientific and engineering disciplines. The flexibility and accuracy afforded by fractional calculus in modeling real-world complexities have opened new avenues for research [2, 3, 14, 16, 23]. As investigations into this mathematical approach continues, its potential for innovative applications across diverse domains remains a compelling driver in contemporary scientific inquiry.

The fractional order diffusion equation has found application in modeling practical sub-diffusive problems in fluid flow processes and finance [6]. In the one-dimensional case, the fundamental solution was first computed in 1996 [25], later extended to multi-dimensional cases [13], and recently simplified [28].

In the literature, various analytical and numeric approaches have been developed for solving such types of fractional-order PDEs. Numerical schemes include a finite difference scheme with nonuniform time steps [38, 49], an implicit finite-difference scheme [34], fractional Adams methods [2, 11], a nonstandard finite difference method [48] and a higher-order numerical scheme [37].

Prior to 1998, there existed no analytical method for resolving equations of this nature. Nowadays, various analytic and semi-analytic techniques exist for solving fractional equations. In 2002, Shawagfeh suggested the Adomian decomposition method (ADM) to solve fractional differential equations [41]. However, it was found to be challenging to compute the Adomian polynomials in ADM [7]. Ji-Huan He proposed the homotopy perturbation method (HPM) for solving such differential equations [17]. Momani and Odibat used HPM to solve various fractional PDEs [32, 33, 35]. Jafari and Momani obtained analytic solutions for fractional diffusion and wave equations using a modified HPM [21]. Authors in [45] solved fractional heat and wave-like equations with variable coefficients using the homotopy analysis method. Dubey et al. in [10] applied

Taylor series expansion approach for solving fractional order heat-like and wave-like equations. Recently, Kumar solved the fractional order multi-dimensional diffusion equation using a modified homotopy perturbation method (M-HPM) based on Sumudu transform and HPM [24]. However, researchers found it challenging to estimate the polynomial in this method, and the major drawback of these approaches is their complicated and extensive calculations.

The variational iteration method (VIM) has been extensively studied for many years by numerous authors [18]. Some authors have extended the VIM for solving fractional differential equations [46]. Hesameddini and Latifizadeh presented the reconstruction of the variational iteration method (RVIM) as a modified form of VIM, designed to address FDEs. It has demonstrated itself as the most straightforward analytical method for precisely solving both linear and nonlinear fractional differential equations. The RVIM stands out as a dependable, efficient, effective, and powerful analytical approach [8,19]. Ahmad and colleagues [1] proposed new algorithms for modified variational iteration for solving linear and nonlinear differential equations of integer and fractional order.

In this paper, inspired by the above works, we present an approximate analytical solution of the time fractional heat-like (diffusion) equation of the order $0 < \beta \leq 1$ in a series form which converges to the exact solution rapidly, named the Modified Variational Iteration Method (MVIM). The subsequent sections of this paper are structured as follows: Section two revisits fundamental preliminaries and notations pertaining to fractional calculus theory. Section three introduces the implementation of the MVIM method and presents solutions to several examples simulated using Mathematica software. In Section four, we elucidate the convergence and error estimation of MVIM for the diffusion equation resembling heat-like processes. The paper concludes in Section five.

## 2. Introductory Concepts

Multiple definitions of fractional integrals or derivatives can be found in the existing body of literature. These definitions have been provided by various researchers such as Riemann-Liouville, Grünwald-Letnikov, and Liouville-Caputo, among others. In this context, we will focus solely on revisiting the fundamental definitions and preliminary concepts centered around fractional derivatives and fractional integrals. These definitions serve as the foundation for our comprehensive investigation [26, 36, 47].

**Definition 2.1.** [36] The Riemann-Liouville fractional integral of $f(t)$ of the order $\beta \geq 0$ is defined as:

$$J_\iota^\beta f(\iota) = \begin{cases} f(\iota), \quad if\ \beta = 0, \\ \frac{1}{\Gamma(\beta)} \int_0^\iota (\iota - \tau)^{\beta-1} f(\tau) \mathrm{d}\tau, \qquad if\ \beta > 0, \end{cases} \tag{1}$$

where $\Gamma$ denotes gamma function:

$$\Gamma(\psi) = \int_0^\infty e^{-\iota}\iota^{\psi-1}\mathrm{d}\iota, \quad \psi \in \mathbb{C}, \quad \mathbb{R}e(\psi) > 0.$$

**Definition 2.2.** [36] The fractional derivative of $f$ of the order $\beta \geq 0$, in Liouville-Caputo sense is defined as:

$$(2) \qquad {}_0^C\mathbf{D}_\iota^\beta f(\iota) = J_\iota^{m-1} D_\iota^m f(\iota) = \frac{1}{\Gamma(m-\beta)} \int_0^\iota (\iota-\tau)^{m-\beta-1} f^{(m)}(\tau)\mathrm{d}\tau,$$

for $m-1 < \beta \leq m$, $m \in \mathbb{N}$ , $\iota > 0$, $f \in C^m$.

**Definition 2.3.** [40] Consider a finite interval $[p, q] \subset \mathbb{R}$, and let $\mathbb{P}[p, q]$ denote the integral space comprising functions that are Lebesgue measurable on the interval $[p, q]$, then

$$\Omega([p,q]) = \left\{ g : ||g||_{\Omega([p,q])} = \int_a^b |g(\iota)|\mathrm{d}\iota < \infty \right\}.$$

**Definition 2.4.** [40] Assuming that $g(\iota)$ is continuous over the interval $[p, q]$ and belongs to the Lebesgue space $\Omega([p, q])$, with $\beta > 0$ and $\sigma \geq 0$, the Tempered fractional R-L integral of order $\beta$ is expressed as follows:

$$(3) \qquad {}_a\mathbb{I}_\iota^{\beta,\sigma}(g(\iota)) = e^{-\sigma t}\, {}_aI_\iota^\beta(e^{\sigma\iota}g(\iota)) = \frac{1}{\Gamma(\beta)} \int_a^\iota e^{-\sigma(\iota-\varrho)}(\iota-\varrho)^{\beta-1}g(\varrho)\mathrm{d}\varrho.$$

Here, ${}_aI_\iota^\beta$ denotes the fractional R-L integral,

$$(4) \qquad {}_aI_\iota^\beta(g(\iota)) = \frac{1}{\Gamma(\beta)} \int_a^\iota (\iota-\varrho)^{\beta-1}g(\varrho)\mathrm{d}\varrho, \quad \beta, \iota > 0.$$

*Remark* 2.5. We observe that, by setting $\sigma = 0$, in (3), we retrieve the fractional integral in R-L concept as (4).

**Definition 2.6.** [40] Suppose $g(\iota)$ belongs to the interval $[p, q]$, is piecewise continuous, and satisfies $m-1 < \beta \leq m$, where $m \in \mathbb{Z}^+$, $\sigma \geq 0$, and $\iota > 0$. In such cases, the definition of the tempered fractional R-L derivative is expressed as:

$$(5) \qquad \begin{aligned} {}_a^{RL}\mathbf{D}_\iota^{\beta,\sigma}[g(\iota)] =& e^{\sigma\iota}\, {}_a^{RL}\mathbf{D}_\iota^\beta[e^{\sigma\iota}g(\iota)] \\ =& \frac{e^{-\sigma\iota}}{\Gamma(m-\beta)} \times \frac{d^m}{d\iota^m} \int_0^\iota (\iota-\varrho)^{m-\beta-1}e^{\sigma\varrho}g(\varrho)\mathrm{d}\varrho, \end{aligned}$$

where ${}_a^{RL}\mathbf{D}_\iota^\beta$ illustrate the fractional derivative in $R-L$'s concept

$$(6) \qquad {}_a^{RL}\mathbf{D}_\iota^\beta[e^{\sigma\iota}g(\iota)] = \frac{1}{\Gamma(m-\beta)} \frac{d^m}{d\iota^m} \int_a^\iota (\iota-\varrho)^{m-\beta-1}e^{\sigma\iota}g(\varrho)\mathrm{d}\varrho.$$

Variants of the tempered fractional R-L derivative in [40] defined as follow:

$$(7)\qquad {}^{RL}_{a}\mathbf{D}^{\beta}_{\iota}g(\iota) = \begin{cases} {}^{RL}_{a}\mathbf{D}^{\beta,\sigma}_{\iota}g(\iota) - \sigma^{\beta}g(\iota), & 0 < \beta \leq 1, \\ {}^{RL}_{a}\mathbf{D}^{\beta,\sigma}_{\iota}g(\iota) - \beta\sigma^{\beta-1}\frac{d}{d\iota}g(t) - \sigma^{\beta}g(\iota), & 1 < \beta < 2. \end{cases}$$

**Definition 2.7.** [40] Let $g(\iota) \in [p, q]$ is piecewise continuous $m-1 < \beta \leq m$ where $m \in \mathbb{Z}^{+}, \sigma \geq 0$ and $\iota > 0$ then the tempered fractional derivative in Liouville-Caputo sense is

$$(8)\qquad \begin{aligned} {}^{C}_{a}\mathbf{D}^{\beta,\sigma}_{\iota}[g(\iota)] =& e^{-\sigma\iota}\,{}^{C}_{a}\mathbf{D}^{\beta}_{\iota}[e^{\sigma\iota}g(\iota)] \\ =& \frac{e^{\sigma\iota}}{\Gamma(m-\beta)}\int_{a}^{\iota}(\iota-\varrho)^{m-\beta-1}e^{\sigma\varrho}\frac{d^{m}}{d\varrho^{m}}g(\varrho)\mathrm{d}\varrho, \end{aligned}$$

where ${}^{C}_{a}\mathbf{D}^{\beta}_{\iota}$ shows the fractional derivative in Liouville-Caputo concept

$$(9)\qquad {}^{C}_{a}\mathbf{D}^{\beta}_{\iota}[e^{\sigma\iota}g(\iota)] = \frac{1}{\Gamma(m-\beta)}\int_{a}^{\iota}(\iota-\varrho)^{m-\beta-1}e^{\sigma\varrho}\frac{d^{m}}{d\iota^{m}}g(\varrho)\mathrm{d}\varrho.$$

## 3. Definition and properties of $\mathbb{P}$-transform

In this section, we will examine the $\mathbb{P}$-transform and discuss some of its properties. Saifullah et al. [40] introduced a novel technique called the Tempered fractional $\mathbb{P}$-transform.

**Definition 3.1.** [40] Consider a function $\nu : [0,\infty) \times \mathbb{R}$ of exponential order defined over the set $\varpi$ (the set of functions). The $\mathbb{P}$-transform of $\nu$ over $\varpi = \left\{\nu(t) : \exists\delta_1, \delta_2, \text{where} \quad |\nu(t)| < \aleph e^{\frac{|\iota|}{\delta_i}}, \text{for} \quad \iota \in (-1)^{i} \times [0,\infty)\right\}$ is defined as:

$$(10)\qquad \mathbb{P}\nu(\iota) = \nu(s,u) = u\int_{0}^{\infty} e^{(\frac{-s\iota}{u})}\nu(\iota)\mathrm{d}\iota = u^{2}\int_{0}^{\infty} e^{-s\iota}\nu(u\iota)\mathrm{d}\iota.$$

**Lemma 3.2.** *[40] If we consider $e^{-\sigma_\iota}$ as a weight function and $\iota^{\beta}$ within the set $\varpi$, then*

$$\mathbb{P}\left\{\frac{e^{-\sigma\iota}\iota^{\beta}}{\Gamma(1+\beta)}\right\} = \frac{u^{\beta+2}}{(s+\sigma u)^{\beta+1}},$$

The $\mathbb{P}$-transform has also the following properties:

(1) The linearity property,

$$\mathbb{P}\left\{f(\iota) + Cg(\iota)\right\} = \mathbb{P}\left\{f(\iota)\right\} + C\mathbb{P}\left\{g(\iota)\right\}, \quad C \in \mathbb{R}.$$

(2) The $\mathbb{P}$-transform of the convolution product is defined as

$$\mathbb{P}\left\{(h * g)(\tau)\right\} = \mathbb{P}\left\{\int_{0}^{\tau} h(\varrho)g(\tau-\varrho)\mathrm{d}\varrho\right\} = \frac{1}{u}H(s,u)G(s,u).$$

(3) The $\mathbb{P}$-transform of tempered fractional integral

$$\begin{aligned}\mathbb{P}\left\{{}_a\mathbb{I}_\iota^{\beta,\sigma}(g(\iota))\right\} &= \mathbb{P}\left\{\frac{1}{\Gamma(\beta)}\int_a^\iota e^{-\sigma(\iota-\varrho)}(\iota-\varrho)^{\beta-1}g(\varrho)\mathrm{d}\varrho\right\}\\ &= \frac{1}{\Gamma(\beta)}\mathbb{P}\left[e^{-\sigma\iota}\iota^{\beta-1}*f(\iota)\right]\\ &= \frac{1}{u\Gamma(\beta)}\mathbb{P}\left\{e^{-\sigma\iota}\iota^{\beta-1}\right\}\mathbb{P}\left\{f(\iota)\right\} = \left[\frac{u}{s+\sigma u}\right]^\beta \nu(s,u),\end{aligned}$$

(4) The $\mathbb{P}$-transform of tempered fractional derivative (Liouville-Caputo sense)

$$\mathbb{P}\left\{{}_a^C\mathbf{D}_\iota^{\beta,\sigma}g(\iota)\right\} = \left[\frac{s+\sigma u}{u}\right]^\beta \mathbb{P}[g(\iota)] - \sum_{l=0}^{n-1} u\left[\frac{s+\sigma u}{u}\right]^{\beta-l-1}\left(D^l(e^{\sigma\iota}g(\iota))\right)\Big|_{\iota=0}.$$

(5) The $\mathbb{P}$-transform of R-L Tempered fractional derivative

$$\mathbb{P}\left\{{}_a^{RL}\mathbf{D}_\iota^{\beta,\sigma}g(\iota)\right\} = \left[\frac{s+\sigma u}{u}\right]^\beta \mathbb{P}[g(\iota)] - \sum_{l=0}^{n-1} u\left[\frac{s+\sigma u}{u}\right]^{l}\left(D^{\beta-l-1}(e^{\sigma\iota}g(\iota))\right)\Big|_{\iota=0}.$$

## 4. **General form of multi-dimensional heat-like (diffusion) equation**

This study is concerned with the time fractional multi-dimensional diffusion equation expressed as:

$$(11) \qquad {}_0^C\mathbf{D}_\iota^{\beta,\sigma}u = \nabla.(D(u,r)\nabla u), \quad 0<\beta\le 1,$$

under the initial condition:

$$(12) \qquad u(r,0) = u_0(r), \qquad r\in\mathbb{R}^3.$$

Here, ${}_a^C\mathbf{D}_\iota^{\beta,\sigma}u$ represents the Liouville-Caputo fractional derivative of $u$ with order $\beta$. The function $u(r,\iota)$ signifies the density of the diffusing material at the point $r=(\eta,\gamma,z)$ and time $\iota$, while $D(u,r)$ denotes the diffusion coefficient for $u$ at point $r$. In cases where the diffusion coefficient is not dependent on density (i.e., $D(u,r)=\rho^2$ is a constant), equation (11) simplifies to the fractional order multi-dimensional heat equation, i.e., ${}_0^C\mathbf{D}_\iota^{\beta,\sigma}u = \sigma^2\nabla^2 u$, which represents the heat distribution within a given domain. Specifically, when $\beta=\sigma=1$ and for the constant coefficient diffusion equation (12), it transforms into the classical multi-dimensional diffusion equation, $u_\iota = \rho^2\nabla^2 u$. This classical diffusion equation has found extensive applications in various linear and nonlinear systems across physics, chemistry, ecology, biology, and engineering. It is widely employed to describe diffusive-like behavior, such as the diffusion of alleles in a population within population genetics.

## 5. Modified Variational Iteration Method

In this segment, we present an approximate analytical technique designed for solving multi-dimensional time fractional-order heat-like (diffusion) equations of fractional order ($0 < \beta \leq 1$). To achieve this, we employ the $\mathbb{P}$-transform on both sides of the equation, incorporating an artificial initial condition. Subsequently, through a series of simplifications, we express the transformed solution in relation to other nonlinear terms. Ultimately, by employing the inverse of the $\mathbb{P}$-transform, we derive an integral equation. During this stage, we articulate a recursive formula reminiscent of the VIM method. Essentially, we have modified the VIM method by incorporating the $\mathbb{P}$-transform.

Now consider the following equation

$$\left({}_{0}^{C}\mathbf{D}_{\iota}^{\beta,\sigma}\Phi\right)(\eta,\gamma,\psi,\iota) = g\left(\iota,\eta,\gamma,\psi,\Phi,\frac{\partial^2\Phi}{\partial\eta^2},\frac{\partial^2\Phi}{\partial\gamma^2},\frac{\partial^2\Phi}{\partial\psi^2}\right), \tag{13}$$

with the following initial condition

$$\Phi(\eta,\gamma,\psi,0) = f_0(\eta,\gamma,\psi),$$

where the operator ${}_{0}^{C}\mathbf{D}_{\iota}^{\beta,\sigma}$ is the tempered Caputo fractional derivatives and $0 < \beta \leq 1$.

As previously noted, the application of the $\mathbb{P}$-Transform to equation (13), in terms of the independent variable $\iota$ and under the imposition of an artificial zero initial condition, yields:

$$\begin{aligned}\left[\frac{s+\sigma u}{u}\right]^{\beta}\mathbb{P}\{\Phi(\eta,\gamma,\psi,\iota)\} - u\left[\frac{s+\sigma u}{u}\right]^{\beta-1}\Phi(\eta,\gamma,\psi,0)\\ = \mathbb{P}\left\{g\left(\iota,\eta,\gamma,\psi,\Phi,\frac{\partial^2\Phi}{\partial\eta^2},\frac{\partial^2\Phi}{\partial\gamma^2},\frac{\partial^2\Phi}{\partial\psi^2}\right)\right\},\end{aligned}$$

Therefore

$$\begin{aligned}\mathbb{P}\{\Phi(\eta,\gamma,\psi,\iota)\} &= \left[\frac{u^2}{s+\sigma u}\right]f_0(\eta,\gamma,\psi,\iota)\\ &+ \left[\frac{u}{s+\sigma u}\right]^{\beta} G\left(s,u,\eta,\gamma,\psi,\Phi,\frac{\partial^2\Phi}{\partial\eta^2},\frac{\partial^2\Phi}{\partial\gamma^2},\frac{\partial^2\Phi}{\partial\psi^2}\right),\end{aligned} \tag{14}$$

Subsequently, by employing the inverse $\mathbb{P}$-Transform on both sides of the equation (14) and utilizing the convolution theorem, we obtain:

$$\begin{aligned} u(\eta,\gamma,\psi,t) =& e^{-\sigma t} f_0(\eta,\gamma,\psi,\iota) \\ &+ \mathbb{P}^{-1}\left\{\left[\frac{u}{s+\sigma u}\right]^{\beta} G\left(s,u,\eta,\gamma,\psi,\Phi,\frac{\partial^2\Phi}{\partial\eta^2},\frac{\partial^2\Phi}{\partial\gamma^2},\frac{\partial^2\Phi}{\partial\psi^2}\right)\right\} \\ =& e^{-\sigma\iota} f_0(\eta,\gamma,\psi,\iota) + \frac{e^{-\sigma\iota}\iota^{\beta-1}}{\Gamma(\beta)} * g\left(\iota,\eta,\gamma,\psi,\Phi,\frac{\partial^2\Phi}{\partial\eta^2},\frac{\partial^2\Phi}{\partial\gamma^2},\frac{\partial^2\Phi}{\partial\psi^2}\right) \\ =& e^{-\sigma\iota} f_0(\eta,\gamma,\psi,\iota) \\ &+ \frac{1}{\Gamma(\beta)}\int_0^{\iota} e^{-\sigma(\iota-\xi)}(\iota-\xi)^{\beta-1} g\left(\xi,\eta,\gamma,\psi,\Phi,\frac{\partial^2\Phi}{\partial\eta^2},\frac{\partial^2\Phi}{\partial\eta^2},\frac{\partial^2\Phi}{\partial\psi^2}\right) d\xi, \end{aligned}$$

Based on the actual initial conditions outlined in (13), a recursive formula is formulated as follows:

$$\begin{aligned} \Phi_{n+1}(\eta,\gamma,\psi,\iota) =& f_0(\eta,\gamma,\psi,\iota) \\ &+ \frac{1}{\Gamma(\beta)}\int_0^{\iota} e^{-\sigma(\iota-\xi)}(\iota-\xi)^{\beta-1} g\left(\xi,\eta,\gamma,\psi,\Phi,\frac{\partial^2\Phi_n}{\partial\eta^2},\frac{\partial^2\Phi_n}{\partial\gamma^2},\frac{\partial^2\Phi_n}{\partial\psi^2}\right) d\xi, \end{aligned} \tag{15}$$

Through the aforementioned iterative process, each term is computed based on the preceding term in the iteration formula, enabling a comprehensive evaluation. Consequently, the solution can be expressed as:

$$\Phi(\eta,\gamma,\psi,\iota) = \lim_{n\to\infty} \Phi_n(\eta,\gamma,\psi,\iota).$$

The resulting approximation for the fractional diffusion equation resembling heat-like behavior is:

$$\tilde{\Phi} = \Phi_0 + \Phi_1 + \Phi_2 + \ldots + \Phi_n + \ldots. \tag{16}$$

## 6. Convergence and Error Estimate of MVIM for the fractional (heat-like) diffusion Equation

We will establish the convergence and error estimation of the modified variational Iteration method.

### 6.1. Convergence.

The following theorem outlines the convergence analysis of our proposed method:

**Theorem 6.1.** *Consider a Banach space* $(\mathbb{B}[0,T], |\cdot|)$*, where* $\Phi_n(\eta,\gamma,\psi,\iota)$ *and* $\Phi(\eta,\gamma,z,\iota)$ *are defined. If there exists a constant* $0 < \varsigma < 1$ *such that* $|\Phi_{n+1}| \leq \varsigma|\Phi_n|$*, then the series solution in* (16) *converges to the solution of the fractional heat-like equation* (11).

*Proof.* Let $\{a_n\}$ denote the sequence of partial sums in (16). We aim to establish that $a_n(\eta,\gamma,\psi,\iota)$ forms a Cauchy sequence within $(\mathbb{B}[0,T], ||.||)$. To demonstrate this, consider

$$
\begin{aligned}
||a_{n+1}(\eta,\gamma,\psi,\iota)-a_n(\eta,\gamma,\psi,\iota)|| =&||\Phi_{n+1}(\eta,\gamma,\psi,\iota)|| \\
&\leq \varsigma||\Phi_n(\eta,\gamma,\psi,\iota)|| \\
&\leq \varsigma^2||\Phi_{n-1}(\eta,\gamma,\psi,\iota) \\
&\leq \ldots \leq \varsigma^{n+1}||\Phi_0(\eta,\gamma,\psi,\iota)||
\end{aligned} \tag{17}
$$

Now, for partial sums $a_n$ and $a_m$, where $n, m \in \mathbb{N}$ and $n \geq m$, applying the triangle inequality yields:

(18)

$$
\begin{aligned}
||a_n - a_m|| =& ||(a_n(\eta,\gamma,\psi,t)-a_{n-1}(\eta,\gamma,\psi,\iota)) + (a_{n-1}(\eta,\gamma,\psi,\iota)-a_{n-2}(\eta,\gamma,\psi,\iota)) \\
&+ \ldots + (a_{m+1}(\eta,\gamma,\psi,\iota)-a_m(\eta,\gamma,\psi,\iota))|| \\
&\leq ||a_n(\eta,\gamma,\psi,\iota)-a_{n-1}(\eta,\gamma,\psi,\iota)|| + ||a_{n-1}(\eta,\gamma,\psi,\iota)-a_{n-2}(\eta,\gamma,\psi,\iota)|| \\
&+ \ldots + ||a_{m+1}(\eta,\gamma,\psi,\iota)-a_m(\eta,\gamma,\psi,\iota)||,
\end{aligned}
$$

Considering (17) we get:

$$
\begin{aligned}
||a_n - a_m|| &\leq \varsigma^n||\Phi_0(\eta,\gamma,\psi,\iota)|| + \varsigma^{n-1}||\Phi_0(\eta,\gamma,\psi,\iota)|| + \ldots + \ldots \varsigma^{m+1}|| \\
&\leq \left(\varsigma^n + \varsigma^{n-1} + \ldots + \varsigma^{m+1}\right)||\Phi_0(\eta,\gamma,z,\iota)|| \\
&\leq \varsigma^{m+1}\left(\varsigma^{n-m-1} + \varsigma^{n-m-2} + \ldots + \varsigma + 1\right)||\Phi_0(\eta,\gamma,z,\iota) \\
&\leq \psi^{m+1}\left(\frac{1-\varsigma^{n-m}}{1-\varsigma}\right)||\Phi_0(\eta,\gamma,\psi,\iota)||,
\end{aligned} \tag{19}
$$

As $0 < \varsigma < 1$, it follows that $1 - \varsigma^{n-m} < 1$. Consequently, we derive:

$$
||a_n - a_m|| \leq \frac{\varsigma^{m+1}}{1-\varsigma}\max|\Phi_0(x,\gamma,\psi,\iota)|, \qquad \forall \iota \in [0,T], \tag{20}
$$

Since $\Phi_0$ is bounded, so

$$
\lim_{n,m\to\infty} ||a_n(\eta,\gamma,\psi,\iota) - a_m(\eta,\gamma,\psi,\iota)||= 0. \tag{21}
$$

We have demonstrated that $a_n(\eta,\gamma,\psi,t)$ constitutes a Cauchy sequence within the Banach space $(\mathbb{B}[0,T], ||.||)$. Consequently, the series solution in (16) converges to the solution of (11). □

6.2. **Error Estimate.** The error estimate serves as a valuable criterion for evaluating the effectiveness and accuracy of a numerical method. In this section, we explore the error estimate associated with the MVIM method.

**Theorem 6.2.** *Consider the time fractional-order heat-like or diffusion equations given in* (11)*. The maximum ideal truncation error in its solution, as expressed in* (16)*, is then determined as:*

$$
|\Phi(\eta,\gamma,\psi,\iota) - \sum_{j=0}^{m}\Phi_j(\eta,\gamma,\psi,\iota)| \leq \frac{\varsigma^{m+1}}{1-\varsigma}||\Phi_0(\eta,\gamma,\psi,\iota)||. \tag{22}
$$

*Proof.* Based on (19)

$$||\Phi(\eta,\gamma,\psi,\iota)-a_m|| \leq \varsigma^{m+1}\left(\frac{1-\varsigma^{n-m}}{1-\varsigma}\right)||\Phi_0(\eta,\gamma,\psi,\iota)||, \tag{23}$$

Given that $0<\varsigma<1$, it follows that $1-\varsigma^{n-m}<1$. As a result:

$$|\Phi(\eta,\gamma,\psi,\iota)-\sum_{j=0}^{m}\Phi_j(\eta,\gamma,\psi,t)| \leq \frac{\varsigma^{m+1}}{1-\varsigma}\Phi_0(\eta,\gamma,\psi,\iota). \tag{24}$$

□

Given that we have covered the solution method for the equation and discussed topics related to convergence analysis and error analysis, in the following section, we will solve several examples using the presented method to showcase its effectiveness.

## 7. **Examples**

In this section, we demonstrate the application of our proposed method (MVIM) to solve some examples of fractional heat-like equations.

**Example 7.1.** *We direct our attention towards a one-dimensional diffusion equation of the heat-like variety, wherein the temporal variation is accounted for by means of a derivative of fractional order [42]*

$${}_0^C\mathbf{D}_t^{\beta,\sigma}\Phi(\eta,\iota)=\frac{\eta^2}{2}\frac{\partial^2\Phi}{\partial\eta^2}, \quad \forall\eta\in[0,1], \iota>0, \quad 0<\beta\leq 1, \tag{25}$$

*with initial condition*

$$\Phi(\eta,t)|_{t=0}=\eta^2. \tag{26}$$

Through the application of MVIM to the provided equation, we obtain the following recurrence relation

$$\Phi_{n+1}(\eta,\iota)=e^{-\sigma\iota}f_0(\eta,t)+\frac{1}{\Gamma(\beta)}\int_0^{\iota}e^{-\sigma(\iota-\xi)}(\iota-\xi)^{\beta-1}\left(\frac{\eta^2}{2}\frac{\partial^2\Phi_n}{\partial\eta^2}\right)\mathrm{d}\xi, \tag{27}$$

for $n=0$ in recurrence equation (25):

$$\begin{aligned}\Phi_1(\eta,\iota)&=e^{-\sigma\iota}\eta^2+\frac{\eta^2}{\Gamma(\beta)}\int_0^{\iota}e^{-\sigma(\iota-\xi)}(\iota-\xi)^{\beta-1}e^{-\sigma\xi}\mathrm{d}\xi\\&=e^{-\sigma\iota}\eta^2+\frac{\eta^2}{\Gamma(\beta)}\mathbb{P}^{-1}\left\{\mathbb{P}\left\{e^{-\sigma\iota}\iota^{\beta-1}*1\right\}\right\}\\&=e^{-\sigma\iota}\eta^2\left(1+\frac{\iota^{\beta}}{\Gamma(1+\beta)}\right).\end{aligned}$$

Now, for $n = 1$, we obtain

$$\begin{aligned}\Phi_2(\eta,\iota) &= e^{-\sigma(\iota-\xi)}\eta^2 + \frac{1}{\Gamma(\beta)}\int_0^\iota e^{-\sigma(\iota-\xi)}(\iota-\xi)^{\beta-1}\left(\frac{\eta^2}{2}\frac{\partial^2 u_1}{\partial\eta^2}\right)\mathrm{d}\xi \\ &= e^{-\sigma\iota}\eta^2 + \frac{\eta^2}{\Gamma(\beta)}\int_0^\iota e^{-\sigma(\iota-\xi)}(\iota-\xi)^{\beta-1}e^{-\sigma\iota}\left(1+\frac{\xi^\beta}{\Gamma(1+\beta)}\right)\mathrm{d}\xi \\ &= e^{-\sigma\iota}\eta^2\left(1+\frac{\iota^\beta}{\Gamma(1+\beta)}+\frac{\iota^{2\beta}}{\Gamma(1+2\beta)}\right).\end{aligned}$$

Finally, for $n$ we obtain:

$$\Phi_n(\eta,\iota) = e^{-\sigma\iota}\eta^2\left(1+\frac{\iota^\beta}{\Gamma(1+\beta)}+\frac{\iota^{2\beta}}{\Gamma(1+2\beta)}+\cdots+\frac{\iota^{n\beta}}{\Gamma(1+n\beta)}\right),$$

Based on the Mittag-Leffler function definition for a single parameter, we can obtain the closed-form solution of (25) as

$$\Phi(\eta,\iota) = \lim_{n\to\infty}\Phi_n(\eta,\iota) = e^{-\sigma\iota}\eta^2\sum_{m=0}^{\infty}\frac{\iota^{m\beta}}{\Gamma(m\beta+1)} = e^{-\sigma\iota}\eta^2 E_\beta(\iota^\beta).$$

Setting $\beta = 1$ yields the exact solution for classical model of (25) as follows:

$$\Phi(\eta,\iota) = \lim_{n\to\infty}u_n(\eta,\iota) = e^{-\sigma\iota}\eta^2\sum_{m=0}^{\infty}\frac{\iota^m}{\Gamma(m+1)} = \eta^2 e^{\iota(1-\sigma)}.$$

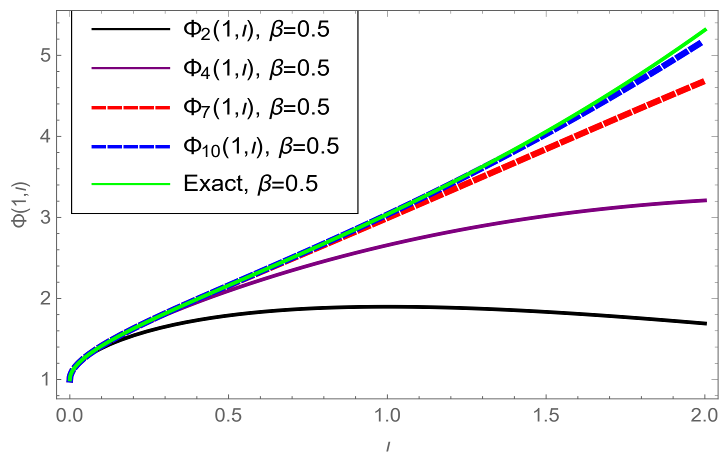
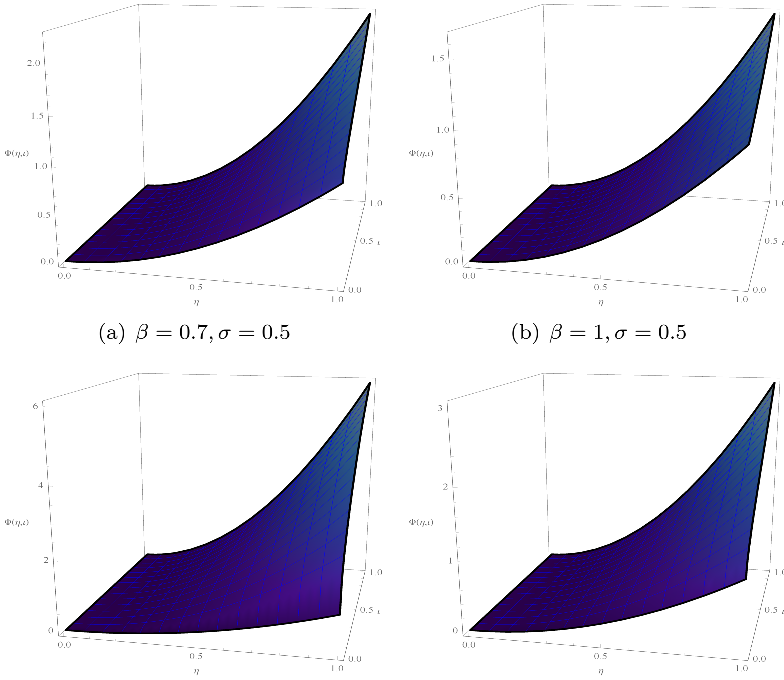



FIGURE 3. Error estimate between approximate solutions and the exact solution for $\beta = 0.5$ in model (25).

Figure 1 portrays the dynamic behavior reminiscent of classical one-dimensional diffusion, similar to processes observed in heat transfer phenomena. These dynamics are governed by equation (25), with a specific parameter value set at



(a) $\beta = 0.7, \sigma = 0.5$ (b) $\beta = 1, \sigma = 0.5$

(c) $\beta = 0.2, \sigma = 0.5$ (d) $\beta = 0.5, \sigma = 0.5$

FIGURE 1. The observed dynamics depict the physical characteristics of one-dimensional classical diffusion, reminiscent of heat transfer, governed by equation (25) with a parameter value of $\sigma = 0.5$.

$\sigma = 0.5$. The observed patterns in Figure 1 provide insight into the spatiotemporal evolution of the system under consideration, shedding light on its underlying physical characteristics. The graph in Figure 3 demonstrates that the numerical method converges to the analytical solution as the number of terms ($n$) used in the calculation increases.

**Example 7.2.** *Examining the time fractional-order diffusion equation in two dimensions, denoted by [42]*

$$ {}_0^C\mathbf{D}_\iota^{\beta,\sigma}\Phi(\eta,\gamma,\iota) = \frac{\partial^2\Phi}{\partial\eta^2} + \frac{\partial^2 u}{\partial\gamma^2}, \quad \forall \eta,\gamma \in [0,1], \iota > 0, \quad 0 < \beta \leq 1, \tag{28} $$

*with an initial concentration exhibiting exponential growth in both $\eta$ and $\gamma$, is the focus. The initial condition is specified as*

$$ \Phi(\eta,\gamma,t)|_{t=0} = e^{\eta+\gamma}. \tag{29} $$

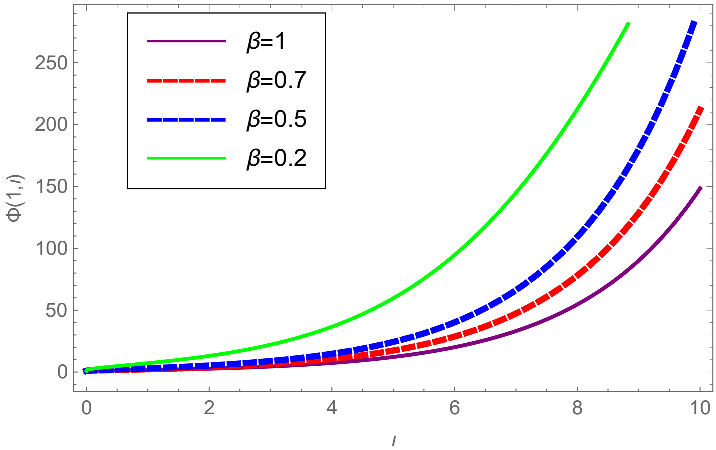



FIGURE 2. Illustration of solutions for time fractional-order (heat-like) diffusion equation (25) with various values of parameters $\beta = 1, 0.7, 0.5, 0.2$, and a fixed auxiliary parameter $\eta = 1$.

Through the application of MVIM to equation (28), we obtain the following recurrence relation:

$$\begin{aligned} \Phi_{n+1}(\eta,\gamma,\iota) =& e^{-\sigma\iota} f_0(\eta,\gamma,\iota) \\ &+ \frac{1}{\Gamma(\beta)} \int_0^{\iota} e^{-\sigma(\iota-\xi)} (\iota-\xi)^{\beta-1} e^{-\sigma\iota} \left( \frac{\partial^2 \Phi_n}{\partial \eta^2} + \frac{\partial^2 \Phi_n}{\partial \gamma^2} \right) \mathrm{d}\xi, \end{aligned} \tag{30}$$

For $n = 0$ in recurrence equation we have

$$\begin{aligned} \Phi_1(\eta,\gamma,\iota) =& e^{-\sigma\iota} e^{\eta+\gamma} + \frac{1}{\Gamma(\beta)} \int_0^{\iota} e^{-\sigma(\iota-\xi)} (\iota-\xi)^{\beta-1} 2 e^{(\eta+\gamma)} \mathrm{d}\xi \\ =& e^{\eta+\gamma-\sigma\iota} \left( 1 + \frac{2\iota^{\beta}}{\Gamma(1+\beta)} \right), \end{aligned}$$

and for $n = 1$, we have

$$\begin{aligned} \Phi_2(\eta,\gamma,\iota) &= e^{\eta+\gamma-\sigma\iota} + \frac{1}{\Gamma(\beta)} \int_0^{\iota} e^{-\sigma(\iota-\xi)} (\iota-\xi)^{\beta-1} \left( \frac{\partial^2 u_1}{\partial \eta^2} + \frac{\partial^2 u_1}{\partial \gamma^2} \right) \mathrm{d}\xi \\ &= e^{\eta+\gamma-\sigma\iota} + \frac{1}{\Gamma(\beta)} \int_0^{\iota} (\iota-\xi)^{\beta-1} 2 e^{\eta+\gamma-\sigma\iota} \left( 1 + \frac{2\xi^{\beta}}{\Gamma(1+\beta)} \right) \mathrm{d}\xi \\ &= e^{\eta+\gamma-\sigma\iota} + \frac{2e^{\eta+\gamma-\sigma\iota}}{\Gamma(\beta)} \int_0^{\iota} (\iota-\xi)^{\beta-1} \left( 1 + \frac{2\xi^{\beta}}{\Gamma(1+\beta)} \right) \mathrm{d}\xi \\ &= e^{\eta+\gamma-\sigma\iota} \left( 1 + \frac{2\iota^{\beta}}{\Gamma(1+\beta)} + \frac{2^2 \iota^{2\beta}}{\Gamma(1+2\beta)} \right). \end{aligned}$$

Here we obtain

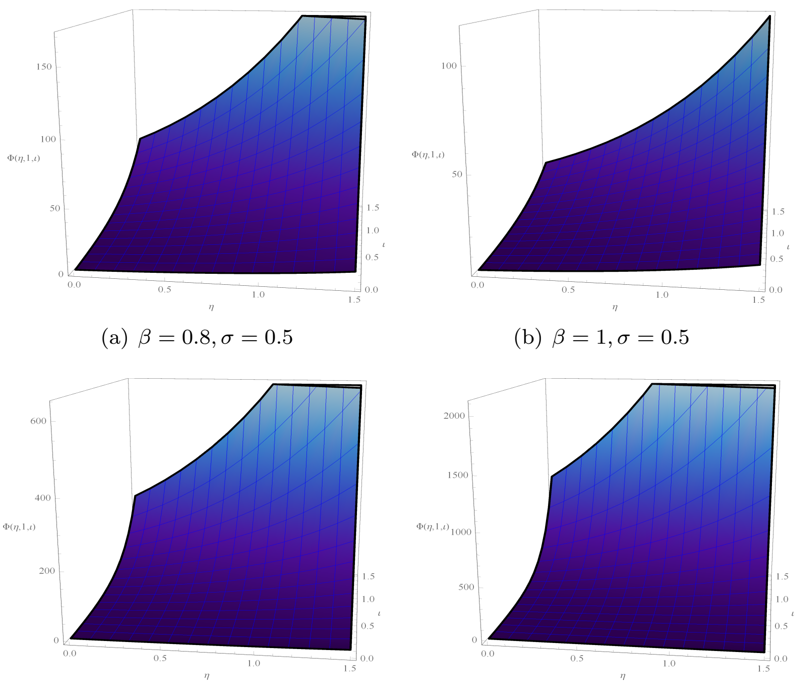
$$\Phi_n(\eta,\gamma,\iota) = e^{\eta+\gamma-\sigma\iota}\left(1 + \frac{2\iota^{\beta}}{\Gamma(1+\beta)} + \frac{2^2\iota^{2\beta}}{\Gamma(1+2\beta)} + \cdots + \frac{2^n\iota^{n\beta}}{\Gamma(1+n\beta)}\right),$$

Therefore by using the definition of Mittag-leffler function in one parameter, the exact solution of the problem (28) is given by

$$\Phi(\eta,\gamma,\iota) = \lim_{n\to\infty}\Phi_n(\eta,\gamma,\iota) = e^{\eta+\gamma-\sigma\iota}\sum_{m=0}^{\infty}\frac{2^m\iota^{m\beta}}{\Gamma(m\beta+1)} = e^{\eta+\gamma-\sigma\iota}E_{\beta}(2\iota^{\beta}).$$

Like the previous example, if we put $\beta = 1$, we obtain the exact solution of classical model as

$$\Phi(\eta,\gamma,\iota) = \lim_{n\to\infty}\Phi_n(\eta,\gamma,\iota) = e^{\eta+\gamma-\sigma\iota}\sum_{m=0}^{\infty}\frac{2^m\iota^{m}}{\Gamma(m+1)} = e^{\eta+\gamma+2\iota-\sigma\iota}.$$



(a) $\beta = 0.8, \sigma = 0.5$ (b) $\beta = 1, \sigma = 0.5$

(c) $\beta = 0.6, \sigma = 0.5$ (d) $\beta = 0.5, \sigma = 0.5$

FIGURE 4. The solution of two-dimensional diffusion equation (28) for different values of parameters in $\gamma = 1$.

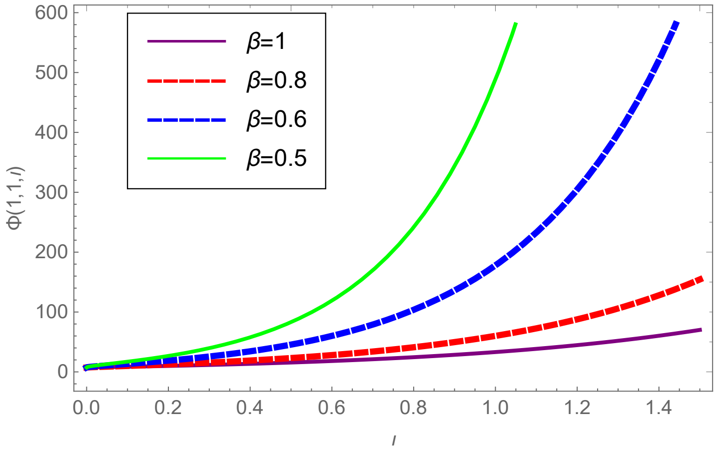



FIGURE 5. The solution for multi-dimensional, time fractional-order heat-like (diffusion) equation (28) with various fractional parameter $\beta = 1, 0.8, 0.6, 0.5$ and parameters $\eta = 1$ and $\gamma = 1$.

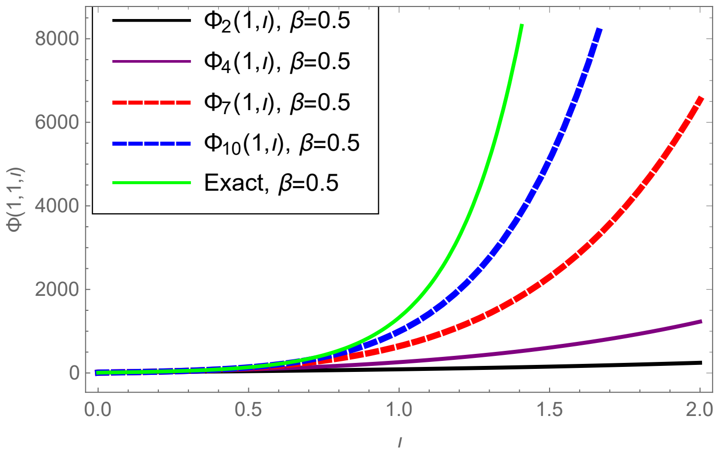



FIGURE 6. Error estimate between approximate solutions and the exact solution for $\beta = 0.5$ in model (28).

Figure 4 illustrates the solution of equation (28) for $\gamma = 1$. This figure shows how the solution behaves with respect to the two variables $t$ and $\eta$. Figure 5 shows the solution at position $(\eta, \gamma) = (1, 1)$ over time $t$ for different values of the fractional order $\beta$. We can see that the solutions exhibit similar characteristics for different values of $\beta$. Figure 6 plots the approximate solution based on the series solution of equation (28) to demonstrate that the approximate solution converges to the exact solution.

**Example 7.3.** *Consider the time fractional-order heat-like equation in three-dimensional [42].*

(31)

$$ {}_{0}^{C}\mathbf{D}_{\iota}^{\beta,\sigma}\Phi(\eta,\gamma,\psi,\iota)=\frac{\partial^2\Phi}{\partial\eta^2}+\frac{\partial^2\Phi}{\partial\eta^2}+\frac{\partial^2\Phi}{\partial\psi^2},\quad \forall\eta,\gamma,\psi\in[0,1],\quad \iota>0,\quad 0<\beta\leq 1, $$

*with initial condition*

$$ \Phi(\eta,\gamma,\psi,t)|_{t=0}=(1-\gamma)e^{\eta+\psi}. \tag{32} $$

Through the application of MVIM to equation (31), we obtain the following recurrence relation:

(33)

$$ \begin{aligned}\Phi_{n+1}(\eta,\gamma,\psi,\iota)=&(1-\gamma)e^{\eta+\psi-\sigma\iota}\\ &+\frac{1}{\Gamma(\beta)}\int_0^{\iota}e^{\sigma(\iota-\xi)}(\iota-\xi)^{\beta-1}\left(\frac{\partial^2\Phi_n}{\partial\eta^2}+\frac{\partial^2\Phi_n}{\partial\gamma^2}+\frac{\partial^2\Phi_n}{\partial\psi^2}\right)\mathrm{d}\xi,\end{aligned} $$

In this step, by setting $n=0$ in recurrence equation (33) we have

$$ \begin{aligned}\Phi_1(\eta,\gamma,\psi,t)&=(1-\gamma)e^{\eta+\psi-\sigma\iota}+\frac{1}{\Gamma(\beta)}\int_0^{\iota}e^{-\sigma(\iota-\xi)}(\iota-\xi)^{\beta-1}2(1-\gamma)e^{\eta+\psi}\mathrm{d}\xi\\ &=(1-\gamma)e^{\eta+\psi-\sigma\iota}\left(1+\frac{2\iota^{\beta}}{\Gamma(1+\beta)}\right),\end{aligned} $$

for $n=1$

$$ \begin{aligned}\Phi_2(\eta,\gamma,\psi,\iota)=&(1-\gamma)e^{\eta+\psi-\sigma\iota}\\ &+\frac{1}{\Gamma(\beta)}\int_0^{\iota}e^{-\sigma(\iota-\xi)}(\iota-\xi)^{\beta-1}\left(\frac{\partial^2\Phi_1}{\partial\eta^2}+\frac{\partial^2\Phi_1}{\partial\gamma^2}+\frac{\partial^2\Phi_1}{\partial\psi^2}\right)\mathrm{d}\xi\\ =&(1-\gamma)e^{\eta+\psi-\sigma\iota}\\ &+\frac{1}{\Gamma(\beta)}\int_0^{\iota}e^{-\sigma(\iota-\xi)}(\iota-\xi)^{\beta-1}2(1-\gamma)e^{\eta+\psi-\sigma\iota}\left(1+\frac{2\iota^{\beta}}{\Gamma(1+\beta)}\right)\mathrm{d}\xi\\ =&(1-\gamma)e^{\eta+\psi-\sigma\iota}\\ &+\frac{2(1-\gamma)e^{\eta+\psi}}{\Gamma(\beta)}\int_0^{\iota}e^{-\sigma(\iota-\xi)}(\iota-\xi)^{\beta-1}e^{-\sigma\iota}\left(1+\frac{2\iota^{\beta}}{\Gamma(1+\beta)}\right)\mathrm{d}\xi\\ =&(1-\gamma)e^{\eta+\psi-\sigma\iota}\left(1+\frac{2\iota^{\beta}}{\Gamma(1+\beta)}+\frac{2^2\iota^{2\beta}}{\Gamma(1+2\beta)}\right),\end{aligned} $$

and for $n$ we obtain:

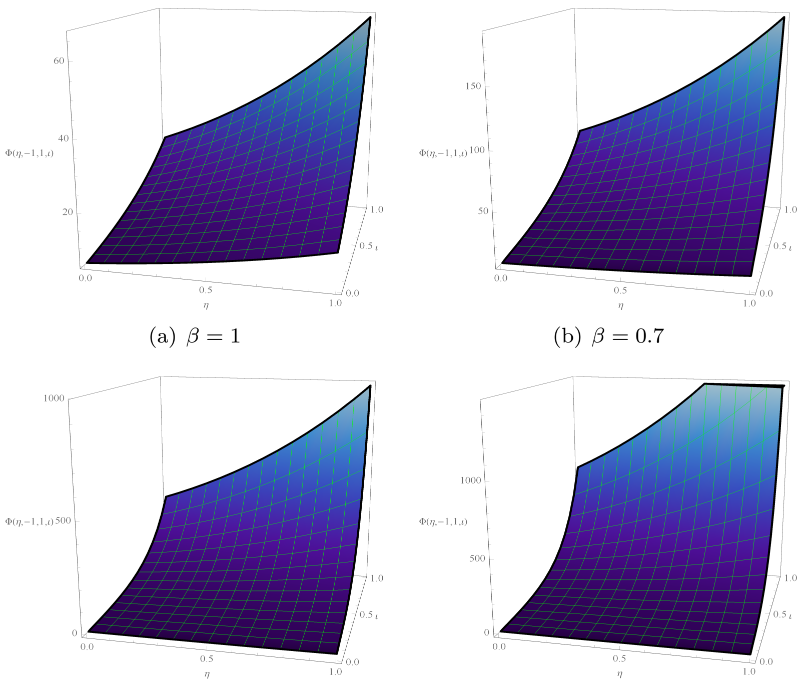
$$ \Phi_n(\eta,\gamma,\psi,\iota)=(1-\gamma)e^{\eta+\psi-\sigma\iota}\left(1+\frac{2\iota^{\beta}}{\Gamma(1+\beta)}+\frac{2^2\iota^{2\beta}}{\Gamma(1+2\beta)}+\cdots+\frac{2^n\iota^{n\beta}}{\Gamma(1+n\beta)}\right), $$

Therefore by using the definition of Mittag-leffler function in one parameter, the solution of the problem (33) is given by

$$\begin{aligned}\Phi(\eta,\gamma,\psi,\iota) = \lim_{n\to\infty}\Phi_n(\eta,y,\psi,\iota) &= (1-\gamma)e^{\eta+\psi-\sigma\iota}\sum_{m=0}^{\infty}\frac{2^m\iota^{m\beta}}{\Gamma(m\beta+1)}\\ &= (1-\gamma)e^{\eta+\psi-\sigma\iota}E_\beta(2\iota^\beta).\end{aligned}$$

Putting $\beta = 1$, implies that the exact solution of counterpart classical problem is

$$\begin{aligned}\Phi(\eta,\gamma,\psi,\iota) = \lim_{n\to\infty}\Phi_n(\eta,\gamma,\psi,\iota) &= (1-\gamma)e^{\eta+\psi-\sigma\iota}\sum_{m=0}^{\infty}\frac{2^m\iota^{m}}{\Gamma(m+1)}\\ &= (1-\gamma)e^{\eta+\psi+2\iota-\sigma\iota}.\end{aligned}$$



(a) $\beta = 1$ (b) $\beta = 0.7$

(c) $\beta = 0.5$ (d) $\beta = 0.45$

FIGURE 7. The solution of three-dimensional diffusion equation (33) for $\gamma = -1, \psi = 1, \sigma = 0.5$ and different values of $\beta$.

Figure 7 presents the solution to equation (33) under the specific conditions where $\gamma = -1$, $\psi = 1$, and $\sigma = 0.5$. This illustration highlights the behavior of

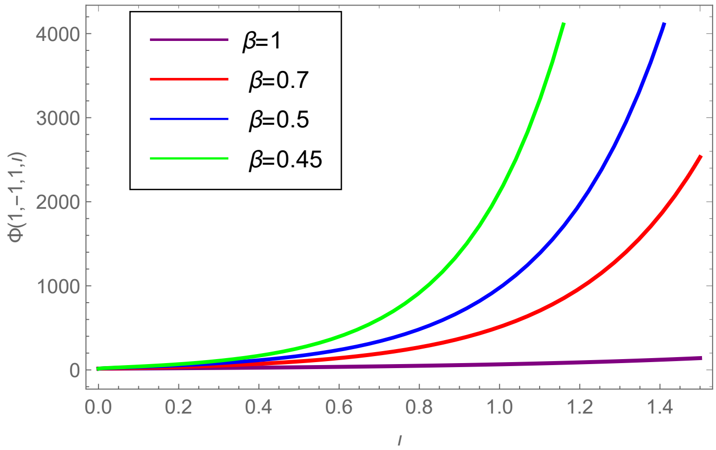



FIGURE 8. The solution for three-dimensional time fractional-order heat-like (diffusion) equations with different values of fractional parameter $\beta = 0.45, 0.5, 0.7, 1$ and auxiliary parameter $\eta = 1, \psi = 1$ and $\gamma = -1$.

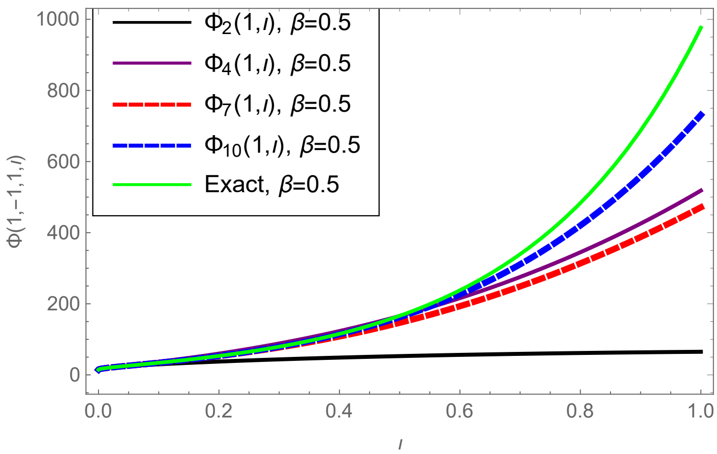



FIGURE 9. Error estimate between approximate solutions and the exact solution for $\beta = 0.5$ in model (13).

the solution in relation to the variables $t$ and $\eta$. The graphical representation provides insights into how the solution evolves over these parameters, offering a clear visualization of the dynamics involved. In Figure 8, the solution is depicted at the fixed position $(\eta, \gamma) = (1, -1, 1)$ over time $t$, showcasing the influence of different values of the fractional order $\beta$. This figure reveals that while the solutions for varying $\beta$ values display similar overall characteristics, subtle differences can be observed in their specific trajectories. This comparison emphasizes the impact of the fractional order on the solution's behavior. Figure 9 compares the approximate solution derived from the series solution of equation (28) with the exact solution. This plot demonstrates the convergence of the approximate solution towards the exact one, validating the accuracy and reliability of the series solution approach. The close alignment of the approximate and exact solutions underscores the effectiveness of this method in solving the given equation.

## 8. Conclusion

In this study, we addressed the time-fractional heat-like differential equation using the tempered fractional derivative. Initially, we enhanced the VIM method through the utilization of the $\mathbb{P}$-transformation and introduced a novel approach termed Modified Variational Iteration Method (MVIM). Within this methodology, we formulated a recursive formula based on the differential equation. The solution series sentences were then easily constructed at each step, leveraging the properties of the tempered fractional integral. Furthermore, this article encompasses the presentation of the convergence and error analysis for the proposed method. Ultimately, we applied our technique to solve one, two, and three-dimensional diffusion equations in three examples, demonstrating the accuracy and effectiveness of the approach. Figures 3, 6, and 9 illustrate the convergence of the approximate solution to the exact solution, confirming the precision and dependability of the series solution obtained from our proposed method. The strong correlation between the approximate and exact solutions highlights the efficacy of this approach in accurately solving the given equation. It is noteworthy that the analytical solutions we proposed were derived without resorting to any deformation, perturbation, discretization, or other limiting conditions. Notably, the MVIM solutions we put forward align with the solutions obtained by Kumar et al [24], Momani [31] and Singh and Srivastava [42].

Future work can expand on this study by applying the MVIM to nonlinear and higher-dimensional fractional differential equations, assessing its performance in more complex scenarios. Real-world applications, such as anomalous diffusion in biological tissues and viscoelastic materials, can be explored to validate the method's practical relevance. Comparative studies with other advanced methods like ADM and HPM will help evaluate its efficiency and accuracy. Developing adaptive algorithms and robust software implementations can enhance the usability and performance of MVIM. Further research could also focus on detailed error analysis, long-term behavior studies, parameter sensitivity, and hybrid methods, while advancing the theoretical foundations of MVIM to ensure broader applicability and deeper understanding.

## 9. Author Contributions

M.H. Akrami: Conceptualization, Methodology, Writing- Reviewing and Editing, Supervision, Validation.
A. Poya: Conceptualization, Methodology, Software, Software, Data curation, Writing- Original draft preparation, Software.
M.A. Zirak: Conceptualization, Methodology, Visualization, Software .

## 10. Data Availability Statement

Not applicable.

## 11. Aknowledgement

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## 12. Ethical considerations

Not applicable.

## 13. Funding

Not applicable.

## 14. Conflict of interest

The authors declare no conflict of interest.

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