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ZIP AND WEAK ZIP ALGEBRAS IN A CONGRUENCE-MODULAR VARIETY

G. Georgescu

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Abstract. The zip (commutative) rings, introduced by Faith and Zelmanowitz, generated a fruitful line of investigation in ring theory. Recently, Dube, Blose and Taherifar developed an abstract theory of zippedness by means of frames. Starting from some ideas contained in their papers, we define and study the zip and weak zip algebras in a semidegenerate congruence-modular variety $\mathcal{V}$. We obtain generalizations of some results existing in the literature of zip rings and zipped frames. For example, we prove that a neo-commutative algebra $A \in \mathcal{V}$ is a weak zip algebra if and only if the frame $RCon(A)$ of radical congruences of A is a zipped frame (in the sense of Dube and Blose). We study the way in which the reticulation functor preserves the zippedness property. Using the reticulation and a Hochster's theorem we prove that a neo-commutative algebra $A \in \mathcal{V}$ is a weak zip algebra if and only if the minimal prime spectrum $Min(A)$ of A is a finite space.

Keywords: Semidegenerate congruence-modular variety, Neo-commutative algebra, Admissible morphisms, Zipped frames, Zip and weak zip algebras.
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1. Introduction

The zip (commutative) rings were introduced by Faith in [9] and Zelmanowitz in [29]. This type of rings was defined by means of the notion of faithful ideal: a commutative ideal I of a commutative ring R is faithful if its annihilator is the zero ideal. R is said to be a zip ring if any faithful ideal of R includes a finitely generated faithful ideal. Using the notion of a weakly faithful ideal in a commutative ring, Ouyang defined in [25], [26] a more general class of rings, named weak zip rings.

Dube and Blose proved in [7] a nice characterization of reduced zip rings: "a reduced ring R is a zip ring if and only if the frame $RId(R)$ of its radical ideals has the property that every dense element of this frame is above a compact dense element". Based on this property, the two authors defined the zipped

georgescu.capreni@yahoo.com, ORCID: 0000-0003-2294-6289
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frames and developed an abstract zippedness theory in the frame setting. The paper contains abstract versions of several results on zip rings. The general theory of zipped frames is applied to study the weak zip rings.

A new topological characterization of reduced zip rings is obtained in the recent paper [8] of Dube and Taherifar: a reduced ring R is a zip ring if and only if the minimal prime spectrum $Min(R)$ is finite. A new class of rings is introduced in [8]: a ring R is a czip ring if each faithful ideal of R includes a countably generated faithful ideal. The countably zipped frames are introduced and it is proven that a reduced ring is a czip ring if and only if the frame $RId(R)$ is countably zipped.

The paper [12] concerns the zipped and weakly zipped quantales, structures that generalize the zipped frames. Meanwhile, the weakly zipped quantales provide an abstract theory for weak zipped rings.

In this paper we shall define the zip and weak zip algebras in a congruence-modular variety $\mathcal{V}$. Due to the commutator theory [10], we can define a notion of prime congruence of an algebra A of $\mathcal{V}$ and a notion of radical congruence of A (see [1]). The most important results of the paper are obtained for neo-commutative algebras of a semidegenerate congruence-modular variety $\mathcal{V}$. These algebras generalize the neo-commutative rings, defined by Kaplansky in [21]. A ring R is neo-commutative if the product of two finitely generated ideals is finitely generated. Then an algebra in $A \in \mathcal{V}$ is said to be neo-commutative if the set of compact congruences of A is closed under the commutator operation.

We do not work with all the morphisms of algebras in $\mathcal{V}$. We will consider only the admissible morphisms, i.e. the morphisms $u : A \to B$ such that the inverse image of any prime congruence of B is a prime congruence of A (cf. [14]). The category $N\mathcal{V}$ of neo-commutative algebras of $\mathcal{V}$ and their admissible morphisms is the most appropriate framework for our results.

Now we shall describe the content of the paper. Section 2 presents the definition of the commutator operation in a congruence-modular variety $\mathcal{V}$ and some of its basic properties (cf. [10]). We remind the definition of a semidegenerate variety (cf. [22]) and we define the neo-commutative algebras. Then we define the prime congruences of an algebra A of $\mathcal{V}$ and the topology of the prime spectrum $Spec(A)$ (cf. [1]). Section 3 contains some notions and elementary facts in the frame theory (cf. [20], [27]). Particularly, we recall from [7] the definition of zipped frames and some results in their theory (cf. [7], [8]).

Section 4 deals with the frame of radical congruences and the admissible morphisms. We recall from [1], [14] a result that describes the radical $\rho_A(\theta)$ of a congruence θ of a neo-commutative algebra $A \in \mathcal{V}$, where $\mathcal{V}$ is a semidegenerate congruence-modular variety. We define a covariant functor $(\cdot)^\rho$ from $N\mathcal{V}$ to the category $CohFrm$ of coherent frames and coherent frame morphisms. We study how this functor is used to establish some connections between properties of admissible morphisms and properties of the corresponding frame morphisms.

Starting from the notion of weak annihilator of an ideal of a commutative ring [26], [25], in Section 5 we define the weak annihilator congruences of an algebra $A \in \mathcal{V}$, where $\mathcal{V}$ is a congruence-modular variety. We prove that the set $Annih_w(A)$ of weak annihilator congruences of A is a complete Boolean algebra that coincides with the Booleanization of the frame $RCon(A)$.

The zip algebras and the weak zip algebras in a congruence-modular variety are defined in Section 6. The main theorem of this section says that for each neo-commutative algebra A in a semidegenerate congruence-modular variety, A is a weak zip algebra if and only if $RCon(A)$ is a zipped frame. We mention that this result is a generalization of Theorem 5.4 of [7]: a commutative ring R is a weak zip ring iff the frame $RId(R)$ is zipped.

The reticulation $L(A)$ of a neo-commutative algebra A in a semidegenerate congruence-modular variety $\mathcal{V}$ was defined in [13]. $L(A)$ is a bounded distributive lattice whose prime spectrum is homeomorphic with the prime spectrum of the algebra A. In fact, the reticulation defines a covariant functor from $N\mathcal{V}$ to the category of bounded distributive lattices (see [14]). This functor is a vehicle for transferring results from lattices to algebras and vice-versa (see [15]). In Section 7 we study how the reticulation functor preserves the zippedness. We prove that an algebra $A \in N\mathcal{V}$ is a weak zip algebra if and only if $L(A)$ is an ideal zip lattice. Some consequences of these results are obtained. For example, we prove that $A \in N\mathcal{V}$ is a weak zip algebra if and only if the minimal spectrum $Min(A)$ is a finite space. This topological characterization of the weak zip algebras extends a result of Dube and Taherifar for zip rings (see [8]).

2. Preliminaries

Throughout this paper we shall assume that the algebras have a fixed signature τ. We refer to [6] as the standard text of universal algebra.

Let A be an algebra and $Con(A)$ the complete lattice of its congruences; Δ_A is the bottom element of $Con(A)$ and ∇_A is the top element of $Con(A)$. If $X \subseteq A^2$ then $Cg_A(X)$ will be the congruence of A generated by X; if $X = \{(a, b)\}$ with $a, b \in A$ then $Cg_A(a, b)$ will denote the (principal) congruence generated by $\{(a, b)\}$. $Con(A)$ is an algebraic lattice: the finitely generated congruences of A are its compact elements. $K(A)$ will denote the set of compact congruences of A. We observe that $K(A)$ is closed under finite joins of $Con(A)$ and $\Delta_A \in K(A)$.

For any $\theta \in Con(A)$, A/θ is the quotient algebra of A w.r.t. θ; if $a \in A$ then a/θ is the congruence class of a modulo θ. We shall denote by $p_\theta : A \to A/\theta$ the canonical surjective τ - morphism defined by $p_\theta(a) = a/\theta$, for all $a \in A$. For any subset X of A^2 we have $(Cg_A(X) \vee \theta)/\theta = Cg_{A/\theta}(X/\theta)$; in particular, $(Cg_A(a, b) \vee \theta)/\theta = Cg_{A/\theta}(a/\theta, b/\theta)$, for all $a, b \in A$.

Thus $K(A/\theta) = \{(\alpha \vee \theta)/\theta | \alpha \in K(A)\} = p_\theta(K(A))$.

Let $\mathcal{V}$ be a congruence-modular variety of τ - algebras. Following [10], p.31, the commutator is the greatest operation $[\cdot, \cdot]_A$ on the congruence lattices

$Con(A)$ of members A of $\mathcal{V}$ such that for any surjective morphism $f : A \to B$ of $\mathcal{V}$ and for any $\alpha, \beta \in Con(A)$, the following conditions hold

$$[\alpha, \beta]_A \subseteq \alpha \cap \beta. \tag{1}$$

$$[\alpha, \beta]_A \vee Ker(f) = f^{-1}([f(\alpha \vee Ker(f)), f(\beta \vee Ker(f))]_B). \tag{2}$$

For all congruences α, β, θ of A, by using (2) we obtain the equality:

$$([\alpha, \beta]_A \vee \theta)/\theta = [(\alpha \vee \theta)/\theta, (\beta \vee \theta)/\theta]_{A/\theta}. \tag{3}$$

The commutator operation is commutative, increasing in each argument and distributive with respect to arbitrary joins. If there is no danger of confusion then we write $[\alpha, \beta]$ instead of $[\alpha, \beta]_A$.

Proposition 2.1. *[10] For any congruence-modular variety $\mathcal{V}$ the following are equivalent:*

(1) $\mathcal{V}$ has the Horn - Fraser property: if A, B are members of $\mathcal{V}$ then the lattices $Con(A \times B)$ and $Con(A) \times Con(B)$ are isomorphic;
(2) $[\nabla_A, \nabla_A] = \nabla_A$, for all $A \in \mathcal{V}$;
(3) $[\theta, \nabla_A] = \theta$, for all $A \in \mathcal{V}$ and $\theta \in Con(A)$.

Following [22], a variety $\mathcal{V}$ is semidegenerate if no nontrivial algebra in $\mathcal{V}$ has one - element subalgebras. By [22], a variety $\mathcal{V}$ is semidegenerate if and only if for any algebra A in $\mathcal{V}$, the congruence ∇_A is compact.

Proposition 2.2. *[1] If $\mathcal{V}$ is a semidegenerate congruence-modular variety then for each algebra A in $\mathcal{V}$ we have $[\nabla_A, \nabla_A] = \nabla_A$.*

Let A be a semidegenerate congruence-modular algebra. Therefore one can define on the complete lattice $Con(A)$ a residuation operation (= implication) $\alpha \to \beta = \bigvee\{\gamma | [\alpha, \gamma] \subseteq \beta\}$ and an annihilator operation $\alpha^\perp = \alpha^{\perp_A} = \alpha \to \Delta_A = \bigvee\{\gamma | [\alpha, \gamma] = \Delta_A\}$. The implication $\to$ fulfills the usual residuation property: for all $\alpha, \beta, \gamma \in Con(A)$, $\alpha \subseteq \beta \to \gamma$ if and only if $[\alpha, \beta] \subseteq \gamma$. We shall use without mention some elementary facts of residuation theory [11].

Remark 2.3. By using Propositions 2.1 and 2.2 we remark that the structure $(Con(A), \vee, \wedge, [\cdot, \cdot], \to, \Delta_A, \nabla_A)$ is a commutative and integral complete l - groupoid (see [5]). In fact, $(Con(A), \vee, \cap, [\cdot, \cdot]_A, \Delta_A, \nabla_A)$ is a multiplicative - ideal structure (= mi - structure) in the sense of [16]. Thus all the results contained in [16] hold for the particular mi - structure $Con(A)$.

For the rest of the section we fix an algebra A in a semidegenerate congruence-modular variety $\mathcal{V}$.

Following [21], a ring R is neo-commutative if the product of two finitely generated ideals of R is a finitely generated ideal. In [17], this notion was generalized to a universal algebra framework: an algebra A of the semidegenerate congruence-modular variety $\mathcal{V}$ is said to be neo-commutative if $K(A)$ is closed under commutator operation. We say that the algebra A has principal commutators if the set $PCon(A)$ of principal congruences of A is closed under commutator operation (see [1]). If A has principal commutators then A is neo-commutative.

Lemma 2.4. *Assume that A is neo-commutative. If $\theta \in Con(A)$ then A/θ is a neo-commutative algebra of $\mathcal{V}$.*

Proof. Recall that any compact congruence of A/θ has the form $(\alpha \vee \theta)/\theta$, where α is a compact congruence of A. Using (3) it follows that $K(A/\theta)$ is closed under the commutator operation. □

For all congruences $\alpha, \beta \in Con(A)$ and for any integer $n \geq 1$ we define by induction the congruence $[\alpha,\beta]^n$: $[\alpha,\beta]^1 = [\alpha,\beta]$ and $[\alpha,\beta]^{n+1} = [[\alpha,\beta]^n, [\alpha,\beta]^n]$. By convention, we set $[\alpha,\alpha]^0 = \alpha$.

Lemma 2.5. *[13] Assume $\alpha, \beta \in Con(A)$. Then $[\alpha,\beta]^{n+1} = [[\alpha,\beta],[\alpha,\beta]]^n$, for any integer $n \geq 0$.*

Let $u : A \to B$ be an arbitrary morphism in $\mathcal{V}$ and $u^* : Con(B) \to Con(A)$, $u^\bullet : Con(A) \to Con(B)$ are the maps defined by $u^*(\beta) = u^{-1}(\beta)$ and $u^\bullet(\alpha) = Cg_B(f(\alpha))$, for all $\alpha \in Con(A)$ and $\beta \in Con(B)$. Thus $u^\bullet$ is the left adjoint of u^*: for all $\alpha \in Con(A)$, $\beta \in Con(B)$, we have $u^\bullet(\alpha) \subseteq \beta$ iff $\alpha \subseteq u^*(\beta)$.

Following [10], p.82 or [1], p.582, a congruence $\phi \in Con(A) - \{\nabla_A\}$ is *prime* if for all $\alpha, \beta \in Con(A)$, $[\alpha,\beta] \subseteq \phi$ implies $\alpha \subseteq \phi$ or $\beta \subseteq \phi$. Let us introduce the following notations: $Spec(A)$ is the set of prime congruences and $Max(A)$ is the set of maximal elements of $Con(A)$. If $\theta \in Con(A) - \{\nabla_A\}$ then there exists $\phi \in Max(A)$ such that $\theta \subseteq \phi$ (because ∇_A is a compact congruence). By [1], the inclusion $Max(A) \subseteq Spec(A)$ holds. We shall denote by $Min(A)$ the set of minimal prime congruences of A. Then $Min(A) \subseteq Spec(A)$ and any prime congruence of A includes a minimal prime congruence.

For any $\theta \in Con(A)$ we denote $V_A(\theta) = V(\theta) = \{\phi \in Spec(A) | \theta \subseteq \phi\}$ and $D_A(\theta) = D(\theta) = Spec(A) - V(\theta)$. If $\alpha, \beta \in Con(A)$ then $D(\alpha) \cap D(\beta) = D([\alpha,\beta])$ and $V(\alpha) \cup V(\beta) = V([\alpha,\beta])$. For any family $(\theta_i)_{i\in I}$ of congruences we have $\bigcup_{i\in I} D(\theta_i) = D(\bigvee_{i\in I}\theta_i)$ and $\bigcap_{i\in I} V(\theta_i) = V(\bigvee_{i\in I}\theta_i)$. Thus $Spec(A)$ becomes a topological space whose open sets are $D(\theta), \theta \in Con(A)$. We remark that this topology extends the Zariski topology (defined on the prime spectra of commutative rings) and the Stone topology (defined on the prime spectra of bounded distributive lattices). The properties of $Spec(A)$ were intensively studied by Agliano in [1]. We mention that the family $(D(\alpha))_{\alpha\in K(A)}$ is a basis of open sets for the topology of $Spec(A)$. We remark that the sets $Max(A)$ and $Min(A)$ can be considered as subspaces of $Spec(A)$.

3. Frames

The monographies [20] and [27] are the standard references on frames.

Let L be a frame and $K(L)$ the set of its compact elements. L is said to be algebraic if any element a of L is a join of compact elements (in particular, $a = \bigvee\{c \in K(L)|c \leq a\}$). An algebraic frame L is coherent if the top element 1 is compact and $K(L)$ is closed under finite meets. The polar of an element a of L is defined by $a^{\perp} = a^{\perp_A} = \bigvee\{b \in L|a \wedge b = 0\}$. The set $Pol(L)$ of polars of L is a complete Boolean algebra, named the Booleanization of L (cf. [3]). An element $a \in L$ is dense if $a^{\perp} = 0$. According to Definition 2.2 of [7], an algebraic frame L is a zipped frame if for any dense element $a \in L$ there exists a compact dense element c such that $c \leq a$.

If L, M are frames then a function $f : L \to M$ is a frame morphism if it preserves the arbitrary joins and the finite meets (in particular, $f(0) = 0$). A frame morphism $f : L \to M$ is coherent if L, M are algebraic frames and $f(K(L)) \subseteq L(M)$. Let us denote by Frm the category of frames and frame morphisms. $CohFrm$ will be the category of coherent frames and coherent frame morphisms. Any frame morphism $f : L \to M$ admits a right adjoint $f_\star : M \to L$: for all $a \in L$ and $b \in M$, $a \leq f_\star(b)$ if and only if $f(a) \leq b$. Then $f_\star$ preserves the arbitrary meets.

A frame morphism $f : L \to M$ is dense if for any element a of L, $f(a) = 0$ implies $a = 0$; f is dense if and only if $f_\star(a) = 0$. By Lemma 1.6 of [24], a frame morphism $f : L \to M$ is injective if and only if $f_\star : M \to L$ is surjective. A frame morphism $f : L \to M$ is $\star$-dense if for any element b of M, $f_\star(b) = 0$ implies $b = 0$.

Several results existing in the literature of zip rings can be generalized to zipped frames (see [7], [8]). We recall some of them:

• If $h : L \to M$ is a dense and $\star$-dense coherent frame morphism then L is a zipped frame iff M is a zipped frame;

• Any algebraic frame satisfying the ascending chain condition on polars is zipped.

4. The frame of radical congruences

Let A be an algebra in a congruence-modular variety $\mathcal{V}$. According to [1], p.582, the *radical* $\rho(\theta) = \rho_A(\theta)$ of a congruence $\theta \in Con(A)$ is defined by $\rho_A(\theta) = \bigwedge\{\phi \in Spec(A)|\theta \subseteq \phi\}$; if $\theta = \rho(\theta)$ then θ is a radical congruence. We shall denote by $RCon(A)$ the set of radical congruences of A. The algebra A is *semiprime* if $\rho(\Delta_A) = \Delta_A$.

Lemma 4.1. *[1], [13] For all congruences $\alpha, \beta \in Con(A)$ the following hold:*

(1) $\alpha \subseteq \rho(\alpha)$;
(2) $\rho(\alpha \cap \beta) = \rho([\alpha, \beta]) = \rho(\alpha) \cap \rho(\beta)$;
(3) $\rho(\alpha) = \nabla_A$ iff $\alpha = \nabla_A$;
(4) $\rho(\alpha \vee \beta) = \rho(\rho(\alpha) \vee \rho(\beta))$;
(5) $\rho(\rho(\alpha)) = \rho(\alpha)$;
(6) $\rho(\alpha) \vee \rho(\beta) = \nabla_A$ iff $\alpha \vee \beta = \nabla_A$;
(7) $\rho([\alpha, \alpha]^n) = \rho(\alpha)$, for all integers $n \geq 0$.

Recall that for an arbitrary family $(\alpha_i)_{i\in I}$ of congruences of A, the following equality holds: $\rho(\bigvee_{i\in I}\alpha_i)=\rho(\bigvee_{i\in I}\rho(\alpha_i))$. Then one can introduce the arbitrary joins in $RCon(A)$: if $(\alpha_i)_{i\in I}\subseteq RCon(A)$ then we denote $\dot{\bigvee_{i\in I}}\alpha_i=\rho(\bigvee_{i\in I}\alpha_i)$. Thus $(RCon(A),\dot{\bigvee},\cap,\rho(\Delta_A),\nabla_A)$ is a frame (see [1]). It is easy to see that $K(RCon(A))=\{\rho_A(\alpha)|\alpha\in K(A)\}$.

Let us fix a semidegenerate congruence-modular variety $\mathcal{V}$. If A is a neo-commutative algebra of $\mathcal{V}$ then $RCon(A)$ is a coherent frame.

The following proposition extends a well-known result in ring theory (cf. Proposition 1.4 of [2]). It was proved in [1] for algebras with principal commutators (the present form is Lemma 6.13 of [13]).

Proposition 4.2. *Assume that the algebra A is neo-commutative. Then for any congruence θ of A the following equality holds:*
$\rho(\theta)=\bigvee\{\alpha\in K(A)|[\alpha,\alpha]^n\subseteq\theta$, *for some* $n\geq 0\}$.
Particularly, we have $\rho(\Delta_A)=\bigvee\{\alpha\in K(A)|[\alpha,\alpha]^n=\Delta_A$, *for some* $n\geq 0\}$.

Let θ be a congruence of A. By using Proposition 4.2, for each compact congruence α of A, the following equivalence holds: $\alpha\subseteq\rho(\theta)$ if and only if $[\alpha,\alpha]^n\subseteq\theta$, for some integer $n\geq 0$. The algebra A is semiprime if and only if for any $\alpha\in K(A)$ and for any integer $n\geq 0$, $[\alpha,\alpha]^n=\Delta_A$ implies $\alpha=\Delta_A$.

Definition 4.3. [14] A morphism $u:A\to B$ of the variety $\mathcal{V}$ is said to be admissible if $f^{\star}(Spec(B))\subseteq Spec(A)$.

We know from [14] that any surjective morphism of $\mathcal{V}$ is admissible and the composition of admissible morphisms is an admissible morphism. We shall denote by $N\mathcal{V}$ the category whose objects are the neo-commutative algebras of $\mathcal{V}$ and whose morphisms are the admissible morphisms.

Lemma 4.4. *[14] For any $u:A\to B$ of $\mathcal{V}$ the following are equivalent:*

(1) u is an admissible morphism;
(2) For all compact congruences α,β of A we have $\rho_B(u^{\bullet}([\alpha,\beta]_A))=\rho_B([u^{\bullet}(\alpha),u^{\bullet}(\beta)]_B)$;
(3) For all compact congruences α,β of A we have $\rho_B(u^{\bullet}([\alpha,\beta]_A))=\rho_B(u^{\bullet}(\alpha))\cap\rho_B(u^{\bullet}(\beta))$.

Proof. (1) ⇔ (2) By Theorem 4.13 of [14].
(2) ⇔ (3) By Lemma 4.1(2). □

Corollary 4.5. *If u is an admissible morphism of $\mathcal{V}$ then $\rho_B(u^{\bullet}([\alpha,\alpha]_A^n))=\rho_B(u^{\bullet}(\alpha))$, for each integer $n\geq 0$ and for each $\alpha\in K(A)$.*

Proof. In order to prove the equality of the corollary we shall use the induction on n. If $n=1$ then by using Lemma 4.4(3) we obtain $\rho_B(u^{\bullet}([\alpha,\alpha]_A))=\rho_B(u^{\bullet}(\alpha))\cap\rho_B(u^{\bullet}(\alpha))=\rho_B(u^{\bullet}(\alpha))$, for each $\alpha\in K(A)$.

Assume that the equality holds for an integer $n \geq 0$ and for each $\alpha \in K(A)$. We have to prove that $\rho_B(u^\bullet([\alpha,\alpha]_A^{n+1})) = \rho_B(u^\bullet(\alpha))$, for each $\alpha \in K(A)$.

By Lemma 2.5, $[\alpha,\alpha]_A^{n+1} = [[\alpha,\alpha]_A, [\alpha,\alpha]_A]_A^n$, hence, using the induction hypothesis one gets:

$\rho_B(u^\bullet([\alpha,\alpha]_A^{n+1})) = \rho_B(u^\bullet([[\alpha,\alpha]_A, [\alpha,\alpha]_A]_A^n)) = \rho_B(u^\bullet([\alpha,\alpha]_A)) = \rho_B(u^\bullet(\alpha))$.

□

Let $u : A \to B$ be an admissible morphism of $\mathcal{V}$. For any $\theta \in RCon(A)$ denote $u^\rho(\theta) = \rho_B(u^\bullet(\theta))$. Thus $u^\rho(\theta) \in RCon(B)$ for each $\theta \in RCon(A)$, so one gets a function $u^\rho : RCon(A) \to RCon(B)$.

Let us fix two neo-commutative algebras $A, B \in \mathcal{V}$ and an admissible morphism $u : A \to B$ of $\mathcal{V}$. Then one can apply Proposition 4.4 for computing the radicals in the algebras A, B. The following theorem is a universal algebra generalization of a result proven by Martinez in the context of commutative ring theory (see Proposition 4.1 of [24]).

Theorem 4.6. *$u^\rho : RCon(A) \to RCon(B)$ is the unique coherent frame morphism such that the following diagram is commutative:*

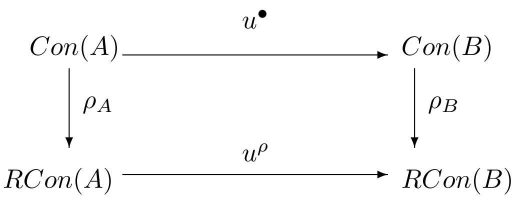

FIGURE 1

Proof. Firstly we shall prove that for each $\theta \in Con(A)$ we have $\rho_B(u^\bullet(\theta)) = \rho_B(u^\bullet(\rho_A(\theta)))$. From $\theta \subseteq \rho_A(\theta)$ we obtain $\rho_B(u^\bullet(\theta)) \subseteq \rho_B(u^\bullet(\rho_A(\theta)))$.

In order to check the converse inclusion $\rho_B(u^\bullet(\rho_A(\theta))) \subseteq \rho_B(u^\bullet(\theta))$ let us consider a compact congruence β of B such that $\beta \subseteq \rho_B(u^\bullet(\rho_A(\theta)))$. By Proposition 4.2 there exists an integer $n \geq 0$ such that $[\beta,\beta]^n \subseteq u^\bullet(\rho_A(\theta))$.

Since $\theta = \bigvee\{\alpha \in K(A) | \alpha \subseteq \theta\}$ and $u^\bullet$ preserves the joins we get the following equality:

$u^\bullet(\rho(\theta)) = \bigvee\{u^\bullet(\alpha) | \alpha \in K(A), \alpha \subseteq \theta\}$.

Since A is a neo-commutative algebra $[\beta,\beta]^n$ is a compact congruence of B so there exists an integer $m \geq 1$ and $c_1, \ldots, c_m \in K(A)$ such that $[\beta,\beta]^n \subseteq \bigvee_{i=1}^m u^\bullet(\alpha_i)$ and $\alpha_i \subseteq \theta$, for $i = 1, \ldots, m$. Recall that the map $u^\bullet$ preserves the joins. If $\alpha = \bigvee_{i=1}^m \alpha_i$ then $\alpha \in K(A)$, $\alpha \subseteq \theta$ and $[\beta,\beta]^n \subseteq u^\bullet(\alpha)$, hence $[\beta,\beta]^n \subseteq u^\bullet(\theta)$. In virtue of Proposition 4.2 we get $\beta \subseteq \rho_B(u^\bullet(\theta))$, hence $\rho_B(u^\bullet(\rho(\theta))) \subseteq \rho_B(u^\bullet(\theta))$. It follows the equality $\rho_B(u^\bullet(\theta)) = \rho_B(u^\bullet(\rho(\theta)))$, therefore the diagram is commutative.

Now we shall prove that u^ρ is a frame morphism. Let $(\theta_i)_{i \in I}$ be a family of radical congruences of A. According to the diagram's commutativity the following equalities hold:

$$u^\rho(\dot{\bigvee_{i\in I}}\theta_i) = u^\rho(\rho_A(\bigvee_{i\in I}\theta_i))) = \rho_B(u^\bullet(\bigvee_{i\in I}\theta_i)) = \rho_B(\bigvee_{i\in I}u^\bullet(\theta_i)) = \dot{\bigvee_{i\in I}}u^\rho(\theta_i).$$

Then u^ρ preserves the arbitrary joins. Let θ, ε be two radical congruences of A, so $\theta = \rho_A(\alpha)$, $\varepsilon = \rho_A(\beta)$ for some $\alpha, \beta \in Con(A)$. Then $\theta \cap \varepsilon = \rho_A([\alpha,\beta]_A)$ (cf. Lemma 4.1(2)). By hypothesis, u is an admissible morphism, hence, by using the diagram's commutativity and Lemma 4.4(c), the following equalities hold:

$u^\rho(\theta\cap\varepsilon) = u^\rho(\rho_A([\alpha,\beta]_A)) = \rho_B(u^\bullet([\alpha,\beta]_A)) = \rho_B(u^\bullet(\alpha)) \cap \rho_B(u^\bullet(\beta)) = u^\rho(\rho_A(\alpha)) \cap u^\rho(\rho_A(\beta)) = u^\rho(\theta)\cap u^\rho(\varepsilon)$.

Then u^ρ preserves the finite meets, so it is a frame morphism. Let θ be a compact element of the coherent frame $RCon(A)$, so $\theta = \rho_A(\alpha)$ for some $\alpha \in K(A)$. We know that $u^\bullet$ preserves the compact congruences, hence $u^\bullet(\alpha)$ is a compact congruence of B. By the commutativity of the diagram we have $u^\rho(\theta) = u^\rho(\rho(\alpha)) = \rho_B(u^\rho(\alpha))$. Thus $u^\rho(\theta)$ is a compact element of the frame $RCon(B)$, hence u^ρ is a coherent frame morphism. The unicity of u^ρ follows by using the diagram's commutativity.

□

Remark 4.7. Let A, B, C be three neo-commutative algebras of the variety $\mathcal{V}$ and $f : A \to B$, $g : B \to C$ two admissible morphisms of $\mathcal{V}$. Using the commutative diagram of Theorem 4.6 we obtain $(g\circ f)^\rho = g^\rho \circ f^\rho$. Thus the assignments $A \mapsto RCon(A)$ and $u \mapsto u^\rho$ define a covariant functor $(\cdot)^\rho : N\mathcal{V} \to CohFrm$.

Lemma 4.8. *Let $u : A \to B$ be a morphism of the category $N\mathcal{V}$. If ε is a radical congruence of B then $u^\star(\varepsilon)$ is a radical congruence of A.*

Proof. We have to prove that $\rho_A(u^\star(\varepsilon)) = u^\star(\varepsilon)$. Let α be a compact congruence of A such that $\alpha \subseteq \rho_A(u^\star(\varepsilon))$, hence there exists an integer $n \geq 0$ such that $[\alpha,\alpha]^n \subseteq u^\star(\varepsilon)$ (by Proposition 4.2). By the adjointness property it follows that $u^\bullet([\alpha,\alpha]^n) \subseteq \varepsilon$, hence $\rho_B(u^\bullet([\alpha,\alpha]^n)) \subseteq \rho_B(\varepsilon) = \varepsilon$.

In accordance with Corollary 4.5, $\rho_B(u^\bullet([\alpha,\alpha]^n)) = \rho_B(u^\bullet(\alpha))$, so we get $u^\bullet(\alpha) \subseteq \rho_B(u^\bullet(\alpha)) \subseteq \varepsilon$. In virtue of the adjointness property we get $\alpha \subseteq u^\star(\varepsilon)$, so $\rho_A(u^\star(\varepsilon)) \subseteq u^\star(\varepsilon)$. The converse inclusion is obvious, so we conclude that $\rho_A(u^\star(\varepsilon)) = u^\star(\varepsilon)$, therefore $u^\star(\varepsilon)$ is a radical congruence of A.

□

According to the previous lemma, for each morphism $u : A \to B$ of $N\mathcal{V}$ one can consider the function $u^\star|_{RCon(B)} : RCon(B) \to RCon(A)$.

Proposition 4.9. *For each morphism $u : A \to B$ of the category $N\mathcal{V}$ we have $(u^\rho)_\star = u^\star|_{RCon(B)}$ (it means that $u^\star|_{RCon(B)}$ is the right adjoint of the frame morphism u^ρ).*

Proof. Let θ be a radical congruence of A and ε a radical congruence of B. Using the adjointness property for $u^\bullet$ and $u^\star$ the following equivalences hold:

$u^\rho(\theta) \subseteq \varepsilon$ iff $\rho_B(u^\bullet(\theta)) \subseteq \varepsilon$ iff $u^\bullet(\theta) \subseteq \varepsilon$ iff $\theta \subseteq u^\star(\varepsilon)$.

We conclude that $u^\star|_{RCon(B)}$ is the right adjoint of the frame morphism u^ρ. □

Corollary 4.10. *Let $u : A \to B$ be a morphism of $N\mathcal{V}$. Then the following assertions are equivalent:*

(1) u^ρ is a dense frame morphism;
(2) $u^\star(\rho_B(\Delta_B)) = \rho_A(\Delta_A)$.

Proof. Recall that $\rho_A(\Delta_A)$ (resp. $\rho_B(\Delta_B)$) is the bottom element of the frame $RCon(A)$ (resp. $RCon(B)$). According to Proposition 4.9, u^ρ is a dense frame morphism iff $(u^\rho)_\star(\rho_B(\Delta_B)) = \rho_A(\Delta_A)$ iff $u^\star(\rho_B(\Delta_B)) = \rho_A(\Delta_A)$. □

Recall from [7] that a ring morphism $f : R \to Q$ is inverse-dense if for each radical ideal J of Q, $f^{-1}(J) = Nil(R)$ implies $J = Nil(Q)$. We extend this notion to a universal algebra setting: the morphism $u : A \to B$ is inverse-dense if for each radical congruence χ of B, $u^\star(\chi) = \rho_A(\Delta_A)$ implies $\chi = \rho_B(\Delta_B)$.

Corollary 4.11. *Let $u : A \to B$ be a morphism of $N\mathcal{V}$. Then the following assertions are equivalent:*

(1) $u : A \to B$ is an inverse-dense morphism;
(2) $u^\rho : RCon(A) \to RCon(B)$ is a $\star$-dense frame morphism.

Proof. The desired equivalence follows by using Proposition 4.9. □

Let us consider the surjective morphism $p_A = p_{\rho_A(\Delta_A)} : A \to A/\rho_A(\Delta_A)$, so $p_A(a) = a/\rho_A(\Delta_A)$, for any $a \in A$. We know that p_A is a morphism of the category $N\mathcal{V}$, so, by Theorem 4.6, we can take the coherent frame morphism $p_A^\rho : RCon(A) \to RCon(A/\rho_A(\Delta_A))$.

Lemma 4.12. *For each $\theta \in RCon(A)$, $p_A^\rho(\theta) = \theta/\rho_A(\Delta_A)$.*

Proof. For each $\theta \in RCon(A)$ we have $p_A^\bullet(\theta) = \theta/\rho_A(\Delta_A)$, hence $p_A^\rho(\theta) = \rho_{A/\rho_A(\Delta_A)}(p_A^\bullet(\theta)) = \rho_{A/\rho_A(\Delta_A)}(\theta/\rho_A(\Delta_A)) = \rho_A(\theta)/\rho_A(\Delta_A) = \theta/\rho_A(\Delta_A)$. □

Corollary 4.13. *p_A^ρ is a lattice isomorphism.*

Given a morphism $u : A \to B$ of $\mathcal{V}$ we define a map $v : A/\rho_A(\Delta_A) \to B/\rho_B(\Delta_B)$ by setting $v(a/\rho_A(\Delta_A)) = u(a)/\rho_B(\Delta_B)$, for any $a \in A$.

Lemma 4.14. *v is the unique morphism of $\mathcal{V}$ such that the following diagram is commutative:*

$$\begin{array}{ccc} A & \xrightarrow{\quad u \quad} & B \\ {\scriptstyle p_A}\downarrow & & \downarrow{\scriptstyle p_B} \\ A/\rho_A(\Delta_A) & \xrightarrow{\quad v \quad} & B/\rho_B(\Delta_B) \end{array}$$

FIGURE 2

Corollary 4.15. *v is an admissible morphism.*

Proof. Firstly, we remark that the map $p_A^\bullet|_{Spec(A)} : Spec(A) \to Spec(A/\rho_A(\Delta_A))$ is a homeomorphism. Let ψ be a prime congruence of the algebra $B/\rho_B(\Delta_B)$. Using the commutative diagram of Lemma 4.14 we get $p_A^\star(v^\star(\psi)) = u^\star(p_B^\star(\psi))$. Denote $\phi = p_A^\star(v^\star(\psi)) = u^\star(p_B^\star(\psi))$, hence $v^\star(\psi) = p_A^\bullet(\phi)$. Since u and p_B are admissible morphisms we get $\phi \in Spec(A)$. It follows that $v^\star(\psi) \in Spec(A/\rho_A(\Delta_A))$, hence v is an admissible morphism.

□

Remark 4.16. $A/\rho_A(\Delta_A)$ and $B/\rho_B(\Delta_B)$ are neo-commutative algebras in $\mathcal{V}$. Therefore, in virtue of Corollary 4.15, it follows that v is a morphism of the category $N\mathcal{V}$.

Lemma 4.17. *If $u : A \to B$ is a morphism of $N\mathcal{V}$ then the following are equivalent:*

(1) u is inverse-dense;
(2) $v^\rho : RCon(A/\rho_A(\Delta_A)) \to RCon(B/\rho_B(\Delta_B))$ is a dense frame morphism.

Proof. It is easy to prove that the following diagram is commutative:

$$\begin{array}{ccc} RCon(A) & \xrightarrow{\quad u^\rho \quad} & RCon(B) \\ \Big\downarrow p_A^\rho & & \Big\downarrow p_B^\rho \\ RCon(A/_{\rho_A(\Delta_A)}) & \xrightarrow{\quad v^\rho \quad} & RCon(B/_{\rho_B(\Delta_B)}) \end{array}$$

FIGURE 3

According to Corollary 4.13 and the commutativity of the diagram, it follows that p_A^ρ and p_B^ρ are lattice isomorphisms. Corollary 4.11 implies that u is inverse-dense iff u^ρ is a $\star$-dense frame morphism iff v^ρ is a $\star$-dense frame morphism.

□

Lemma 4.18. *If $u : A \to B$ is a morphism of $N\mathcal{V}$ then the following are equivalent:*

(1) $u^\star(\rho_B(\Delta_B)) = \rho_A(\Delta_A)$;
(2) u^ρ is a dense frame morphism;
(3) v^ρ is a dense frame morphism.

Proof. (1) ⇔ (2) By Corollary 4.10.

(2) ⇔ (3) By the commutativity of the diagram from the proof of Lemma 4.17. □

5. **Weak annihilator congruences**

This section deals with the weak annihilator congruences for algebras in a congruence-modular variety. This notion generalizes the weak annihilators of ideals in commutative rings (see [25], [26]).

In order to obtain this general notion, we will present the definition of weak annihilator of an ideal I in a commutative ring R. Recall that the annihilator of I is defined by $Ann_R(I) = \{a \in R | ax = 0 \text{ for each } x \in I\}$. According to [25], [26], the weak annihilator of I is defined by

$Ann_{R,w}(I) = \{a \in R | ax \in Nil(R) \text{ for each } x \in I\}$,

where $Nil(R)$ is the nilradical of the ring R. $Ann_R(I)$ and $Ann_{R,w}(I)$ are ideals of the ring R.

Now we shall extend these two notions (the annihilator and the weak annihilator) to the congruences of an algebra A in a congruence-modular variety $\mathcal{V}$. For any congruence θ of the algebra A, the weak annihilator of θ is the congruence $\theta^{\perp_w} = \theta \to \rho_A(\Delta_A)$. If A is a semiprime algebra then $\theta^{\perp_w} = \theta^{\perp}$. A congruence of the form $\theta^{\perp_w}$ is named a weak annihilator congruence. The set of weak annihilator congruences of A will be denoted by $Annih_w(A)$.

Let R be a commutative ring and I an ideal of R. The congruence of R associated with I is denoted by θ_I. Then the weak annihilator congruence of θ_I coincides with the congruence associated with $Ann_{R,w}(I) : \theta_I^{\perp_w} = \theta_{Ann_{R,w}(I)}$.

Theorem 5.1. *Assume that A is a neo-commutative algebra of $\mathcal{V}$ and θ, χ are two congruences of A. Then the following hold*

(1) If $\theta \in RCon(A)$ then $\theta^{\perp_{RCon(A)}} = \theta^{\perp_w}$;
(2) If $\theta \subseteq \chi$ then $\chi^{\perp_w} \subseteq \theta^{\perp_w}$;
(3) $\theta \subseteq \theta^{\perp_w \perp_w}$; $\theta^{\perp_w} = \theta^{\perp_w \perp_w \perp_w}$;
(4) $\theta^{\perp_w} = (\rho_A(\theta))^{\perp_w}$.

Proof. The theorem follows by imitating in every detail the proofs of Propositions 4.1(3) and Lemma 4.2 of [12]. □

The properties (1)-(4) of the previous theorem are generalizations of some results on the weak annihilators in ring theory (see Lemmas 5.2 and 5.3 of [7]; the quantale versions of these lemmas can be found in [12]).

Now we shall use Theorem 5.4 to prove that $Annih_w(A)$ is a complete Boolean algebra. In fact, we shall prove that the complete Boolean algebras $Annih_w(A)$ and $Pol(RCon(A))$ coincide.

Corollary 5.2. *If A is a neo-commutative algebra of $\mathcal{V}$ then $Annih_w(A) = Pol(RCon(A))$.*

Proof. Recall the following facts:

• By definition, the Booleanization $Pol(RCon(A))$ of the frame $RCon(A)$ is $\{\varepsilon^{\perp_{RCon(A)}} | \varepsilon \in RCon(A)\}$;
• For any $\varepsilon \in RCon(A)$ we have $\varepsilon^{\perp_{RCon(A)}} = \varepsilon^{\perp_w}$ (cf. Theorem 5.1(1));
• For any $\theta \in Con(A)$ we have $\theta^{\perp_w} = (\rho_A(\theta))^{\perp_w}$ (cf. Theorem 5.1(4)).

Therefore the following equalities hold: $Annih_w(A) = \{\theta^{\perp_w} | \theta \in Con(A)\} = \{(\rho_A(\theta))^{\perp_w} | \theta \in Con(A)\} = \{\varepsilon^{\perp_w} | \varepsilon \in RCon(A)\} = \{\varepsilon^{\perp_{RCon(A)}} | \varepsilon \in RCon(A)\} = Pol(RCon(A))$.

□

Let $(\theta_i)_{i\in I}$ be a family of weak annihilator congruences of A. Let us denote: $\bigsqcup_{i\in I} \theta_i = (\bigvee_{i\in I} \theta_i)^{\perp_w \perp_w}$. We remark that $\bigsqcup_{i\in I} \theta_i \in Annih_w(A)$. In virtue of the definition of the weak annihilator congruences we can find a family $(\varepsilon_i)_{i\in I}$ of congruences of A such that $\theta_i = \varepsilon_i^{\perp_w}$, for each $i \in I$. Then we get
$\bigcap_{i\in I} \theta_i = \bigcap_{i\in I} \varepsilon_i^{\perp_w} = (\bigvee_{i\in I} \varepsilon_i)^{\perp_w}$.
It follows that $\bigcap_{i\in I} \theta_i \in Annih_w(A)$. We conclude that $Annih_w(A)$ is closed under the infinite operations $\bigsqcup$ and $\bigcap$.

Lemma 5.3. *For each family $(\theta_i)_{i\in I}$ of weak annihilator congruences of the neo-commutative algebra A we have $\bigsqcup_{i\in I} \theta_i = (\dot{\bigvee_{i\in I}} \theta_i)^{\perp_{RCon(A)} \perp_{RCon(A)}}$.*

Proof. In accordance with Theorem 5.1,(1) and (4), the following equalities hold:
$\bigsqcup_{i\in I} \theta_i = (\bigvee_{i\in I} \theta_i)^{\perp_w \perp_w} = (\rho_A(\bigvee_{i\in I} \theta_i))^{\perp_w \perp_w} = (\dot{\bigvee_{i\in I}} \theta_i)^{\perp_w \perp_w} = (\dot{\bigvee_{i\in I}} \theta_i)^{\perp_{RCon(A)} \perp_{RCon(A)}}$.

□

Theorem 5.4. *If A is a neo-commutative algebra of $\mathcal{V}$ then the set $Annih_w(A)$ of weak annihilator congruences of A is endowed with a structure of complete Boolean algebra such that*
- *the meet of any family $(\theta_i)_{i\in I} \subseteq Annih_w(A)$ is $\bigcap_{i\in I} \theta_i$;*
- *the join of any family $(\theta_i)_{i\in I} \subseteq Annih_w(A)$ is $\bigsqcup_{i\in I} \theta_i$.*

Proof. We know that the set $Pol(RCon(A))$ of polars of the frame $RCon(A)$ is a complete Boolean algebra with the property that the meet (resp. the join) of a family $(\theta_i)_{i\in I} \subseteq Pol(RCon(A))$ is $\bigcap_{i\in I} \theta_i$ (resp. $(\dot{\bigvee_{i\in I}} \theta_i)^{\perp_{RCon(A)} \perp_{RCon(A)}}$).
Then the theorem follows by applying Corollary 5.2 and Lemma 5.3.

□

6. Zip and weak zip algebras

This section concerns the principal notions of this paper: the zip algebras and the weak zip algebras. They will be defined by means of the faithful congruences and the weakly faithful congruences, respectively.

Let R be a commutative ring and I an ideal of R. Following [9], [29], [25] we say that
- I is faithful if $Ann_R(I)$ is the zero ideal of R;
- I is weakly faithful if $Ann_{R,w}(I)$ is the nilradical $Nil(R)$ of R;

• R is a zip ring if for each faithful ideal J of R there exists a faithful finitely generated ideal K of R such that $K \subseteq J$;

• R is a weak zip ring if for each weakly faithful ideal J of R there exists a weakly faithful finitely generated ideal K of R such that $K \subseteq J$.

Now we shall generalize these notions to a universal algebra framework. Let A be an algebra in a congruence-modular variety $\mathcal{V}$ and θ a congruence of A. Then we say that

• θ is a faithful congruence if $\theta^{\perp} = \Delta_A$;

• θ is weakly faithful if $\theta^{\perp_w} = \rho(\Delta_A)$;

• A is a zip algebra if for each faithful congruence θ of A there exists a faithful compact congruence α of A such that $\alpha \subseteq \theta$;

• A is a weak zip algebra if for each weakly faithful congruence θ of A there exists a weakly faithful compact congruence α of A such that $\alpha \subseteq \theta$.

A congruence θ of a semiprime algebra A is faithful if and only if it is weakly faithful. Then a semiprime algebra A is a zip algebra if and only if it is a weak zip algebra.

Let us fix a neo-commutative algebra A in a semidegenerate congruence-modular variety $\mathcal{V}$.

Lemma 6.1. *For any $\theta \in Con(A)$ the following assertions are equivalent:*

(1) θ is weakly faithful;

(2) $\rho_A(\theta)$ is a dense element of the frame $RCon(A)$.

Proof. (1) $\Rightarrow$ (2) Assume that θ is weakly faithful, so $\theta^{\perp_w} = \rho_A(\Delta_A)$. $\rho_A(\theta)$ is an element of $RCon(A)$. Then, by using Theorem 5.1,(1) and (4), it follows that $(\rho_A(\theta))^{\perp_{RCon(A)}} = (\rho_A(\theta))^{\perp_w} = \theta^{\perp_w} = \rho_A(\Delta_A)$, hence $\rho_A(\theta)$ is a dense element of the frame $RCon(A)$.

(2) $\Rightarrow$ (1) Assume that $\rho_A(\theta)$ is a dense element of the frame $RCon(A)$. We apply Theorem 5.1,(4) and (1): $\theta^{\perp_w} = (\rho_A(\theta))^{\perp_w} = (\rho_A(\theta))^{\perp_{RCon(A)}} = \rho_A(\Delta_A)$. Then θ is a weakly faithful congruence. □

Corollary 6.2. *If A is semiprime then θ is faithful if and only if $\rho_A(\theta)$ is a dense element of the frame $RCon(A)$.*

Theorem 6.3. *The following assertions are equivalent:*

(1) A is a weak zip algebra;

(2) $RCon(A)$ is a zipped frame.

Proof. (1) $\Rightarrow$ (2) Assume that A is a weak zip algebra. Let ε be a dense element of the frame $RCon(A)$, so $\varepsilon = \rho_A(\theta)$, for some congruence θ of A. By Lemma 6.1, θ is a weakly faithful congruence of A. Then there exists a weakly faithful compact congruence α of A such that $\alpha \subseteq \theta$. Denote $\beta = \rho_A(\alpha)$, so $\beta \in K(RCon(A))$ and β is a dense element of the frame $RCon(A)$ (cf. Lemma 6.1). It is clear that $\beta \subseteq \varepsilon$, so $RCon(A)$ is a zipped frame.

(2) $\Rightarrow$ (1) Assume that $RCon(A)$ is a zipped frame. Let θ be a weakly faithful congruence of A, so $\rho_A(\theta)$ is a dense element of the frame $RCon(A)$

(cf. Lemma 6.1). Thus $\beta \subseteq \rho_A(\theta)$, for some dense compact element β of $RCon(A)$, therefore there exists an integer $n \geq 0$ such that $[\beta, \beta]^n \subseteq \theta$ (by Proposition 4.2). We recall that $K(RCon(A)) = \{\rho_A(\chi) | \chi \in K(A)\}$, so one can find a compact congruence α of A such that $\beta = \rho(\alpha)$, hence β is weakly faithful (cf. Lemma 6.1). We remark that $[\alpha, \alpha]^n \subseteq [\beta, \beta]^n \subseteq \theta$ and $[\alpha, \alpha]^n \in K(A)$ (because A is neo-commutative). According to Lemma 4.1(7) we have $\rho_A([\alpha, \alpha]^n) = \rho_A(\alpha)$. Therefore, by using Lemma 5.1(4) the following hold:

$([\alpha, \alpha]^n)^{\perp_w} = (\rho_A([\alpha, \alpha]^n))^{\perp_w} = \alpha^{\perp_A} = \rho_A(\Delta_A)$.

Thus $[\alpha, \alpha]^n$ is weakly faithful, hence A is a weak zip algebra. □

If we apply Theorem 6.3 to a commutative ring R then we obtain Theorem 5.4 of [7]: R is a weak zip ring if and only if the frame $RId(R)$ of radical ideals of R is a zipped frame.

Corollary 6.4. *If A is semiprime then the following assertions are equivalent:*

(1) A is a zip algebra;

(2) $RCon(A)$ is a zipped frame.

Let S be a subset of $Con(A)$. We say that the algebra A of $\mathcal{V}$ satisfies the ascending chain condition on S if for any ascending chain $\theta_1 \subseteq \theta_2 \subseteq \cdots \subseteq \theta_k \subseteq \cdots$ in S there exists a positive integer n such that $\theta_n = \theta_{n+k}$ for all positive integers k.

Proposition 6.5. *If A satisfies the ascending chain condition on its weak annihilator congruences then A is a weak zip algebra.*

Proof. By hypothesis, A satisfies the ascending chain condition on $Annih_w(A)$. From Corollary 5.2 we know that $Annih_w(A) = Pol(RCon(A))$, so the frame $RCon(A)$ satisfies the ascending chain condition on its polars. Then, using Corollary 3.1 of [7], it follows that $RCon(A)$ is a zipped frame. According to Theorem 6.3, A is a weak zip algebra.

□

$A \in \mathcal{V}$ is said to be a noetherian algebra if it satisfies the ascending chain condition on $Con(A)$. According to Proposition 6.5, any noetherian algebra is a weak zip algebra. In particular, any finite algebra $A \in \mathcal{V}$ is a weak zip algebra.

Let us fix a morphism $u : A \to B$ of the category $N\mathcal{V}$ of neo-commutative algebras of the variety $\mathcal{V}$.

Theorem 6.6. *Assume that $u : A \to B$ is an inverse-dense morphism and $u^\star(\rho_B(\Delta_B)) = \rho_A(\Delta_A)$. The following are equivalent:*

(1) A is a weak zip algebra;

(2) B is a weak zip algebra.

Proof. In virtue of Corollaries 4.10 and 4.11, u^ρ is a dense and $\star$-dense coherent frame morphism. Applying Theorem 4.13 of [7], it follows that the frame $RCon(A)$ is zipped if and only if the frame $RCon(B)$ is zipped. Therefore, by using Theorem 6.3, the following equivalences hold: A is a weak zip algebra iff $RCon(A)$ is a zipped frame iff $RCon(B)$ is a zipped frame iff B is a weak zip algebra.

□

Corollary 6.7. *Assume that $u : A \to B$ is an inverse-dense morphism and $u^\star(\rho_B(\Delta_B)) = \rho_A(\Delta_A)$. If A and B are semiprime then A is a zip algebra if and only if B is a zip algebra.*

7. Preserving the zippedness by reticulation

The reticulation $L(A)$ of a neo-commutative algebra A in a semidegenerate congruence-modular variety, introduced in [13], generalizes the reticulation of a ring [20], [28]. $L(A)$ is a bounded distributive lattice whose ideals are strongly connected with the congruences of A. This allows us to transfer some results from ideals of $L(A)$ to the congruences of A and vice-versa. In this section we shall study how the reticulation relates the weak zip algebras and the ideal zip lattices (a notion defined in a similar way to the zip rings).

Let us fix a semidegenerate congruence-modular variety $\mathcal{V}$ and A a neo-commutative algebra of $\mathcal{V}$. Then the set $K(A)$ of compact congruences of A is closed under the commutator operation and $\nabla_A \in K(A)$, so one can consider the algebraic structure $(K(A), \vee, [\cdot,\cdot], \Delta_A, \nabla_A)$.

We shall recall from [13] the construction of the reticulation $L(A)$ of A. Consider the following equivalence relation on $K(A)$: for all $\alpha, \beta \in K(A)$, $\alpha \equiv \beta$ if and only if $\rho(\alpha) = \rho(\beta)$. Let $\hat{\alpha}$ be the equivalence class of $\alpha \in K(A)$ and $0 = \hat{\Delta_A}, 1 = \hat{\nabla_A}$. Then $\equiv$ is a congruence of the algebraic structure $(K(A), \vee, [\cdot,\cdot], \Delta_A, \nabla_A)$ so the quotient set $L(A) = K(A)/\equiv$ is a bounded distributive lattice, named the reticulation of the algebra A (see [13]). We shall denote by $\lambda_A : K(A) \to L(A)$ the function defined by $\lambda_A(\alpha) = \hat{\alpha}$, for all $\alpha \in K(A)$. We remark that for all $\alpha, \beta \in K(A)$ we have $\lambda_A(\alpha) = \lambda_A(\beta)$ if and only if $\rho(\alpha) = \rho(\beta)$.

Let L be a bounded distributive lattice and $Id(L)$ the set of its ideals. Consider the following topological spaces:

• the prime spectrum $Spec_{Id}(L)$ of L, i.e. the set of prime ideals of L, endowed with the Stone topology;

• the minimal prime spectrum $Min_{Id}(L)$ spectrum of L, i.e. the set of minimal prime ideals of L, endowed with the restriction of the Stone topology.

For all $\theta \in Con(A)$ and $I \in Id(L(A))$ we shall denote

$\theta^* = \{\lambda_A(\alpha) | \alpha \in K(A), \alpha \subseteq \theta\}$ and $I_* = \bigvee\{\alpha \in K(A) | \lambda_A(\alpha) \in I\}$.

Thus θ^* is an ideal of the lattice $L(A)$ and I_* is a congruence of A. In this way one obtains two order - preserving functions $(\cdot)^* : Con(A) \to Id(L(A))$ and $(\cdot)_* : Id(L(A)) \to Con(A)$.

Lemma 7.1. *[13] For all $\theta \in Con(A)$, $\alpha \in K(A)$ and $I \in Id(L(A))$ the following hold:*

(1) $\alpha \subseteq I_*$ *iff* $\lambda_A(\alpha) \in I$*;*
(2) $(\theta^*)_* = \rho(\theta)$ *and* $(I_*)^* = I$*;*
(3) $\theta^* = (\rho(\theta))^*$ *and* $\rho(I_*) = I_*$*;*
(4) If $\theta \in Spec(A)$ *then* $(\theta^*)_* = \theta$ *and* $\theta^* \in Spec_{Id}(L(A))$*.*
(5) If $I \in Spec_{Id}(L(A))$ *then* $I_* \in Spec(A)$*;*

According to the previous lemma one can consider the order - preserving functions $u : Spec(A) \to Spec_{Id}(L(A))$ and $v : Spec_{Id}L((A)) \to Spec(A)$, defined by $u(\phi) = \phi^*$ and $v(P) = P_*$, for all $\phi \in Spec(A)$ and $P \in Spec_{Id}(L(A))$.

The following two propositions emphasize a strong connection between the congruence theory in the algebras of $\mathcal{V}$, the ideal theory in bounded distributive lattices and the frame theory. The results of this section will be proven by using these propositions.

Proposition 7.2 ([13], Proposition 11)**.** *The maps* $u : Spec(A) \to Spec_{Id}(L(A))$ *and* $v : Spec_{Id}(L(A)) \to Spec(A)$ *are homeomorphisms, inverse to one another.*

Proposition 7.3 ([13], Proposition 14)**.** *The maps* $(\cdot)^*|_{RCon(A)} : RCon(A) \to Id(L(A))$ *and* $(\cdot)_* : Id(L(A)) \to RCon(A)$ *are frame isomorphisms, inverse to one another.*

Let L be a bounded distributive lattice. An ideal I of L is dense if its annihilator $Ann(I) = \{a \in L | a \wedge x = 0$ for any $x \in I\}$ is the zero ideal of L. L is an ideal zip lattice if for any dense ideal I of L there exists $a \in I$ such that the principal ideal $(a]$ is a dense ideal of L.

Lemma 7.4. *A bounded distributive lattice L is an ideal zip lattice if and only if the frame* $Id(L)$ *of ideals of L is a zipped frame.*

Theorem 7.5. *If A is a neo-commutative algebra of* $\mathcal{V}$ *then the following are equivalent:*

(1) A is a weak zip algebra;
(2) The reticulation $L(A)$ *of A is an ideal zip lattice.*

Proof. By Lemma 7.1(3), we observe that the following diagram is commutative:

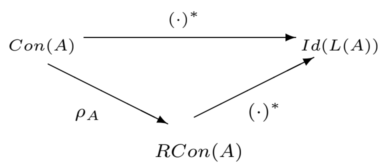

FIGURE 4

We know from Proposition 7.3 that the map $(\cdot)^* : RCon(A) \to Id(L(A))$ is a frame isomorphism, therefore, by using Theorem 6.3 and Lemma 7.4 we get the equivalence of the following properties:

• A is a weak zip algebra;
• $RCon(A)$ is a zipped frame;
• $Id(L(A))$ is a zipped frame;
• $L(A)$ is an ideal zip lattice.

□

Corollary 7.6. *If A is a semiprime neo-commutative algebra of $\mathcal{V}$ then the following are equivalent:*

(1) A is a zip algebra;
(2) $L(A)$ is an ideal zip lattice.

Lemma 7.7. *The reticulations $L(A)$ and $L(A/\rho_A(\Delta_A))$ of the neo-commutative algebras A and $A/\rho_A(\Delta_A)$ are isomorphic lattices.*

Proof. By Proposition 7.6 of [13], the lattices $L(A/\rho_A(\Delta_A))$ and $L(A)/(\rho_A(\Delta_A))^*$ are isomorphic. According to Lemma 7.1(3) we have $(\rho_A(\Delta_A))^* = (\Delta_A)^* = \{\Delta_A\}$, therefore we get the following lattice isomorphisms:
$L(A/\rho_A(\Delta_A)) \simeq L(A)/(\rho_A(\Delta_A))^* = L(A)/\{\Delta_A\} \simeq L(A)$. □

Theorem 7.8. *If A is a neo-commutative algebra of $\mathcal{V}$ then the following are equivalent:*

(1) A is a weak zip algebra;
(2) $A/\rho_A(\Delta_A)$ is a zip algebra.

Proof. It is clear that $A/\rho_A(\Delta_A)$ is a semiprime algebra. Therefore by using Theorem 7.5, Lemma 7.7 and Corollary 7.6 it follows that the following properties are equivalent:
• A is a weak zip algebra;
• $L(A)$ is an ideal zip lattice;
• $L(A/\rho_A(\Delta_A))$ is an ideal zip lattice;
• $A/\rho_A(\Delta_A)$ is a zip algebra. □

An important theorem of Hochster [19] asserts that for any distributive lattice L there exists a reduced commutative ring R such that the lattices L and $L(R)$ are isomorphic.

Applying the Hochster theorem to the reticulation $L(A)$ of the neo-commutative algebra A one can find a reduced commutative ring R such that the reticulations $L(A)$ and $L(R)$ are isomorphic lattices.

Lemma 7.9. *If A is a neo-commutative algebra of $\mathcal{V}$ then the following are equivalent:*

(1) A is a weak zip algebra;
(2) R is a zip ring.

Proof. Taking into account that R is a reduced ring and applying Theorem 7.5 and Corollary 7.6 we obtain the following equivalences: A is a weak zip algebra iff $L(A)$ is a zip lattice iff $L(R)$ is a zip lattice iff R is a zip ring. □

Theorem 7.10. *Let A be a neo-commutative algebra of $\mathcal{V}$. Then the following are equivalent:*

(1) A is a weak zip algebra;
(2) The minimal prime spectrum $Min(A)$ of A is a finite space.

Proof. Recall from [8] the following topological characterization of reduced zip rings: a reduced ring Q is a zip ring if and only if the minimal prime spectrum $Min(Q)$ of Q is finite.

Let R be a reduced ring such that $L(A)$ and $L(R)$ are isomorphic lattices (the existence of R is ensured by the Hochster theorem). Applying Proposition 7.2 to the algebra A and the ring R the following hold:

• $Min(A)$ and $Min_{Id}(L(A))$ are homeomorphic spaces;
• the minimal prime spectrum $Min(R)$ of R and $Min_{Id}(L(R))$ are homeomorphic spaces.

It follows that $Min(A)$ and $Min(R)$ are homeomorphic spaces. Therefore, using Lemma 7.9 and the previous remarks, we obtain the equivalence of the following properties:

• A is a weak zip algebra;
• R is a zip ring;
• $Min(R)$ is a finite space;
• $Min(A)$ is a finite space.

□

Proposition 7.11. *Let A, B be two neo-commutative algebras of $\mathcal{V}$ such that the direct product $A \times B$ has no skew congruence. Then $A \times B$ is a weak zip algebra if and only if A, B are weak zip algebras.*

Proof. By the Hochster theorem one can find two reduced commutative rings R, Q such that $L(A) \simeq L(R)$ and $L(B) \simeq L(Q)$. In accordance with Proposition 5.1 of [13] we get the following lattices isomorphisms:

$L(A \times B) \simeq L(A) \times L(B) \simeq L(R) \times L(Q) \simeq L(R \times Q)$.

By Lemma 7.9, A (resp. B) is a weak zip algebra if and only if R (resp. Q) is a zip ring.

Therefore, by using Theorem 7.5 and Proposition 1.1(b) of [23], the following properties are equivalent:

• $A \times B$ is a weak zip algebra;
• $L(A \times B)$ is an ideal zip lattice;
• $L(R \times Q)$ is an ideal zip lattice;
• $R \times Q$ is a zip ring:
• R, Q are zip rings;
• A, B are weak zip algebras.

□

8. **A final discussion**

This paper proposes a universal algebra setting for extending some definitions and results of zip ring theory. The main notions of this theory can be generalized to algebras in a congruence-modular variety (due to the commutator operation that ensures good properties for prime congruences, radical congruences, etc. (cf. [1]). For developing the general theory we chose as framework the category $N\mathcal{V}$, whose objects are the neo-commutative algebras of a semimodular congruence-modular variety $\mathcal{V}$ and the morphisms are the admissible morphisms of $\mathcal{V}$ (in the sense of [14]).

A reason for this choice is that several proofs use Proposition 4.2, proven in [1] for algebras with principal commutators and in [13] for neo-commutative algebras. The quasi-commutative algebras were introduced in [17] as universal algebra generalizations of the Belluce quasi-commutative rings [4]. They include the neo-commutative algebras and are characterized as algebras of $\mathcal{V}$ that admit a reticulation [18]. We don't know if the property formulated in Proposition 4.2 holds in a quasi-commutative algebra, but it would be interesting if some results of this paper remain valid in this more general context.

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13. **Conflict of interest**

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George Georgescu
Orcid number: 0000-0003-2294-6289
Faculty of Mathematics and Computer Science
Bucharest University
Bucharest, Romania
Email address: georgescu.capreni@yahoo.com