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SOME PROPERTIES OF CAMINA AND n-BAER LIE ALGEBRAS

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Abstract. Let I be a non-zero proper ideal of a Lie algebra L. Then (L, I) is called a Camina pair if $I \subseteq [x, L]$, for all $x \in L \setminus I$. Also, L is called a Camina Lie algebra if (L, L^2) is a Camina pair.

We first give some properties of Camina Lie algebras, and then show that all Camina Lie algebras are soluble.

Also, a new notion of n-Baer Lie algebras is introduced, and we investigate some of its properties, for $n = 1, 2$. A Lie algebra L is said to be 2-Baer if for any one dimensional subalgebra K of L, there exists an ideal I of L such that K is an ideal of I.

Finally, we study three classes of Lie algebras with 2-subideal subalgebras and give some relations among them.

Keywords: Camina Lie algebra, n-Baer Lie algebra, 2-subideal subalgebra, nilpotent Lie algebra.
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1. Introduction

Recall that a non-abelian group G is a Camina group if the conjugacy class of every element $g \in G \setminus G'$ is gG'. It is clear from the earliest papers that the study of Camina groups and the more general objects Camina pairs was motivated by finding a common generalization of Frobenius groups and extra-special groups. It is not difficult to see that extra-special groups and Frobenius groups with an abelian Frobenius complement are Camina groups. Thus, one question that seems reasonable to ask is whether there are any other Camina groups, or can one classify all Camina groups?

In [9], Dark and Scoppola stated that they had completed the classification of all Camina groups. The classification is recorded in the following theorem.

Theorem 1.1. *[9] Let G be a group. Then G is a Camina group if and only if one of the following holds:*
(i) G is a Camina p-group of nilpotence class 2 *or* 3.

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(ii) G is a Frobenius group with a cyclic Frobenius complement.
(iii) G is a Frobenius group whose Frobenius complement is isomorphic to the quaternions.

In fact, the work in [9] is the capstone of several results that combined and lead to the classification. The first result needed is that if G is a Camina group, P is a Sylow p-subgroup for some prime p, and G/G' is a p-group, then P is a Camina group which is proved in Lemma 3.6 of [6]. The second result needed is Theorem 3 of [7] which states that if G is a Camina group such that G/G' is a p-group for some prime p and a Sylow p-subgroup of G has nilpotency class at most $p+1$, then either G is a p-group or G is a Frobenius group whose complement is either cyclic or quaternion. Then Dark and Scopolla proved in [9] that Camina p-groups have nilpotency class 2 or 3. This proves Theorem 1.1 under the hypothesis that G/G' is a p-group for some prime p. Finally, by Theorem 2.1 of [13], we know that if G is a Camina group, then either G is a Frobenius group with Abelian Frobenius complement or G/G' is a p-group for some prime p. Combining all of these results, one obtains the above theorem.

In section 2 of the present article, we give some properties of Camina Lie algebras. Moreover, we show that all Camina Lie algebras are soluble.

Let n be a positive integer. A subgroup H of a group G is called n-subnormal, denoted by $H \lhd_n G$, if there exist distinct subgroups $H_0 = H, H_1, \cdots, H_n = G$ such that

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G.$$

In a group of nilpotency class n, all subgroups are n-subnormal. Conversely, by a well known result of Roseblade [18], a group with all subgroups n-subnormal is nilpotent of class bounded by a function of n. For $n = 1$, having all subgroups n-subnormal, hence normal, is equivalent to having all cyclic subgroups n-subnormal; but this is no longer the case if $n \geqslant 2$ [15].

A group G is called an n-Baer group if all of its cyclic subgroups are n-subnormal. It can be easily seen that every n-Baer group is $(n+1)$-Engel, i.e., $[x,_{n+1} y] = 1$ for all $x, y \in G$, where $[x,_1 y] = [x, y] = x^{-1}x^y$ and $[x,_k y] = [[x,_{k-1} y], y]$, $k \geqslant 2$. We say $x \in G$ is a right n-Engel element, if $[x,_n g] = 1$ for all $g \in G$, and a left n-Engel element, if $[g,_n x] = 1$ for all $g \in G$.

Of course, the class of 1-Baer groups coincides with the familiar Dedekind groups. For the finite case, these groups were classified by Dedekind [10], and for the case of infinite groups by Baer [1].

In parts 3 and 4 we introduce a new notion of n-Baer Lie algebras, and we investigate some of its properties, for $n = 1, 2$. Finally, we study three classes of Lie algebras with 2-subideal subalgebras and give some relations among them.

2. Camina Lie algebras

Let L be a Lie algebra over a field F of characteristic zero. Usually, F is considered to be the real or complex numbers, $\mathbb{R}$ or $\mathbb{C}$. The derived subalgebra $L^2 = [L, L]$ generated by $[x, y]$, for all $x, y \in L$.

An element $x \in L$ acts as a linear transformation on the vector space L by the rule $y \mapsto [x, y]$, which is denoted by ad_x and called *adjoint representation*. Clearly, the adjoint map is a homomorphism and

$$ad_{[x,y]} = ad_x \circ ad_y - ad_y \circ ad_x.$$

Recall that a linear transformation T is nilpotent, if its n^{th} power is zero for some $n \in \mathbb{N}$, i.e.$T^n = 0$. Clearly, $(ad_x)y = [x, y]$ and inductively

$$(ad_x)^2 y = (ad_x \circ ad_x)y = ad_x[x, y] = [x, [x, y]],$$

$$(ad_x)^3 y = (ad_x \circ ad_x \circ ad_x)y = [x, [x, [x, y]]], \cdots .$$

By the definition of a nilpotent Lie algebra, one can easily check that if L is nilpotent, then ad_x is also nilpotent, for all $x \in L$, as $(ad_x)^n = 0$. One notes that the converse is also true. Engel's theorem says that a finite dimensional Lie algebra L is nilpotent if and only if ad_x is nilpotent, for all x in L.

Note that the bilinear map is defined by $\kappa : L \times L \to F$, where $(x, y) \mapsto tr(ad_x \circ ad_y)$ and is called a *Killing form* of L. The Killing form is a special case of the notion of trace form. It is the trace form associated to the adjoint representation. Killing form, named after Wilhelm Killing, is a symmetric bilinear form which plays a basic role in the theory of Lie groups and Lie algebras, as $tr(AB) = tr(BA)$, for any squared matrices A and B. It is also invariant, i.e.

$$\kappa([x, y], z) = \kappa(x, [y, z]),$$

for all $x, y, z \in L$.

In 1894, Killing form was essentially introduced into Lie algebra theory by Élie Cartan [5], in his Thesis.

The following theorem is useful for further investigation

Theorem ([5], Cartan's solubility criterion). A Lie algebra L is soluble if and only if $\kappa(L, L^2) = 0$.

In 1978, Camina [2] introduced and studied the notion of Camina pairs in group theory. Camina pairs are studied by many authors in [3, 14, 16], when G is a finite group. Infinite Camina groups are studied in [12]. For example, all extra-special p-groups are Camina groups. An important property of finite Camina groups is obtained by Dark and Scoppola [9]. They proved that every finite Camina p-group is nilpotent of class at most 3.

In the present article, we introduce and study the notion of Camina pair for Lie algebras (see also [17]).

Definition 2.1. Let L be a finite dimension Lie algebra over a field F and I a non-zero proper ideal of L. Then the pair (L, I) is called Camina pair of Lie algebras, if for any elements $x \in L \setminus I$ and $i \in I$, there exists an element $l \in L$ such that $[x, l] = i$, i.e.

$$I \subseteq \{[x, l] : l \in L\}.$$

Following the notation in [8], the nilpotent Lie algebras in this paper are denoted by $L_{d,k}$, where d is the dimension of the algebra and k is its index among the nilpotent Lie algebras with dimension d.

Example 2.2. Let $L = L_{5,5} = \langle x_1, \cdots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5\rangle$ and I be the one dimensional ideal of L with basis x_5. Then

$$I \subseteq \{[x_1, x_2], [x_1, x_3], [x_2, x_4]\},$$

and so (L, I) is a Camina pair (see [8] for more information).

Now, by considering the above discussion, we introduce and study some new results on Camina Lie algebras.

Lemma 2.3. *If (L, I) is a Camina pair of Lie algebra L, then $Z(L) \subseteq I \subseteq L^2$.*

Proof. Clearly, for every $x \in L \setminus I$

$$I \subseteq \{[x, l] : l \in L\} \subseteq \langle [x, l] : l \in L\rangle \subseteq L^2 = [L, L].$$

Now, assume there exists a non-zero element $x \in Z(L) \setminus I$. Then for each $i \in I$, there exists an element $l \in L$ such that $i = [x, l] = 0$ and so $I = 0$. This contradiction gives the result. □

Definition 2.4. If (L, L^2) is a Camina pair of a Lie algebra L, then L is called a Camina Lie algebra.

Clearly, All Heisenberg Lie algebras with finite or infinite dimensions are Camina.

Remark 2.5. A Lie algebra L is called Heisenberg, when $L^2 = Z(L)$ and $dimL^2 = 1$. All Heisenberg Lie algebras of finite dimensions have odd number of basis $\{x_1, \cdots, x_{2m}, x\}$ say, and the only non-zero multiplication between basis elements are $[x_{2i-1}, x_{2i}] = x$, for $i = 1, \cdots, m$. Similarly, any Heisenberg Lie algebra of infinite dimension with basis $\{x, x_i\}$, is defined by the relations $[x_{2i-1}, x_{2i}] = x$, for all $i = 1, 2, \cdots$. Clearly, any Heisenberg Lie algebra is nilpotent. However, it is not true that all Camina Lie algebras are nilpotent. For example, any 2-dimensional non-abelian Lie algebra is Camina Lie algebra, which is not nilpotent.

Lemma 2.6. *Let L be a Camina Lie algebra and I be an ideal of L contained in L^2. Then L/I is a Camina Lie algebra.*

Proof. Clearly,

$$(\frac{L}{I})^2 = [\frac{L}{I}, \frac{L}{I}] = \frac{L^2 + I}{I} = \frac{L^2}{I},$$

as $I \subseteq L^2$. Let $x + I \in \frac{L}{I} \setminus \frac{L^2}{I}$, then for every $l' \in L^2$, there exists $l \in L$ such that $[x, l] = l'$, as L is a Camina Lie algebra. It is clear that $l' + I = [x, l] + I = [x + I, l + I]$. Thus for every $x + I \in \frac{L}{I} \setminus \frac{L^2}{I}$ and $l' + I \in \frac{L'}{I}$, there exists $l + I \in \frac{L}{I}$ such that $[x + I, l + I] = l' + I$. Hence L/I is a Camina Lie algebra. □

Using the above lemma, we have the following

Corollary 2.7. *If L is a Camina Lie algebra, then $L/Z(L)$ is also Camina Lie algebra.*

Lemma 2.8. *Let (L, I) be a Camina pair of Lie algebras and J be an ideal of L contained in I, then $(L/J, I/J)$ is also a Camina pair.*

Proof. We need to show that for each $x + J \in \frac{L}{J} \setminus \frac{I}{J}$ and $i + J \in \frac{I}{J}$, there exists $l + J \in \frac{L}{J}$ such that $[x + J, l + J] = i + J$. As (L, I) is Camina pair, then for any $x + J \in \frac{L}{J} \setminus \frac{I}{J}$ and $i \in I$, there exists $l \in L$ with $[x, l] = i$. Obviously, $i + J = [x, l] + J = [x + J, l + J]$ and this gives the result. □

Theorem 2.9. *Let L be a non-abelian Camina Lie algebra with finite $dim(L/Z(L))$. Then $dim(L)$ is also finite.*

Proof. Assume $dim(L/Z(L)) = n$, then a well known theorem implies that $dim(L^2) \leqslant \frac{1}{2}n(n-1)$. On the other hand, $Z(L) \subseteq L^2$, as L is Camina Lie algebra. Thus $Z(L)$ is of finite dimension and so $dim(L)$ is also finite. □

Let L be a Lie algebra, the lower central series of L is defined as follows:

$$L = L^1 \supseteq L^2 \supseteq \cdots \supseteq L^n \supseteq \cdots ,$$

where $L^2 = L'$ is the derived algebra of L and $L^n = [L^{n-1}, L]$. Also, the upper central series of L is defined as

$$0 = Z_0(L) \subseteq Z_1(L) \subseteq \cdots \subseteq Z_n(L) \subseteq \cdots ,$$

where $Z_1(L) = Z(L)$ is the centre of L and $Z_{n+1}(L)/Z_n(L) = Z(L/Z_n(L))$. A Lie algebra L is nilpotent if there exists a non-negative integer i such that $L^{i+1} = 0$ (or $Z_i(L) = L$). The smallest integer i, for which $L^{i+1} = 0$ (or $Z_i(L) = L$) is called the *nil-index* of L. Clearly, the Lie algebras with nil-index 1 are abelian.

Theorem 2.10. *Let L be a nilpotent Lie algebra with nil-index i. If (L, I) is Camina pair, then $I = L^r$ and $I = Z_{i-r+1}(L)$, for some $1 < r \leqslant i$.*

Proof. Clearly, $Z(L) \subseteq I$ when (L, I) is a Camina pair. Now, we show that $I = Z_{i-r+1}(L)$ for some $1 < r \leqslant i$. If $Z(L) = I$, then the result is obtained. Now assume that $Z(L) \subsetneqq I$, then Lemma 2.6 implies that $(L/Z(L), I/Z(L))$ is a Camina pair. Therefore $Z(L/Z(L)) \subseteq I/Z(L)$ and hence $Z_2(L)/Z(L) \subseteq I/Z(L)$. Continuing the same trend, we have $I = Z_{i-r+1}(L)$, for some $1 < r \leqslant i$. Using induction on r, we obtain $L^r \subseteq Z_{i-r+1}(L) = I$. Now, if $L^r \subsetneqq Z_{i-r+1}(L) = I$, then $(L/L^r, I/L^r)$ is Camina pair and so $L^{r-1}/L^r \subseteq Z(L/L^r) \subseteq I/L^r$. Hence, $L^{r-1} \subseteq I = Z_{i-r+1}(L)$ implies that $L^i \subseteq Z_0(L) = 0$, which is a contradiction. Therefore $L^r = Z_{i-r+1}(L) = I$. □

Corollary 2.11. (i) *Let L be a nilpotent Lie algebra with nil-index i. If $(L, Z(L))$ is a Camina pair, then $L^i = Z(L)$.*
(ii) *Let L be a nilpotent Lie algebra with nil-index 2 and $dim L^2 = 1$. Then (L, I) is Camina pair if and only if $I = L^2$ and L is a Heisenberg Lie algebra.*

Proof. (i) The result follows by Theorem 2.8.

(ii) Let L be a nilpotent Lie algebra with nil-index 2, $dim L^2 = 1$ and (L, I) be a Camina pair. Then Theorem 2.8 implies that $I = L^2 = Z(L)$ and hence L is Heisenberg Lie algebra. The converse is obvious. □

Let L be a nilpotent Lie algebra with nil-index 2. Theorem 2.8 implies that, if L is a Camina Lie algebra, then $Z(L) = L^2$. In general, the converse of this statement is not true. For example

$$L_{5,8} = \langle x_1, \cdots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5 \rangle$$

is not Camina Lie algebra, while $Z(L) = L^2$ (see [8] for more details).

Now, we state and prove our final result of this section, as follows:

Theorem 2.12. *Every Camina Lie algebra is soluble.*

Proof. Let (L, L^2) be a Camina pair, then for all $x \in L \setminus L^2$ we have

$$L^2 \subseteq \{[x, l] : l \in L\}.$$

Hence for any $y \in L^2$, there exists $l_y \in L$ such that $y = [x, l_y]$. Now, for every $x_0 \in L$ and $y \in L^2$ the property of Killing form implies that

$$\begin{aligned} K(x_0, y) = K(x_0, [x, l_y]) &= K([x_0, x], l_y) \\ &= K([x, l_1], l_y), \ (since\ [x_0, x] \in L^2) \\ &= K(x, [l_1, l_y]), \ (since\ [l_1, l_y] \in L^2) \\ &= K(x, [x, l_2]) \\ &= K([x, x], l_2) = 0. \end{aligned}$$

Hence $K(L, L^2) = 0$ and Cartan's solubility criterion gives the result. □

3. Lie algebras with 2-subideal subalgebras

Clearly, subnormality property in groups is a natural generalization of normality. It receives no attention from group theorists until 1939, when Wielandt's fundamental paper [19] has been appeared.

Let K be a subalgebra of a Lie algebra L. We call K is n-subideal of L and denoted by $K \lhd_n L$, if there exist distinct subalgebras $K_1, K_2, \cdots, K_n$ such that

$$K \lhd K_1 \lhd K_2 \lhd \cdots \lhd K_n = L,$$

for some $n \in \mathbb{N}$.

In this section, we introduce a new notion of n-Baer Lie algebras and it is shown that some of the known results of n-Baer groups can be proved in n-Baer Lie algebras. We remind that G is called *n-Baer group*, if all of its cyclic subgroups are n-subnormal.

A natural set up for Lie algebras is as follows.

Definition 3.1. A Lie algebra L is called n-Baer Lie algebra, if all of its one dimensional subalgebras are n-subideal.

Clearly, in a nilpotent group of class n, all subgroups are n-subnormal. Conversely, by a well-known result of Roseblade [18], a group with all subgroups n-subnormal is nilpotent of class bounded by a function of n.

The next lemma describes the closure properties of the class of n-Baer Lie algebras.

Lemma 3.2. *Every subalgebra K of n-Baer Lie algebra L is at most n-Baer.*

Proof. Assume that K is a subalgebra of an n-Baer Lie algebra L. Then for every one dimensional subalgebra H of L which is contained in K, there exist distinct subalgebras $L_1, L_2, \cdots, L_n$ such that

$$H \lhd L_1 \lhd L_2 \lhd \cdots \lhd L_n = L.$$

Clearly, the following series is n-subideal series for K and so K is n-Baer Lie algebra.

$$H = H \cap K \lhd L_1 \cap K \lhd L_2 \cap K \lhd \cdots \lhd L \cap K = K.$$

□

Now, we prove some structural results for 1 and 2-Baer Lie algebras.

Definition 3.3. A Lie algebra L is called 1-Baer or *Dedekind Lie algebra* if all of its one dimensional subalgebras are ideal in L.

We remind that a group in which all of its subgroups are normal, called a *Dedekind group*. Such finite groups were classified by Dedekind in 1897 [10], and the infinite case by Baer in [1]. Dedekind groups are either abelian or the direct product of the Quaternion group of order 8 by a periodic abelian group with no elements of order 4. Clearly, every abelian Lie algebra is Dedekind.

Let L be a Lie algebra over a field F of characteristic zero and let $D(L)$ be the derivation algebra of L, which is the Lie algebra of all derivations of L.

Now, using the above discussion we give an important property of Dedekind Lie algebras.

Proposition 3.4. *Let L be Dedekind Lie algebra, then L is nilpotent of class at most* 2*.*

Proof. By the definition, any one dimensional subalgebra K of L is an ideal of L. Hence, each element of L induces an inner derivation of K. So there exists a homomorphism from L into $\mathrm{Inn}(K)$. The kernel of this homomorphism, which contains L^2, is the centralizer of k in L, for any element $k \in K$. So $[L^2, k] = 0$, which gives the result. □

The following interesting lemma is a useful property for 2-Baer Lie algebras.

Lemma 3.5. *Let H be a one dimensional subalgebra of a Lie algebra L. Then L is 2-Baer Lie algebra if and only if $[L, h, h] \subseteq H$, for all non-zero elements h of H.*

Proof. Assume that L is 2-Baer Lie algebra, then there exists some subalgebra K of L such that $H \lhd K \lhd L$ is 2-subideal series for H. Thus $[K, h] \subseteq H$ and $[L, k] \subseteq K$, for all $h \in H$ and $k \in K$. Hence $[L, h, h] \subseteq H$.

Conversely, assume $[L, h, h] \subseteq H$, for every non-zero element h in H. Then H is an ideal of $[L, h]$. On the other hand, $H \lhd \langle [l, h] \rangle \lhd L$, where $\langle [l, h] \rangle$ is an ideal of L, and so L is 2-Baer Lie algebra. □

Considering the discussion in the previous part and Lemma 3.5, we have the following result.

Theorem 3.6. *Every 2-Baer Lie algebra is 3-Engel.*

Proof. Assume L is 2-Baer Lie algebra, then Lemma 3.5 implies that

$$[L, x, x, x] = 0,$$

for all $x \in L$. Hence L is 3-Engel Lie algebra and $(ad_x)^3 = 0$. □

Corollary 3.7. *If L is a finite dimensional 2-Baer Lie algebra, then L is nilpotent.*

We denote the class of all 2-Baer Lie algebras, the class of Lie algebras in which every abelian subalgebra is 2-subideal, and the class of all Lie algebras in which every subalgebra is 2-subideal by $\mathcal{L}_B$, $\mathcal{L}_A$ and $\mathcal{L}_S$, respectively. It is obvious that $\mathcal{L}_S \subseteq \mathcal{L}_A \subseteq \mathcal{L}_B$.

In the rest of this section, we show that for 2-dimensional Lie algebras the properties $\mathcal{L}_B$, $\mathcal{L}_A$ and $\mathcal{L}_S$ are equivalent.

The next result and Lemma 3.5, play an important role in proving our main theorem of this section.

Proposition 3.8. *Let $\mathcal{C}$ be a class of Lie algebras, which is closed under forming subalgebras. If $K \in \mathcal{C}$ is a subalgebra of a Lie algebra L, then K is 2-subideal in L if and only if $[L, x, y] \subseteq \langle x, y \rangle$, for all $x, y \in K$.*

Proof. Assume that every subalgebra of L in the class $\mathcal{C}$ is 2-subideal. Then for all $x, y \in K$, there exists an ideal I of L such that $\langle x, y \rangle \lhd I \lhd L$. Clearly $[l, x] \in I$, for all $l \in L$, and $[[l, x], y] \in \langle x, y \rangle$, as $\langle x, y \rangle$ is ideal of I. Hence $[L, x, y] \subseteq \langle x, y \rangle$, which proves the necessary part of the result.

Conversely, suppose $[L, x, y] \subseteq \langle x, y \rangle$, for all $x, y \in K$. In order to show that every subalgebra K of L, which is in $\mathcal{C}$, is 2-subideal, we must show that $K \lhd [L, K]$, which is obvious by the assumption. On the other hand, consider the ideal generated by $[l, k]$, and so the following 2-subideal series completes the proof

$$K \lhd \langle [l, k] \rangle \lhd L.$$

□

The following corollary is an immediate consequence of the above proposition.

Corollary 3.9. *Let L be a Lie algebra. Then for all $x, y, l \in L$,*
(i) $L \in \mathcal{L}_S$ if and only if $[l, x, y] \subseteq \langle x, y\rangle$;
(ii) $L \in \mathcal{L}_A$ if and only if $[l, x, y] \subseteq \langle x, y\rangle$, with $[x, y] = 0$.

Proof. Both parts follow easily using Proposition 3.8, by taking $\mathcal{C}$ to be the class of all Lie algebras, and the class of all abelian Lie algebras, respectively. □

Recall that $\mathcal{L}_S \subseteq \mathcal{L}_A \subseteq \mathcal{L}_B$. The following main theorem, gives the exact relations among $\mathcal{L}_S$, $\mathcal{L}_A$ and $\mathcal{L}_B$.

Theorem 3.10. *Let L be a Lie algebra over a field F of characteristic $\neq 2$. Then $L \in \mathcal{L}_B$ is equivalent to $L \in \mathcal{L}_A$.*

Proof. Clearly $\mathcal{L}_A$ is contained in $\mathcal{L}_B$. Assume that L is a 2-Baer Lie algebra. Then by Lemma 3.5, for every one dimensional subalgebra H of L and any non-zero element $h \in H$, we have $[L, h, h] \subseteq H$. In view of Corollary 3.9 (ii), it suffices to prove that for any 2-dimensional abelian subalgebra $K = \langle x, y\rangle$ of L, $[l, x, y] \in K$, for all $l \in L$. On the other hand, the Jacobi identity yields

$$[[l, x], y] + [[x, y], l] + [[y, l], x] = 0.$$

Now, the property $[x, y] = 0$ implies that $[l, x, y] = [l, y, x]$. By Lemma 3.5,

$$[l, x + y, x + y] = [l, x, x] + [l, x, y] + [l, y, x] + [l, y, y],$$

and hence, we obtain

$$2[l, x, y] = [l, x + y, x + y] - [l, x, x] - [l, y, y] \in \langle x, y\rangle = K.$$

Therefore the assumption implies that $[l, x, y] \in K$ and so $L \in \mathcal{L}_A$. □

4. Generalized 2-Baer Lie algebras

In this final section, we concentrate on the generalized version of n-Baer Lie algebras, specially 2-Baer Lie algebras, which is motivated by the works of D. Cappitt, L.C. Kappe and Tortora in group theory (see [4] and [11] for more details).

For any Lie algebra L, let $T_n(L) = \langle x \in L : \langle x\rangle \ntrianglelefteq_n L\rangle$. Clearly, $T_n(L)$ is trivial, if all of one dimensional subalgebras of L are n-subideal, i.e. L is n-Baer Lie algebra.

Definition 4.1. If L is a Lie algebra with $T_n(L) \neq L$, then L is called a generalized n-Baer Lie algebra. In addition, if $T_n(L)$ is non-trivial, then L is called a generalized T_n-Lie algebra.

The class of generalized n-Baer and T_n-Lie algebras will be denoted by $\mathcal{GB}_n$ and $\mathcal{GT}_n$, respectively.

Here, we provide some structural results for generalized 2-Baer Lie algebras.

Lemma 4.2. *Let $L \in \mathcal{GB}_2$.*
(i) If $x \in L \setminus T_2(L)$ and K is a subalgebra of L containing x, then $K \in \mathcal{GB}_2$.
(ii) If I is an ideal of L contained in $T_2(L)$, then $L/I \in \mathcal{GB}_2$.

Proof. (i) Let $L \in \mathcal{GB}_2$, then for the one dimensional subalgebra $\langle x\rangle$ of L, there exists an ideal I such that $\langle x\rangle \lhd I \lhd L$, where $x \in L \setminus T_2(L)$. Thus

$$\langle x\rangle = \langle x\rangle \cap K \lhd I \cap K \lhd L \cap K = K,$$

which gives the result.

(ii) Clearly, $T_2(L/I) \subseteq T_2(L)/I$. Assume that $T_2(L/I) = L/I$, then we have

$$\frac{T_2(L)}{I} \subseteq \frac{L}{I} = T_2(\frac{L}{I}) \subseteq \frac{T_2(L)}{I},$$

and so $L = T_2(L)$, which is a contradiction. Therefore $T_2(L/I) \neq L/I$, and hence $L/I \in \mathcal{GB}_2$. □

Theorem 4.3. *Let $L \in \mathcal{GB}_2$ and $x \in L \setminus T_2(L)$. Then x is a left 3-Engel element.*

Proof. Let $x \in L\setminus T_2(L)$, then there exists an ideal I in L such that $\langle x\rangle \trianglelefteq I \trianglelefteq L$. Therefore $[I, x] \subseteq \langle x\rangle$ and $[L, i] \subseteq I$ for every $i \in I$, which implies that $[L, i, x] \subseteq \langle x\rangle$ and so $[L, x, x, x] = 0$. □

In the following we give couple of examples of 2-Baer and generalized 2-Baer Lie algebras.

Example 4.4. (i) Let $L = L_{5,5} = \langle x_1, \cdots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5\rangle$ and $\langle x_1\rangle, \langle x_2\rangle, \langle x_3\rangle$ and $\langle x_4\rangle$ are one dimensional subalgebras of L, then we have the following series

$$\langle x_1\rangle \lhd \langle x_4, x_5\rangle \lhd L;$$
$$\langle x_2\rangle \lhd \langle x_3, x_5\rangle \lhd L;$$
$$\langle x_3\rangle \lhd \langle x_2, x_3, x_5\rangle \lhd L;$$
$$\langle x_4\rangle \lhd \langle x_1, x_3, x_5\rangle \lhd L.$$

Note that $\langle x_5\rangle$ is an ideal of L and $\langle x_5\rangle \lhd \langle x_3, x_5\rangle \lhd L$, so L is 2-Baer Lie algebra.

(ii) Let $L = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_2, [x_1, x_4] = x_4\rangle$. Then

$$\langle x_2\rangle \lhd \langle x_2, x_3\rangle \lhd L;$$
$$\langle x_3\rangle \lhd \langle x_2, x_3, x_4\rangle \lhd L,$$

imply that $x_2, x_3 \in T_2(L)$. Also $\langle x_4\rangle \lhd L$ and $x_1 \notin T_2(L)$. Hence, L is generalized 2-Baer Lie algebra.

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