Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

THE PRIME SPECTRUM OF A BCI-ALGEBRA

A. Najafi ✉ and A. Borumand Saeid

Special issue dedicated to Professor Esfandiar Eslami

Article type: Research Article
(Received: 16 May 2024, Received in revised form 26 August 2024)
(Accepted: 27 September 2024, Published Online: 30 September 2024)

Abstract. The aim of the present paper is to define the prime spectrum of a BCI-algebra as a generalization of prime spectrum BCK-algebras with respect to prime ideals. The notions of prime spectrum BCI-algebras using prime ideals, and some properties of these concepts are studied. Finally, we attempt to generalize some useful theorems about prime spectra in BCI-algebras instead of commutative BCK-algebras.

Keywords: BCI-algebras, prime spectrum, prime ideals, prime BCI-algebras.
2020 MSC: 03G25, 06F35.

1. Introduction

The class of BCI-algebras was introduced by Imai and Iséki as algebraic semantics for non-classical logic having only implication in 1966 [5]. Iséki, introduced the concept of prime ideal in commutative BCK-algebras and Palasinski, generalized this definition for any lower BCK-semilattices [7, 11]. Many authors have studied the properties of this ideal in lower BCK-semilattices (see [7, 8, 9, 10]).
M. Palasinski [11] and C. S. Hoo and P. V. Murty introduced topologies on the set of all prime ideals of a commutative BCK-algebra in different way, respectively [8]. M. Aslam, E. Y. Deeba and A. B. Thaheem developed this theory and applied it to investigate some properties of commutative BCK-algebras [1].
R. A. Borzooei and O. Zahiri generalized the concept of prime ideals for BCI-algebras. They verified some properties of these ideals in BCI-algebras such as the relationship between prime and maximal ideals [2]. After that, the present authors used this notion to define a prime BCI-algebra and considered semiprime BCI-algebras using semiprime ideals [12].
In this paper, we focus on the prime BCI-algebras, denoted pBCI, which has ties to many other algebraic structures including MV-algebras, lattice ordered, Abelian groups, commutative integral residuated lattices, and others.
Now, we give a definition of a topology on BCI-algebras and consider properties

✉ ardavan.najafi@iau.ac.ir, ORCID: 0000-0003-3754-651X
https://doi.org/10.22103/jmmr.2024.23433.1647

Publisher: Shahid Bahonar University of Kerman
How to cite: A. Najafi, A. Borumand Saeid, *The prime spectrum of a BCI-algebra*, J. Mahani Math. Res. 2024; 13(4): 143-151.

of prime BCI-algebra from the point of view of this topology. Finally, we attempt to generalize some useful theorems about prime spectra in BCI-algebras instead of commutative BCK-algebras.

2. Preliminaries

By a BCI-algebra, we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfies the following axioms, for all $x, y, z \in X$ [4,5],
(BCI1) $((x * y) * (x * z)) * (z * y) = 0$,
(BCI2) $(x * (x * y)) * y = 0$,
(BCI3) $x * x = 0$,
(BCI4) $x * y = y * x = 0$ implies $x = y$.
Recall that given a BCI-algebra X, the BCI-ordering $\leq$ on X is defined by $x \leq y$ if and only if $x * y = 0$ for any $x, y \in X$. The set $\{x \in X : 0 * x = 0\}$ is called BCK-part of BCI-algebra X and is denoted by $B(X)$. If $X = B(X)$, then we say X is a BCK-algebra. A BCK-algebra X is called to be a BCK chain if $x \leq y$ or $y \leq x$ for any $x, y \in X$. A BCK-algebra X is said to be a lower BCK-semilattice if X is a lower semilattice with respect to BCK-order $\leq$. A BCI-algebra X has the following properties for all $x, y, z \in X$,
(BCI5) $x * 0 = x$,
(BCI6) $(x * y) * z = (x * z) * y$,
(BCI7) $0 * (x * y) = (0 * x) * (0 * y)$,
(BCI8) $x * (x * (x * y)) = x * y$.
An ideal I of X is a subset of X such that (i) $0 \in I$ and (ii) $x, y * x \in I$ imply $y \in I$ for any $x, y \in X$. We say that an ideal I is proper if $I \neq X$. Sometimes, $\{0\}$ is called the zero ideal of X, denoted by O in brevity. A subalgebra Y of X is a nonempty subset of X such that Y is closed under the BCI-operation $*$ on X. If A is both an ideal and a subalgebra of X, we call it a closed ideal of X. An ideal I is called a maximal ideal of X if I is a proper ideal of X and it is not a proper subset of any proper ideal of X. Let I be an ideal of a BCI-algebra X, then the relation θ defined by $(x, y) \in \theta$ if and only if $x * y \in I$ and $y * x \in I$ is a congruence relation on X. We usually denote x/I for $[x] = \{y \in X : (x, y) \in \theta\}$. Moreover, $0/I$ is a closed ideal of BCI-algebra X. In fact, it is the greatest closed ideal contained in I. Let $(X, *, 0)$ and $(Y, ., 0)$ be two BCI-algebras, the map $f : X \longrightarrow Y$ is called a homomorphism, if $f(x * y) = f(x).f(y)$ for all $x, y \in X$. A BCK/BCI-algebra X is said to be simple if it has no non-zero proper ideal. A BCI-algebra X is called commutative BCI-algebra if $x \leq y$ implies $x = x \wedge y$, where $x \wedge y = y * (y * x)$. A BCK-algebra X is said to be implicative if $x * (y * x) = x$ for all $x, y \in X$. Let S be a subset of BCI-algebra X. We call the least ideal of X, containing S, the generated ideal of X by S, denoted by $(S]$. (see [4, 6, 7, 10])
Note: From now on, in this paper, we let $(X, *, 0)$ or simply X be a BCI-algebra, unless otherwise specified.

Definition 2.1. [2, 7] i) A proper ideal I of X is called an *irreducible ideal* if $A \cap B = I$ implies $A = I$ or $B = I$, for any ideals A and B of X.
ii) A proper ideal P of X is called *prime* if $A \cap B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for all ideals A and B of X. If X is a lower BCK-semilattice, then this definition is equivalent with $x \wedge y \in P$ implying $x \in P$ or $y \in P$.
iii) If M is a maximal ideal of BCK-algebra X, then M is a prime ideal of X.

The set of all ideals of X is denoted by $Id(X)$, the set of all prime ideals of X is denoted by $Spec(X)$, called the spectrum of X [1].
As lattices, $Id(X) \cong Con(X)$, where $Con(X)$ is the set of all congruence of a BCK-algebra X [13]. Let X be a nonempty set. A topology τ for X is a collection of subsets of X such that $\emptyset, X \in \tau$, and τ is closed under arbitrary unions and finite intersections. We say (X, τ) is a topological space. Members of τ are called open sets. A topological space fulfilling the T_0-Space if for any two points x, y in X, there is an open set U such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. A ring of sets is a nonempty set R of subsets of a set S such that if $A, B \in R$, then $A \cup B \in R$ and $A \cap B \in R$. Let (X, τ) be a topological space and V be a subset of X, then a point $x \in X$, is said to be an exterior point of V if there exists an open set U, such that $x \in U \in V^c$, the set of all exterior points of V is said to be the exterior of V and is denoted by $Ext(V)$. V is said to be a dense subset of X if for every $x \in X$ every neighborhood U of x, $U \cap V \neq \emptyset$. (see [3]).

Definition 2.2. [12] X is said to be a prime BCI-algebra if the zero ideal is a prime ideal, that is, $I \cap J = O$ implies that $I = O$ or $J = O$, for all proper ideals I and J of X.

Example 2.3. *Let $X = \{0, a, b, c\}$ ($Y = \{0, a, b, c, d\}$) be a BCI-algebra in which $*$ operation is defined by the following table, respectively.*

$*$	0	a	b	c
0	0	0	b	b
a	a	0	b	b
b	b	b	0	0
c	c	b	a	0

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	0
b	b	a	0	b	0
c	c	c	c	0	c
d	d	d	d	d	0

By routine calculations, it was observed that $Id(X) = \{\{0\}, \{0, a\}, X\}$. So, for ideals I, J of X if we have $I \cap J = \{0\}$, then $I = \{0\}$ or $J = \{0\}$. Hence X is a prime BCI-algebra. Also $Id(Y) = \{\{0\}, \{0, c\}, \{0, a, b\}, \{0, a, b, c\}, \{0, a, b, d\}, X\}$. For ideals $I = \{0, a, b\}$ and $J = \{0, c\}$, note that $I \cap J = \{0\}$ but $I \neq \{0\}$ and $J \neq \{0\}$. Hence Y is not a prime BCI-algebra.

Proposition 2.4. [2] *Every simple BCI-algebra X is a prime BCI-algebra.*

Proposition 2.5. [2] *Let X be a BCK-algebra. Then the following assertions hold:*

i) Any proper ideal I of X can be expressed as the intersection of all prime ideals of X containing I.
ii) If M is a maximal ideal of X, then M is a prime ideal of X.
iii) Let $f : X \longrightarrow Y$ be a BCI-homomorphism of BCI-algebras, such that $f(X)$ is an ideal of Y. If I is a prime ideal of Y and $f^{-1}(I) \neq X$, then $f^{-1}(I)$ is the prime ideal of X.

3. Spectrum of a prime BCI-algebra

In this section, we consider the spectra of the prime BCI-algebras. In the last years, Palasinski first explored a topological representation for commutative BCK-algebras in [11], the next, Hoo and Murty defined the spectrum of a bounded commutative BCK-algebra in [8], the topological space they define is not the same as Palasinski's. The spectrum defined by Hoo and Murty was used as the standard. Later, Meng and Jun proved that the spectrum of a bounded commutative BCK-algebra is a spectral space. Aslam, Deeba, and Thaheem studied spectra of commutative BCK-algebras both with and without the assumption boundedness in [1]. The present work should be introduced as a generalization and continuation of these several works.

Definition 3.1. Let S be a subset of X, we define

$$\sigma(S) = \{P \in Spec(X) : S \nsubseteq P\}.$$

Suppose that X is a prime BCI-algebra. Since $O = \{0\} \subseteq I$, for all ideals I of X, $\sigma(O) = \{P \in Spec(X) : O \nsubseteq P\} = \emptyset$ and $\sigma(X) = \{P \in Spec(X) : X \nsubseteq P\} = Spec(X)$. For short denote $\sigma(a)$ instead of $\sigma(\{a\})$. Also $\sigma(I) = \sigma((I])$ and, in particular, $\sigma(a) = \sigma((a])$, for $a \in X$.

Lemma 3.2. *Suppose A and B are subsets of X and I, J are two ideals of X.*
i) If $A \subseteq B$, then $\sigma(A) \subseteq \sigma(B)$.
ii) $\sigma(I \cap J) = \sigma(I) \cap \sigma(J)$.
iii) $\sigma(I \cup J) = \sigma(I) \cup \sigma(J)$.

Proof. i) Let $A \subseteq B$ and $P \in \sigma(A)$. Then $A \nsubseteq P$. As $A \subseteq B$, $B \nsubseteq P$. Hence $P \in \sigma(B)$. Therefore $\sigma(A) \subseteq \sigma(B)$.
ii)

$$\begin{aligned} P \in \sigma(I) \cap \sigma(J) &\Longleftrightarrow P \in \sigma(I) \wedge P \in \sigma(J) \\ &\Longleftrightarrow I \nsubseteq P \wedge J \nsubseteq P \\ &\Longleftrightarrow I \cap J \nsubseteq P \\ &\Longleftrightarrow P \in \sigma(I \cap J) \end{aligned}$$

The third relation, as P is a prime ideal is obtained.
iii) Since $I \subseteq I \cup J$ and $J \subseteq I \cup J$, $\sigma(I) \subseteq \sigma(I \cup J)$ and $\sigma(J) \subseteq \sigma(I \cup J)$. Hence $\sigma(I) \cup \sigma(J) \subseteq \sigma(I \cup J)$. Now, if $P \in \sigma(I \cup J)$, then $I \cup J \nsubseteq P$. So at least occurs one of the two states $I \nsubseteq P$ or $J \nsubseteq P$. If $I \nsubseteq P$, then $P \in \sigma(I)$ and if $J \nsubseteq P$, then $P \in \sigma(J)$. Therefore $P \in \sigma(I) \cup \sigma(J)$. □

Theorem 3.3. *Let I be an ideal of X, then*
i) If X is a BCK-algebra, then $\sigma(I) = Spec(X)$ if and only if $I = X$.
ii) $\sigma(I) = \cup_{a\in I}\sigma(a)$.
Moreover, if X is a prime BCI-algebra
iii) $\sigma(I) = \emptyset$ iff $I = O$.

Proof. i) Suppose that $\sigma(I) = Spec(X)$. Then $\{P \in Spec(X) : I \nsubseteq P\} = Spec(X)$. Therefore, for every $P \in Spec(X)$, we have $I \nsubseteq P$. Since any maximal ideal of a BCK-algebra is a prime ideal, $I = X$.
Conversely, if $I = X$, then $\sigma(I) = \sigma(X) = \{P \in Spec(X) : X \nsubseteq P\} = Spec(X)$.
ii) Let $P \in \sigma(I)$. Then $I \nsubseteq P$. Therefore there exists $a \in I$ such that $a \notin P$. That means $P \in \sigma(a)$ and so $P \in \cup_{a\in I}\sigma(a)$.
Conversely, if $P \in \cup_{a\in I}\sigma(a)$, then $P \in \sigma(a)$ for some $a \in I$. Then $a \notin P$ and hence $I \nsubseteq P$. Therefore $P \in \sigma(I)$.
iii) Suppose that $\sigma(I) = \emptyset$. Then $\{P \in Spec(X) : I \nsubseteq P\} = \emptyset$. Hence $I \subseteq P$ for all $P \in Spec(X)$. Since X is prime, $O \in Spec(X)$. Whence, $I \subseteq O$. Therefore $I = O$.
Conversely, if $I = O$, then $\sigma(I) = \sigma(O) = \{P \in Spec(X) : O \nsubseteq P\} = \emptyset$. □

Proposition 3.4. *The family*

$$T(X) = \{\sigma(I) : I \in Id(X)\}$$

satisfies the following conditions:
i) $T(X)$ is a ring of sets.
For a prime BCI-algebra X
ii) $T(X)$ forms a topology on $Spec(X)$.
iii) $T(X)$ forms a lattice.

Proof. i) It is obvious from Lemma 3.2 parts (ii) and (iii).
ii) Empty set and $Spec(X)$ itself belong to $T(X)$. By Lemma 3.2 part (ii) the intersection of any finite number of members of $T(X)$ belongs to $T(X)$. Further, for any family $(\sigma(A_\alpha))_{\alpha\in\Omega}$, we obtain

$$\begin{aligned}\cup_{\alpha\in\Omega}\sigma(A_\alpha)) &= \{P \in Spec(X) : \exists\alpha \in \Omega \;\; A_\alpha \nsubseteq P\}\\ &= \{P \in Spec(X) : \cup_{\alpha\in\Omega}A_\alpha \nsubseteq P\}\\ &= \{P \in Spec(X) : < \cup_{\alpha\in\Omega}A_\alpha > \nsubseteq P\}\\ &= \sigma(< \cup_{\alpha\in\Omega}A_\alpha >)\end{aligned}$$

which implies that $\cup_{\alpha\in\Omega}\sigma(A_\alpha) \in T(X)$. So the ordered pair $(Spec(X), T(X))$ is a topological space of X.
In the above proof, the third equality is obtained as follows:
It can be clearly seen that $\{P \in Spec(X) : \cup_{\alpha\in\Omega}A_\alpha \nsubseteq P\} \subseteq \{P \in Spec(X) : < \cup_{\alpha\in\Omega}A_\alpha > \nsubseteq P\}$. Now, if $\{P \in Spec(X) : < \cup_{\alpha\in\Omega}A_\alpha > \nsubseteq P\} \nsubseteq \{P \in Spec(X) : \cup_{\alpha\in\Omega}A_\alpha \nsubseteq P\}$, then there exists $P \in Spec(X)$ such that $P \in \{P \in Spec(X) : < \cup_{\alpha\in\Omega}A_\alpha > \nsubseteq P\}$ but $P \notin \{P \in Spec(X) : \cup_{\alpha\in\Omega}A_\alpha \nsubseteq P\}$. Hence $\cup_{\alpha\in\Omega}A_\alpha \subseteq P$

and so $< \cup_{\alpha\in\Omega} A_\alpha > \subseteq P$. This is a contradiction.
iii) It is straightforward by Lemma 3.2 parts (ii) and (iii). □

The topological space $(Spec(X), T(X))$ is called the spectral space of X. It is easy to see that for any ideal I of prime BCI-algebras X, $\sigma(I) = \cup_{a\in I}\sigma(a)$, therefore $\{\sigma(a) : a \in X\}$ forms a base for $T(X)$.
In the following, we provide a few examples of the spectrum for finite and infinite BCI-algebras.

Example 3.5. *i) Let $X = \{0, a, b, c, d\}$ be a BCI-algebra in which $*$ is defined by the following table*

$*$	0	a	b	c	d
0	0	0	0	c	c
a	a	0	0	c	c
b	b	b	0	d	c
c	c	c	c	0	0
d	d	d	c	b	0

Routin calculations shows that O, $I = \{0, a\}$, $J = \{0, a, b\}$, $K = \{0, a, c\}$ are all proper ideals of X. The ideals O, J, K are prime ideals. Therefore, $Spec(X) = \{O, J, K\}$ and the associated topology $T(X) = \{\sigma(O), \sigma(I), \sigma(J), \sigma(K), \sigma(X)\} = \{\emptyset, \{O\}, \{O, J\}, \{O, K\}, \{O, J, K\} = Spec(X)\}$.
ii) Suppose that $X_1, X_2, ..., X_m$ be chains. Let

$$\begin{aligned} X_1 =& \{0, x_{1_1}, x_{1_2}, ..., x_{1_k}, ...\} \\ X_2 =& \{0, x_{2_1}, x_{2_2}, ..., x_{2_k}, ...\} \\ &. \\ &. \\ &. \\ X_m =& \{0, x_{m_1}, x_{m_2}, ..., x_{m_k}, ...\} \end{aligned}$$

Where $0 < x_{i_1} < x_{i_2} < x_{i_3} < ... < x_{i_k} < ...$ and $i = 1, 2, ..., m$ and for $i \neq j$ we have $X_i \cap X_j = \{0\}$. Also every pair of nonzero elements of distinct chains is incomparable.
We put $X = \bigcap_{i=1}^m X_i$ and for any $x, y \in X$ define

$$x * y = \begin{cases} 0 & \text{if } x \leq y \\ x_{i_{k-l}} & \text{if } y < x \text{ and } x = x_{i_k}, y = x_{i_l} \\ x & \text{otherwise} \end{cases}$$

By simple calculations we can show that X is an infinite BCI-algebra and X_i is an ideal of X for any $i = 1, 2, ..., m$. We claim that X_i are only proper ideals of X. Let I be an ideal of X which is different from X_i for *each $1 \leq i \leq m$. Suppose that $I \cap X_i \neq \{0\}$, for some i. We may assume that x_{i_p} is the least of all elements of X_i that belong to I, as X_i is a chain. Since $x_{i_{p-1}} < x_{i_p}$, $x_{i_p} \in I$. Hence $x_{i_{p-1}} \in I$, that is a contradiction. So X_i are only proper ideals of X such that are prime ideal, because X_i are maximal ideals. Hence $Spec(X) = \{X_1, X_2, ..., X_m\}$ and the associated topology $T(X)$ is $T(X) = \{\sigma(X_1), ..., \sigma(X_m), \sigma(X)\} = \{Spec(X) - X_1, Spec(X) - X_2, ..., Spec(X) - X_m, Spec(X)\}$.*

Lemma 3.6. *$Spec(X)$ is a T_0 space.*

Proof. Suppose that $P_1, P_2 \in Spec(X)$ such that $P_1 \neq P_2$. Without loss of generality let $P_1 \nsubseteq P_2$, then there is some $a \in P_1$ such that $a \notin P_2$. Then $P_1 \notin \sigma(a)$ and $P_2 \in \sigma(a)$, so $\sigma(a)$ is an open set separating P_1 and P_2. □

Corollary 3.7. *If X is a simple BCI-algebra, then $Spec(X)$ is a T_0 space.*

Proposition 3.8. *For a BCK-algebra X, $T(X) \cong Id(X)$.*

Proof. We define the mapping $f : Id(X) \longrightarrow T(X)$ given by $f(I) = \sigma(I)$, for an ideal I of X and we show that f is a lattice isomorphism. f is a homomorphism by Lemma 3.2 parts (ii) and (iii). Obvious, f is well defined and onto by definition. We only show that f is one-to-one. Let $f(I) = f(J)$ for any $I, J \in Id(X)$. Then $\sigma(I) = \sigma(J)$ and hence $(\sigma(I))^c = (\sigma(J))^c$, where $(\sigma(I))^c = Spec(X) - \sigma(I)$. That means $\{P \in Spec(X) : I \subseteq P\} = \{P \in Spec(X) : J \subseteq P\}$. So $\bigcap\{P \in Spec(X) : I \subseteq P\} = \bigcap\{P \in Spec(X) : J \subseteq P\}$. But $\bigcap\{P \in Spec(X) : I \subseteq P\} = I$ whence it follows that $I = J$ and consequently f is one-to-one. □

Corollary 3.9. *$T(X) \cong Con(X)$, where $Con(X)$ is the set of all congruence of a BCK-algebra X.*

Proof. By Proposition 3.8 we saw $T(X) \cong Id(X)$. Also, in any BCK-algebra X we have $Id(X) \cong Con(X)$ [13]. By transitivity of $\cong$ we get $T(X) \cong Con(X)$. □

Lemma 3.10. *For prime BCI-algebras X and Y every isomorphism $f : X \longrightarrow Y$ induces an isomorphism $f^* : Spec(Y) \longrightarrow Spec(X)$ between the corresponding prime spectra which is given by $f^*(P) = f^{-1}(P)$.*

Proof. Let $f : X \longrightarrow Y$ be an isomorphism from prime BCI-algebras X and Y. Suppose that $P \in Spec(Y)$, then $P \neq Y$ and hence $f^{-1}(P) \neq f^{-1}(Y)$. That is $f^{-1}(P) \neq X$. From Proposition 2.5 part (iii) we deduce $f^*(P) = f^{-1}(P) \in Spec(X)$. Now, let P_1 and P_2 be elements of $Spec(Y)$, then $f^*(P_1 \cap P_2) = f^{-1}(P_1 \cap P_2) = f^{-1}(P_1) \cap f^{-1}(P_2) = f^*(P_1) \cap f^*(P_2)$. In the same way, we can show that $f^*(P_1 \cup P_2) = f^*(P_1) \cup f^*(P_2)$. So f^* is a homomorphism. Obviously, f is well defined and onto by definition. Since $f(O) = O$, $Ker(f^*) = \{P \in Spec(Y) : f^*(P) = O\} = \{P \in Spec(Y) : f^{-1}(P) = O\} = \{P \in Spec(Y) : f(O) = P\} = \{O\}$. Therefore f^* is one to one. Hence f^* is an isomorphism. □

The following proposition describes the exterior of a subset of $Spec(X)$.

Proposition 3.11. *For any $V \subseteq Spec(X)$, $\sigma(\bigcap_{P\in V} P) = Ext(V)$, the exterior of V.*

Proof. Let $P^{'} \in \sigma(\bigcap_{P\in V} P)$. Therefore $\bigcap_{P\in V} P \nsubseteq P^{'}$. Hence there is an element $x \in \bigcap_{p\in V} P$ such that $x \notin P^{'}$. In this case we claim $\sigma(x) \cap V = \varnothing$. For proof of these claim, we let $x \in \bigcap_{P\in V} P$. Then $x \in P$, for every $P \in V$. Hence $P \notin \sigma(x)$, for all $P \in V$ and so $\sigma(x) \cap V = \varnothing$. Now, since $x \notin P^{'}$, $P^{'} \in \sigma(x)$. Consequently, we have an open set $\sigma(x)$ that contains $P^{'}$ and $\sigma(x) \cap V = \varnothing$ which implies $P^{'} \in Ext(V)$. So $\sigma(\bigcap_{P\in V} P) \subseteq Ext(V)$.
Conversely, let $P^{'} \in Ext(V)$. Then there exists a basic open set $\sigma(x)$ such that $P^{'} \in \sigma(x)$ and $\sigma(x) \cap V = \varnothing$. Now, if $x \notin \bigcap_{p\in V} P$, then $x \notin P_0$ for some $P_0 \in V$. That is $P_0 \in \sigma(x)$ and so $P_0 \in \sigma(x) \cap V$. It is a contradiction. So $x \in \bigcap_{P\in V} P$. Thus we deduce $\bigcap_{P\in V} P \nsubseteq P^{'}$. Hence $P^{'} \in \sigma(\bigcap_{P\in V} P)$. This shows that $Ext(V) \subseteq \sigma(\bigcap_{P\in V} P)$. Therefore $Ext(V) = \sigma(\bigcap_{P\in V} P)$. □

Proposition 3.12. *: $V \subseteq Spec(X)$ is dense if and only if $\bigcap_{P\in V} P = \{0\}$.*

Proof. V is dense iff $\bar{V} = spec(X)$. Since $\bar{V} = Spec(X) - Ext(V)$, $\bar{V} = Spec(X)$ iff $Ext(V) = \varnothing$ iff $\sigma(\bigcap_{p\in V} P) = \varnothing$, iff $\bigcap_{p\in V} P = \{0\}$. □

4. Conclusions

In the present paper, we have introduced the concepts of the prime spectrum of a BCI-algebra as a generalization of prime spectrum commutative BCK-algebras with respect to prime ideals and investigated some of their properties. To develop the theory of BCI-algebras, one of the most encouraging ideas could be investigating the prime spectrum and finding a relation diagram between subclasses of BCI-algebras. For instance, the set $T(X)$ for BCK-algebra X is strictly isomorphism with $Con(X)$. Therefore, we think that the results presented in this paper and the forthcoming works can pave the way for a bright future of the theory of the BCI-algebras.

References

[1] Aslam, J., Deeb, E., & Thaheem, A. B. (1993). On Spectral properties of BCK-algebra. Mathematica Japonica, 38, 1121-1128.
[2] Borzooei, R. A., & Zahiri, O. (2012). Prime Ideals in BCI and BCK-algebras. Annals of the University of Craiova, Mathematics and Computer Science Series, 39 (2), 266-276. https://doi.org/10.52846/ami.v39i2.469
[3] Dickmann, M., Schwartz, N., Tressl, M. (2019). Spectral Spaces, Cambridge University Press. https://doi.org/10.1017/9781316543870
[4] Huang, Y. (2006). BCI-algebras. Science Press. Beijing, China.
[5] Imai, Y., & Iséki, K. (1966). On axiom system of propositional calculi. XIV, Japan Academy, 42, 19-22. https://doi.org/10.3792/pja/1195522169
[6] Iséki, K. (1966). An algebra related with a propositional calculus. Japan Academy, 42, 26-29. https://doi.org/10.3792/pja/1195522171

[7] Iséki, K. (1975). On some ideals in BCK-algebras. Math Seminar Notes 3. 42, 65-70. https://doi.org/10.1515/dema-1994-3-409
[8] Hoo, C. S., & Murty, P. V. R. (1987). The ideals of a bounded commutative BCK-algebras. Math. Japon. 32, 723 - 733.
[9] Meng, J., & Jun, Y., Xin, X. (1998). Prime ideal in commutative BCK-algebras. Discussiones Mathematicae, 18, 5-15. https://doi.org/10.7151/dmgaa
[10] Meng, J., & Jun, Y. (1994). BCK-algebras. Kyung Moon Sa Co. Seoul.
[11] Palasinski, M. (1981). Ideal in BCK-algebras which are lower lattices. Bulletin of the Section of Logic, 10(1), 48-50.
[12] Najafi, A., & Borumand Saeid, A. (2022). Semi prime BCI-algebras. Scientific Bulletin Series A, Applied Mathematics and Physics, 11, 19-28. https://doi.org/10.3233/ifs-162148
[13] Romanowska, A. & Traczyk, T. (1980). On commutative BCK-algebras. Math. Japonica, 25(5), 567–583.

Ardavan Najafi
Orcid number: 0000-0003-3754-651X
Department of Mathematics
Behbahan Branch Islamic Azad University
Behbahan, Iran
Email address: ardavan.najafi@iau.ac.ir

Arsham Borumand Saeid
Orcid number: 0000-0001-9495-6027
Department of Pure Mathematics
Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
Email address: arsham@uk.ac.ir