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SOME RESULTS ON SEMI MAXIMAL FILTERS IN $BL-$ALGEBRAS

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Abstract. In this article, we present an equivalent definition for the concept of the semi maximal filter in $BL-$algebras and some of their properties are studied. At first, we focus on elucidating the relationship between semi maximal filters and minimal prime filters. By conducting this analysis, some classifications for semi maximal filters are given.

Keywords: $BL-$algebra, (semi maximal, minimal prime) filter.
2020 MSC: Primary 06D35, 06B10.

1. Introduction

$BL-$algebras were initially introduced by Hájek as a means to investigate many-valued logic using algebraic methods [4]. He provided an algebraic proof of the completeness theorem for "Basic Logic" (BL), which is derived from the continuous triangular norms commonly used in the framework of fuzzy logic. The theory of filters plays a crucial role in the study of $BL-$algebras. From a logical perspective, different filters correspond to distinct sets of provable formulas. Hájek introduced the concepts of filters and prime filters in $BL-$algebras, utilizing prime filters to establish the completeness of basic logic BL [4]. Turunnen further explored the properties of filters and prime filters in $BL-$algebras [10, 11]. Turunnen introduced the notion of Boolean filters in $BL-$algebras and derived several characterizations of these filters [12]. Additionally, it was proven that a $BL-$algebra is bipartite if and only if it possesses a proper Boolean filter. Haveshki et al. continued the algebraic analysis of $BL-$algebras and introduced the concept of positive implicative filters in $BL-$algebras [5]. S. Motamed et al. introduced the notion of radical of filters in $BL-$algebras and stated and proved some theorems relating this notion to other types of filters in $BL-$algebras. Additionally, they introduced the concept of semi-maximal filters in $BL-$algebras. The reader can find more information in articles [7, 9].

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The goal of this paper is to provide an equivalent definition for the concept of semi maximal filters in $BL-$algebras and to examine the various properties associated with these filters. We show that every semi maximal filter of L can be expressed as the intersection of the minimal prime filters that contain it. By analyzing the properties of semi-maximal filters, the understanding of this concept in $BL-$algebra is enhanced and new connections are obtained in this field.

2. **Preliminaries**

We recollect some definitions and results which will be used in the sequel:

Definition 2.1. [4] A BL-algebra is an algebra $(L, \wedge, \vee, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ equipped with an order $\leq$ satisfying the following:
(BL_1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
(BL_2) $(L, \odot, 1)$ is a commutative monoid,
(BL_3) $\odot$ and $\to$ form an adjoint pair i.e., $z \leq x \to y$ if and only if $x \odot z \leq y$, for all $x, y, z \in L$,
(BL_4) $x \wedge y = x \odot (x \to y)$, for all $x, y \in L$,
(BL_5) $(x \to y) \vee (y \to x) = 1$, for all $x, y \in L$.

Throughout the paper, we denote L as a $BL-$algebra.

Definition 2.2. [12] A filter is a non-empty subset F of L satisfying the following conditions:
$(F1)$ If $a \in F,\ b \in L$ and $a \leq b$, then $b \in F$,
$(F2)$ If $a, b \in F$, then $a \odot b \in F$.
We denote by $F(L)$ the set of all filters of L.

Definition 2.3. [2,12] Let F be a filter of L.
• If $F \neq L$, then F is called a proper filter of L.
• A proper filter F of L is called a prime filter if for all $a, b \in L, a \vee b \in F$, satisfies $a \in F$ or $b \in F$.
We denote by $Spec(L)$ the set of all prime filters of a $BL-$algebra L.
• A filter P of L is called a minimal prime filter of L when:
(i) $P \in Spec(L)$;
(ii) *If there exists* $Q \in Spec(L)$ *such that* $Q \subseteq P$, *then* $P = Q$.
We denote by $Min(L)$ the set of all minimal prime filters of L.
• A proper filter F of L is called maximal if and only if for each filter $J \neq F$, if $F \subseteq J$, implies $J = L$.
We denote by $Max(L)$ the set of all maximal filters of L.
A $BL-$algebra is called local if it has a unique maximal filter.
The intersection of all maximal filters of L is called the radical of L and it is denoted by $Rad(L)$. The intersection of all maximal filters of L which contain the filter F is called the radical of F and it is denoted by $Rad(F)$. In this paper, we display $Rad(F)$ with M_F.

Note: Prime filter F of L is called minimal prime filter over filter I, if
(i) $I \subseteq F$;
(ii) If there exists $Q \in Spec(L)$ such that $I \subseteq Q \subseteq F$, then $F = Q$.
We denote by $Min(I)$ the set of all minimal prime filters over filter I.

Corollary 2.4. *[8] Every prime filter of L is contained in a unique maximal filter of L.*

Note: Let $a \in L$. We put:

$$M_a = \cap \{M |\ M \in Max(L),\ a \in M\} \qquad M(a) = \{M |\ M \in Max(L),\ a \in M\}$$
$$P_a = \cap \{P |\ P \in Min(L),\ a \in P\}$$

Theorem 2.5. *[7] A $BL-$algebra L is called semi simple if and only if $Rad(L) = \{1\}$.*

Definition 2.6. [7] Let F be a proper filter of L. If $Rad(F) = F$ (or $M_F = F$), then F is called a semi maximal filter of L.

Theorem 2.7. *[7] Every maximal filter of L is a semi maximal filter.*

Definition 2.8. [8] A non-empty subset S of L is called a $\vee-$closed system in L if $0 \in S$ and $a, b \in S$ implies that $a \vee b \in S$.

Theorem 2.9. *[8] Let S be a $\vee-$closed system of L and $F \in F(L)$ such that $F \cap S = \emptyset$. Then there exists a prime filter P of L such that $F \subseteq P$ and $P \cap S = \emptyset$.*

Definition 2.10. [1] A proper filter F of L is called a $Z^{\circ}-$filter if $P_a \subseteq F$, for each $a \in F$.

Definition 2.11. [10] Let X be a non-empty subset of L. $Co - Ann_L(X)$ is the co- annihilator of X defined by:

$$co - Ann_L(X) = \{a \in L : a \vee x = 1, \forall x \in X\}$$

Theorem 2.12. *[3] Let $P \in Min(L)$ and F be a finitely generated filter. Then $F \subseteq P$ if and only if $co - Ann_L(F) \nsubseteq P$.*

Definition 2.13. [8] An element $a \in L$ is called archimedean if there is $n \in \mathbb{N}$, such that $a \vee (a^n)^* = 1$.

Theorem 2.14. *[8] L is Hyperarchimedean if and only if $Spec(L) = Max(L)$.*

Remark 2.15. [8] For every $F \in F(L)$;
(i) $F = \cap \{P \in Spec(L) \mid F \subseteq P\}$.
(ii) $\cap \{P \in Spec(L)\} = \{1\}$.

3. Semi maximal filters in $BL-$algebras

We begin this section with a theorem that in fact provides an equivalent definition for the concept of semi maximal filters. Also, we provide several equivalent definitions of semi maximal filters and investigate their relationship with other types of filters. Building on these insights, also the properties of semi maximal filter in special classes of BL-algebras, such as Hyperarchimedean and semi simple $BL-$algebras are studied.

Theorem 3.1. *A proper filter F of L is a semi maximal filter if and only if for all $a \in F$, $M_a \subseteq F$.*

Proof. Suppose that F is a semi maximal filter of L. By contrary, let there exists $a \in F$ such that $M_a \nsubseteq F$. Always $M_F \subseteq M_a$, for all $a \in F$. Thus $M_F \subseteq M_a \nsubseteq F$. Since F is semi maximal, so $M_F = F$. Hence $F \nsubseteq F$, which is a contradiction.

Conversely, it is clear that $F \subseteq M_F$. We now show that $M_F \subseteq F$. Obviously, for all $a \in F$ we have $M_F \subseteq M_a$. Also, since $M_a \subseteq F$, we can conclude that $M_F \subseteq F$. Therefore, $M_F = F$. □

Example 3.2. *(i) Let $L = \{0, a, b, c, d, 1\}$. where $0 < a, b < c < 1$ and $0 < b < d < 1$. Define $\odot$ and $\rightarrow$ as follows:*

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

The relationships between the members are depicted in the following diagram:

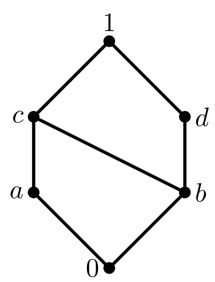

Figure 1

Then $(L, \odot, \rightarrow, 0, 1)$ *is a BL-algebra [8]. It has four filters:*
$F_0 = \{1\}, F_1 = L, F_2 = \{1, a, c\}, F_3 = \{1, d\}$. *Obviously,* $Max(L) = \{F_2, F_3\}$ *and* $M_1 = F_0$, $M_a = F_2$, $M_b = L$, $M_c = F_2$, $M_d = F_3$. *Then* F_0, F_2 *and* F_3 *are semi maximal filters.*
(ii) *Let* $L = \overline{\mathbb{Z}} \cup \{-\infty\} \cup \{0, a, b, 1\}$, *where* $\overline{\mathbb{Z}}$ *is the set of negative integer numbers and* $-\infty < \cdots < -2 < -1 < 0 < a, b < 1$. *Operations* $\odot$ *and* $\rightarrow$ *are defined as follows:*

$\odot$	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	1
$-\infty$	$-\infty$	$\cdots$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	$\cdots$	-6	-5	-4	-3	-3	-3	-3
-2	$-\infty$	$\cdots$	-5	-4	-3	-2	-2	-2	-2
-1	$-\infty$	$\cdots$	-4	-3	-2	-1	-1	-1	-1
0	$-\infty$	$\cdots$	-3	-2	-1	0	0	0	0
a	$-\infty$	$\cdots$	-3	-2	-1	0	a	0	a
b	$-\infty$	$\cdots$	-3	-2	-1	0	0	b	b
1	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	1

$\rightarrow$	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	1
$-\infty$	1	$\cdots$	1	1	1	1	1	1	1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	$\cdots$	1	1	1	1	1	1	1
-2	$-\infty$	$\cdots$	-1	1	1	1	1	1	1
-1	$-\infty$	$\cdots$	-1	1	1	1	1	1	1
0	$-\infty$	$\cdots$	-3	-2	-1	1	1	1	1
a	$-\infty$	$\cdots$	-3	-2	-1	b	1	b	1
b	$-\infty$	$\cdots$	-3	-2	-1	a	a	1	1
1	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	1

The relationships between the members are depicted in the following diagram:

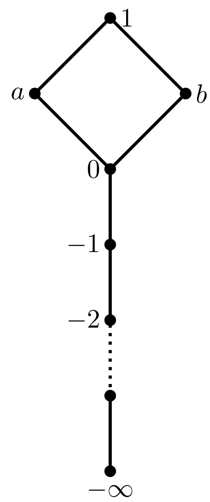

Figure 2

Then $(L, \wedge, \vee, \odot, \rightarrow, -\infty, 1)$ *is a* BL*-algebra [6]. It has filters:*
$F_0 = \{1\}$, $F_1 = L$, $F_2 = \{1, b, a, 0\}$, $F_3 = \{1, a\}$, $F_4 = \{1, b\}$ *and* $F_5 = \{\ldots, -3, -2, -1, 0, a, b, 1\} \setminus \{-\infty\}$*. Obviously,* $F_2 \in Max(L)$ *and* $M_a = F_2 \nsubseteq F_3$*. Hence* F_3 *is not a semi maximal filter.*

The following theorem provides equivalent definitions for a semi maximal filter in a $BL-$algebra.

Theorem 3.3. *Let* F *be a proper filter of* L*. Then the following statements are equivalent:*
(i) F *is a semi maximal filter in* L;
(ii) $M(a) \subseteq M(b)$ *and* $a \in F$*, imply that* $b \in F$;
(iii) $M(a) = M(b)$ *and* $a \in F$*, imply that* $b \in F$.

Proof. $i \Rightarrow ii)$ Let $M(a) \subseteq M(b)$ and $a \in F$. Then, we have $M_b \subseteq M_a$ and since $M_a \subseteq F$, it follows that $M_b \subseteq F$. Therefore $b \in F$.
$ii \Rightarrow i)$ Suppose there exists $a \in F$ such that $M_a \nsubseteq F$. Then there exists $b \in M_a$ such that $b \notin F$. It is clear that $M_b \subseteq M_a$, so $M(a) \subseteq M(b)$. By the hypothesis, we have $b \in F$, which is a contradiction.
$ii \Rightarrow iii)$ It is clear.
$iii \Rightarrow ii)$ Let $M(a) \subseteq M(b)$ and $a \in F$. Then, it is obvious that $a \vee b \in F$ and $M(a) = M(a \vee b)$. Therefore $b \in F$. □

Theorem 3.4. *Let* $f : L_1 \longrightarrow L_2$ *be a* $BL-$*homomorphism. Then every semi maximal filter of* L_2 *contracts to a semi maximal filter of* L_1 *if and only if every maximal filter of* L_2 *contracts to a semi maximal filter.*

Proposition 3.5. *Let* F *be a proper filter of* L *and* $P \in Spec(L)$ *such that* $F \subseteq P$*. Then there exists* $Q \in Min(F)$ *such that* $Q \subseteq P$*.*

Proof. Set $\Sigma = \{H \in Spec(L) \mid F \subseteq H \subseteq P\}$. Since $P \in \Sigma$, we have $\Sigma \neq \emptyset$. We define a partial order $\leq$ on Σ by

$$\forall \quad H, K \in \Sigma, \quad H \leq K \Longleftrightarrow K \subseteq H$$

Let $\beta = \{P_i\}_{i\in I}$ be a non-empty chain of elements of Σ. Put $E = \bigcap_{i\in I} P_i$. It can be shown that $E \in Spec(L)$. Since $\beta \neq \emptyset$, we have $E \neq L$. Let $x \vee y \in E$ and $x \notin E$. Then, there exists $i \in I$ such that $x \notin P_i$. For any $P_j \in \beta$, then we consider two cases:
Case 1: If $P_i \subseteq P_j$, then $x \vee y \in P_i$ and $x \notin P_i$, so $y \in P_i \subseteq P_j$. Thus $y \in P_j$.
Case 2: If $P_j \subseteq P_i$, then $x \notin P_j$ and $x \vee y \in P_j$, thus $y \in P_j$.
In both cases, we have $y \in P_j$, for all $j \in I$, implies that $y \in E$. Clearly, $F \subseteq E \subseteq P$. Hence E is an upper bound of chain β in E, so by Zorn's Lemma E has a maximal elements Q such that $Q \in \Sigma$. Now, It can be shown that $Q \in Min(L)$ such that $F \subseteq Q$. Let T be an element of $Spec(L)$ such that $F \subseteq T \subseteq Q$. Then $T \in \Sigma$ and $Q \leq T$, which is a contradiction. □

Theorem 3.6. *Every semi maximal filter of L is the intersection of the minimal prime filters containing it.*

Proof. It is clear that $F \subseteq \bigcap_{P\in Min(F)} P$. Since F is a semi maximal filter, we have $M_x \subseteq F$, for each $x \in F$. Obviously, $M_F \subseteq F$. On the other hand by Corollary 2.4 and Proposition 3.5, we have $\bigcap_{P\in Min(F)} P \subseteq M_F$. This implies that $\bigcap_{P\in Min(F)} P \subseteq F$. Combining the two inclusions, we can conclude that $F = \bigcap_{P\in Min(F)} P$. □

Theorem 3.7. *Let F be a semi maximal filter of L. Then every minimal prime filter over F is a semi maximal filter.*

Proof. Let Q be a minimal prime filter over F and suppose there exists $y \in Q$ such that $M(y) \subseteq M(a)$ and $a \in L \setminus Q$. Define $S = (L \setminus Q) \cup \{x \vee y | \ x \in L \setminus Q\}$. It is clear that S is a $\vee-$closed system of L. We claim that $I \cap S = \emptyset$. Let $x \vee y \in F$ such that $x \notin Q$. We show that $M(x \vee y) \subseteq M(x \vee a)$. If $M \in Max(L)$ such that $x \vee y \in M$, then $x \in M$ or $y \in M$. Thus, $M(x \vee y) \subseteq M(x \vee a)$, which implies that $x \vee a \in F$. Therefore, $x \vee a \in Q$, so $x \in Q$ or $a \in Q$, which is a contradiction. Hence $F \cap S = \emptyset$. By Theorem 2.9, there exists $H \in Spec(L)$ such that $H \cap S = \emptyset$ and $F \subseteq H$. Clearly, $H \subsetneq Q$ which is a contradiction. □

Note: In general, the trivial filter is not a semi maximal filter. As shown in Example 3.2 (ii), F_0 is not a semi maximal filter. In the following, we will provide the conditions under which the trivial filter is a semi maximal filter.

Lemma 3.8. *Let L be a semi simple $BL-$algebra. Then*
(i) *The trivial filter is a semi maximal filter of L.*
(ii) *Every minimal prime filter of L is a semi maximal filter.*

Proof. (i) We have $\cap\ \{M|\ M \in Max(L)\} = Rad(L)$. Since L is semi simple, so $Rad(L) = \{1\}$. Therefore, the trivial filter is a semi maximal filter.
(ii) In Theorem 3.7, consider the case where $I = \{1\}$. Then, every minimal prime filter of L is a semi maximal filter. □

Theorem 3.9. *Let L be a semi simple $BL-$algebra and P be a prime filter of L. Then either P is a semi maximal filter or the maximal semi maximal filters contained in P are prime semi maximal filters.*

Proof. Define $\Omega = \{K \in F(L) : K \subseteq P$ and K is a $Z - filter\}$. By Lemma 3.8 (i), $K_0 = \{1\} \in \Omega$, then $\Omega \neq \emptyset$. We define a partial order $\leq$ on Ω as follows:

$$\forall K^{'}, K^{''} \in \Omega \quad ; \quad K^{'} \leq K^{''} \Longleftrightarrow K^{'} \subseteq K^{''}.$$

Let $\beta = \{K_j\}_{j\in J}$ be a non-empty chain of elements of Ω. Define $Q = \bigcup_{j\in J} K_j$. Clearly, $Q \in \Omega$ as well. By Zorn's Lemma Ω has a maximal element K^* such that $K^* \in \Omega$. We consider two cases:
Case 1: if $K^* = P$, then P is a semi maximal filter.
Case 2: if $K^* \subsetneq P$, then by Proposition 3.5, there exists a minimal prime filter $P^{'} \in Min(L)$ such that $K^* \subseteq P^{'} \subsetneq P$. By Lemma 3.8 (ii), $P^{'}$ is a semi maximal filter and therefore $K^* = P^{'}$ is a prime filter. □

Proposition 3.10. *Let F be a semi maximal filter of L and K a subset of L such that $K \nsubseteq F$. Then the set $(F : K) = \{x \in L \mid x \vee k \in F, \forall k \in K\}$ is also a semi maximal filter.*

Proof. Since $K \nsubseteq F$, it follows that $(F : K)$ is a proper filter. Let $M(b) \subseteq M(a)$ and $b \in (F : K)$. Then $b \vee k \in F$, for all $k \in K$. We first prove that $M(b \vee k) \subseteq M(a \vee k)$, for all $k \in K$. Suppose that $M \in M(b \vee k)$ and $M \notin M(a \vee k)$, then $b \vee k \in M$ and $a \vee k \notin M$. Hence $b \in M$ and $a \notin M$, so $M \in M(b)$ and $M \notin M(a)$, This contradicts the assumption that $M(b) \subseteq M(a)$. Therefore, $M(b \vee k) \subseteq M(a \vee k)$. By Theorem 3.3, since $M(b \vee k) \subseteq M(a \vee k)$ for all $k \in K$, we have $a \vee k \in F$. Therefore, $a \in (F : K)$ and by Theorem 3.3, $(F : K)$ is a semi maximal filter. □

Corollary 3.11. *If L is a semi simple $BL-$algebra and X be a subset of L such that $X \neq \{1\}$, then $co - Ann_L(X)$ is a semi maximal filter.*

Proof. By Lemma 3.8, we have $\{1\}$ is a semi maximal filter. Now, let's put $F := \{1\}$. By Proposition 3.10, we know that $(\{1\} : X)$ is a semi maximal filter. Obviously, $(\{1\} : X) = co - Ann_L(X)$. Hence $co - Ann_L(X)$ is a semi maximal filter. □

Remark 3.12. Note that a semi maximal filter is not necessarily a $Z^{\circ}-$filter and vice versa. This can be seen in Example 3.2 (ii):
(i) F_2 is a semi maximal filter but not a $Z^{\circ}-$filter.
(ii) F_3 is a $Z^{\circ}-$filter but not a semi maximal filter.

In the following proposition, we investigate the conditions under which a semi maximal filter is a $Z^{\circ}-$filter. This will provide a deeper understanding of the relationship between these two types of filters in BL-algebras.

Proposition 3.13. *If $M_a \subseteq P_a$, for all $a \in L$, then every $Z^{\circ}-$filter of L is a semi maximal filter.*

Proof. Suppose that F is a $Z^{\circ}-$filter and $a \in F$. By the hypothesis that $M_a \subseteq P_a$ for all $a \in L$, we have $P_a \subseteq F$. Therefore, $M_a \subseteq P_a \subseteq F$ and thus F is a semi maximal filter. □

Theorem 3.14. *Let L be a semi simple $BL-$algebra. Then for any $a \in L$, the following holds:*

$$M_a \subseteq P_a.$$

Proof. Proof by contradiction, suppose there exists an element $b \in M_a$ but $b \notin P_a$. Since $b \notin P_a$, it means that there exists $P \in Min(L)$ such that $b \notin P$ and $a \in P$. By Theorem 3.14, $co - Ann_L(a) \nsubseteq P$, so there exists $x \in L$ such that $x \vee a = 1$ but $x \notin P$. Since $b \notin P$ and $x \notin P$, thus $b \vee x \notin P$, this means $b \vee x \neq 1$. However, there exists $M \in Max(L)$ such that $b \vee x \notin M$ so $b \notin M$ and $x \notin M$. Also, $x \vee a = 1$, hence $x \vee a \in M$ implies that $a \in M$, which is a contradiction. □

Corollary 3.15. *Let L be a semi simple $BL-$algebra. Then every $Z^{\circ}-$filter is also a semi maximal filter.*

Theorem 3.16. *Let L be a Hyperarchimedean $BL-$algebra. Then every proper filter in L is both a semi maximal filter and a $Z^{\circ}-$filter.*

Proof. Suppose that F is a proper filter of L and $a \in F$. By Theorem 2.14, for any $a \in L$, we have

$$P_a = M_a = \cap\{P \in Spec(L) \mid a \in P\}$$

Now, by Remark 2.15, it is clear that both P_a and M_a are included in the proper filter F. □

Next, we show a summary of the relationship between semi maximal filters in a diagram. The word (S) refers to semi simple $BL-$algebra.

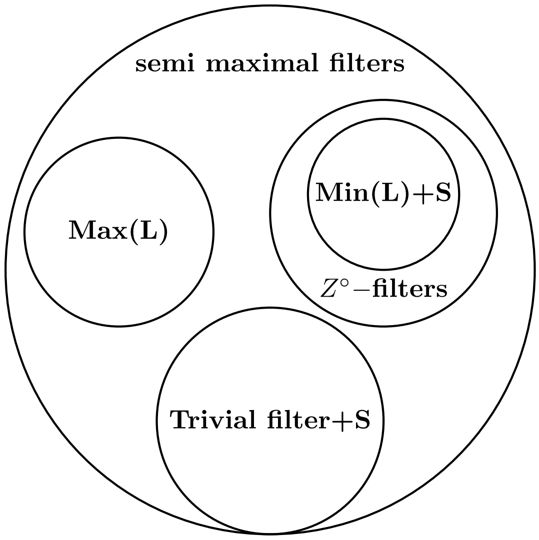

Figure 3: The relationships between semi maximal filters and the other filters in $BL-$algebras.

4. Conclusions and future research

In this paper, a more comprehensive study of semi maximal filters be obtained. We have concluded that if F is a semi maximal filter of a $BL-$algebra L, then every minimal prime filter over F is a semi maximal filter. However, we have also proved that every semi maximal filter F is the intersection of the minimal prime filters containing it. This result highlights the relationship between semi maximal filters and minimal prime filters, providing a useful characterization of semi maximal filters within the context of $BL-$algebras.
For future research, we suggest the following approach.
Let $S(L) = \{F \in F(L) : F \text{ is a semi maximal filter }\}$ and $U(F) = \{S \in S(L) : F \nsubseteq S\}$, for every filter F of L. These sets $U(F)$ can be used to define a topology on L. This new topology can then be compared to the Zariski topology and the inverse topology on $BL-$algebras.

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