Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

# EXPLORING THE PROPERTIES OF THE ZERO-DIVISOR GRAPH OF DIRECT PRODUCT OF $*$-RINGS

M. Nazim ✉, N. U. Rehman, and S. A. Mir

*Special Issue: First Joint IIIMT-Algebra Forum Conference 2023*
Article type: Research Article
(Received: 01 May 2024, Received in revised form 01 September 2024)
(Accepted: 01 November 2024, Published Online: 07 November 2024)

Abstract. In this paper, we delve into the study of zero-divisor graphs in rings equipped with an involution. Specifically, we focus on abelian Rickart $*$-rings. Our investigation revolves around characterizing the diameter of a zero-divisor graph in the context of the direct product $\mathcal{S}_1 \oplus \mathcal{S}_2$, in relation to the diameters observed in the zero-divisor graphs of the constituent $*$-rings $\mathcal{S}_1$ and $\mathcal{S}_2$.

*Keywords*: $*$-ring, Rickart $*$-ring, Zero-divisor graph.
*2020 MSC*: 16W10, 05C25.

## 1. Introduction

In 1988, Beck [6] embarked on a pioneering investigation into the intriguing idea of coloring a commutative ring $R$ by constructing its corresponding zero-divisor graph. This graph, a simple graph, assigns vertices to the elements of the ring $R$, where two distinct elements $x$ and $y$ are linked if and only if their product $xy$ equals zero. Beck's focus primarily centered on establishing connections between the clique number and the chromatic number of this graph. Following Beck's groundbreaking work, numerous scholars have been inspired to delve deeper into the intricate relationship between commutative rings and their associated zero-divisor graphs. However, a significant advancement came in 1999 when Anderson and Livingston [3] introduced a modification by considering the zero-divisor graph whose vertices exclusively consisting of the non-zero zero-divisors of the commutative ring. Furthermore, the exploration of the zero-divisor graph of a commutative ring has been a subject of extensive inquiry by several prominent researchers, including [1–4, 9–13]. Expanding beyond the realm of commutative rings, Redmond [14] in 2002 introduced and scrutinized the concept of the zero-divisor graph for non-commutative rings. Additionally, DeMeyer and Schneider extended this concept to semigroups in their work [8].

Consider a graph $G$ with vertex set $V(G)$. The distance between any two vertices $u$ and $v$ within $G$, denoted as $d(u, v)$, represents the shortest path

---

✉ mnazim1882@gmail.com, ORCID: 0000-0001-8817-4336
https://doi.org/10.22103/jmmr.2024.23353.1639

Publisher: Shahid Bahonar University of Kerman
How to cite: M. Nazim, N. U. Rehman, S. A. Mir, *Exploring the properties of the zero-divisor graph of direct product of $*$-rings*, J. Mahani Math. Res. 2024; 13(5): 11-20.

length from $u$ to $v$. If no such path exists, $d(u, v)$ is assigned $\infty$. The diameter of $G$, denoted $diam(G)$, is the maximum distance observed among all pairs of vertices: $diam(G) = \sup\{d(u, v) : u, v \in V(G)\}$. A cycle in $G$ refers to a closed path, while the girth of $G$, denoted $gr(G)$, signifies the length of the shortest cycle within $G$. If $G$ lacks cycles entirely, $gr(G)$ is defined as $\infty$.

A graph earns the title of a *complete graph* when every vertex is directly connected to all others. A complete graph with $n$ vertices is denoted by $K^n$. On the other hand, if a graph $G$ can partition its vertex set into two distinct, non-overlapping subsets $U_1$ and $U_2$ in such a manner that vertices $u$ and $v$ are adjacent if and only if $u \in U_1$ and $v \in U_2$, then $G$ is identified as a *complete bipartite graph*. Specifically, a complete bipartite graph showcases disjoint vertex sets, with sizes $m$ and $n$ respectively, denoted as $K^{m,\ n}$. In cases where one or both of the disjoint vertex sets are infinite, we denote the graph as $K^{n,\ \infty}$ or $K^{\infty,\ \infty}$. Furthermore, a complete bipartite graph following the pattern $K^{1,n}$ is often referred to as a *star graph*.

*This paper centers on exploring the fundamental properties exhibited by zero-divisor graphs associated with rings featuring involution. An associative ring* $\mathcal{S}$ *is endowed with an involution, denoted by '*$*$*,' if it satisfies the conditions* $(x + y)^* = x^* + y^*$, $(xy)^* = y^*x^*$ *and* $(x^*)^* = x$ *for all* $x, y \in \mathcal{S}$. *Such a ring, equipped with the involution operation, is termed a* $*$*-ring. Notably, the identity mapping qualifies as an involution if and only if the ring is commutative. In the context of* $*$*-rings, an element* $e$ *is deemed a projection if it satisfies* $e = e^2$ *and* $e = e^*$. *For a nonempty subset* $B$ *of* $\mathcal{S}$, *the right annihilator of* $B$ *in* $\mathcal{S}$, *denoted* $r(B)$, *comprises elements* $a \in \mathcal{S}$ *such that* $ba = 0$ *for all* $b \in B$. *A key concept in the study of* $*$*-rings is that of Rickart* $*$*-rings, where every element's right annihilator is generated, as a right ideal, by a projection in* $\mathcal{S}$. *It's noteworthy that every Rickart* $*$*-ring includes unity. For each element* $a$ *within a Rickart* $*$*-ring, there exists a unique projection* $e$ *satisfying* $ae = a$ *and* $ax = 0$ *if and only if* $ex = 0$. *This projection is referred to as the right projection of* $a$, *denoted by* $RP(a)$. *Specifically,* $r(\{a\}) = (1 - RP(a))\mathcal{S}$. *Analogously, the left annihilator* $l(\{a\})$ *and the left projection* $LP(a)$ *are defined for each element* $a$ *within a Rickart* $*$*-ring* $\mathcal{S}$. *Moreover, the set of projections* $P(\mathcal{S})$ *within a Rickart* $*$*-ring* $\mathcal{S}$ *forms a lattice, denoted by* $L(P(\mathcal{S}))$, *under the partial order '*$e \leq f$ *if and only if* $e = fe = ef$*'. Notably, the lattice operations are defined as* $e \vee f = f + RP(e(1 - f))$ *and* $e \wedge f = e - LP(e(1 - f))$. *Further insights into Rickart* $*$*-rings can be found in Berberian [7].*

*Patil and Waphare [5] introduced a novel concept in the realm of* $*$*-rings, defining the zero-divisor graph associated with such structures. Here is the essence of their construction: Consider a* $*$*-ring* $\mathcal{S}$. *They devised a graph that encapsulates the structure of* $\mathcal{S}$, *with vertices representing nonzero left zero-divisors, defined as* $\{x(\neq 0) \in \mathcal{S} : xy = 0$ *for some nonzero* $y \in \mathcal{S}\}$. *Notably, two distinct vertices* $x$ *and* $y$ *are connected in this graph if and only* $xy^* = 0$. *Termed as the zero-divisor graph of the* $*$*-ring* $\mathcal{S}$, *this graph is symbolized by* $\Gamma^*(\mathcal{S})$.

*In this paper, we have established a set of theorems that provide insights into the diameter of the zero-divisor graph associated with the direct product* $\mathcal{S}_1 \oplus \mathcal{S}_2$*, relative to the diameters of the zero-divisor graphs of* $\mathcal{S}_1$ *and* $\mathcal{S}_2$*. Here,* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *represent* $*$*-rings, while* $\mathcal{S}_1 \oplus \mathcal{S}_2$ *forms a* $*$*-ring with componentwise involution. Additionally, we have derived several properties concerning* $*$*-rings whose zero-divisor graphs exhibit diameter-two characteristics.*

## 2. ***Preliminaries***

*In [5], Patil and Waphare proposed an extension of the concept of zero-divisor graphs for* $*$*-rings, which generalizes the traditional notion established for commutative rings when the identity map is regarded as an involution.*

**Definition 2.1.** Let $\mathcal{S}$ be a $*$-ring. We associate a simple undirected graph $\Gamma^*(\mathcal{S})$ to $\mathcal{S}$ whose vertex set is $V^*(\Gamma^*(\mathcal{S}))$, where $V(\Gamma^*(\mathcal{S})) = \{x \in \mathcal{S} : xy = 0$ for some nonzero $y \in \mathcal{S}\}$ and $V^*(\Gamma^*(\mathcal{S})) = V(\Gamma^*(\mathcal{S})) \setminus \{0\}$ and two distinct vertices $x$ and $y$ are adjacent if and only if $xy^* = 0$.

*Before starting the characterization of the diameter of direct product of* $*$*-rings, we will collect some known results, which will be use in sequel. This theorem due to Patil and Waphare [5], which gives a sufficient condition for* $a^* \in V^*(\Gamma^*(\mathcal{S}))$ *whenever* $a \in V^*(\Gamma^*(\mathcal{S}))$*.*

**Theorem 2.2.** *[5, Theorem 2.1] Let* $\mathcal{S}$ *be a left Artinian* $*$*-ring with unity and* $a \in \mathcal{S}$*. Then* $a \in V^*(\Gamma^*(\mathcal{S}))$ *if and only if* $a^* \in V^*(\Gamma^*(\mathcal{S}))$*.*

*Now, we state a result which gives a characterization for* $\Gamma^*(\mathcal{S})$ *to be complete.*

**Theorem 2.3.** *[5, Theorem 2.2] Let* $\mathcal{S}$ *be a left Artinian* $*$*-ring with unity. Then* $\Gamma^*(\mathcal{S})$ *is complete if and only if either* $\mathcal{S} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *or* $V^*(\Gamma^*(\mathcal{S}))^2 = 0$*.*

**Theorem 2.4.** *[5, Proposition 3.3] Let* $\mathcal{S}$ *be an abelian Rickart* $*$*-ring. Then* $\Gamma^*(\mathcal{S})$ *is connected and* $diam(\Gamma^*(\mathcal{S})) \leq 3$*.*

*We begin with the following lemma.*

**Lemma 2.5.** *Let* $\mathcal{S}$ *be a left Artinian abelian Rickart* $*$*-ring and* $diam(\Gamma^*(\mathcal{S})) = 1$*. Then* $(\mathcal{S})^2 \neq 0$ *implies* $\mathcal{S} \neq V(\Gamma^*(\mathcal{S}))$*.*

*Proof.* Assume that $diam(\Gamma^*(\mathcal{S})) = 1$. Then $\Gamma^*(\mathcal{S})$ is complete and hence by Theorem 2.3, we get $\mathcal{S} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $V(\Gamma^*(\mathcal{S}))^2 = 0$. If $\mathcal{S} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, then $\mathcal{S} \neq V(\Gamma^*(\mathcal{S}))$. On the other hand, assume that $V(\Gamma^*(\mathcal{S}))^2 = 0$ and if$\mathcal{S} = V(\Gamma^*(\mathcal{S}))$, then $(\mathcal{S})^2 = 0$, a contradiction. □

*As a component of our examination into the direct products of* $*$*-rings, we found it beneficial to derive certain outcomes concerning diameter-two* $*$*-rings, similar to those derived by Anderson and Livingston [3] for diameter-one rings.*

**Lemma 2.6.** *Let $\mathcal{S}$ be a left Artinian abelian Rickart $*$-ring such that $diam(\Gamma^*(\mathcal{S})) = 2$ and $V(\Gamma^*(\mathcal{S}))$ is a subring in $\mathcal{S}$ (not necessarily proper) of $\mathcal{S}$. Then for all $a, b \in V(\Gamma^*(\mathcal{S}))$, there exists a nonzero $c$ such that $ac = bc = 0$.*

*Proof.* Let $a, b \in V(\Gamma^*(\mathcal{S}))$. If either $a = b = 0$ or $a = b$, the proof is trivial, as the condition $diam(\Gamma^*(\mathcal{S})) = 2$ guarantees the existence of the required element $c$. Hence, we consider the scenario where $a$ and $b$ are distinct and nonzero. If $a$ and $b$ are adjacent, meaning $ab^* \neq 0$, then by the property $diam(\Gamma^*(\mathcal{S})) = 2$, there must exist an element $c \in V^*(\Gamma^*(\mathcal{S}))$ such that $ac^* = bc^* = 0$. Thus, we assume that $a$ and $b$ are nonadjacent that is, $ab^* = 0$. Consider $a+b$ and observe that $a+b \neq a$ and $a+b \neq b$. If $a+b = 0$ then $a = -b$ and hence $bb^* = 0$. Thus, $c = b^*$ suffices. Therefore, we assume $a + b \neq 0$. Since $V(\Gamma^*(\mathcal{S}))$ is a subring of $\mathcal{S}$, we have $a + b \in V^*(\Gamma^*(\mathcal{S}))$. Also, we assume that $aa^* \neq 0$ and $bb^* \neq 0$ else choose $c = a^*$ or $c = b^*$, respectively. Let $P = \{a' \in V^*(\Gamma^*(\mathcal{S})) : aa' = 0\}$ and $Q = \{b' \in V^*(\Gamma^*(\mathcal{S})) : bb' = 0\}$. Observe that $b^* \in P$ and $a^* \in Q$, hence $P$ and $Q$ are nonempty. If $P \cap Q \neq \emptyset$, then choose $c \in P \cap Q$. Assume $P \cap Q = \emptyset$ and consider $a + b$. Since $aa^* \neq 0$ we have $(a + b)^* \notin P$ and similarly, $(a + b)^* \notin Q$. Since $diam(\Gamma^*(\mathcal{S})) = 2$, there exists $d^* \in P$ such that $a - d - (a + b)$ is a path in $\Gamma^*(\mathcal{S})$. Then $0 = (a + b)d^* = ad^* + bd^* = cd^*$ and so $d^* \in Q$, a contradiction. □

*This exploration into diameter-two graphs yields intriguing results that stand on their own merit. Many of these results revolve around the characterization of $\Gamma^*(\mathcal{S})$ as either a complete bipartite graph or one that closely resembles it.*

**Lemma 2.7.** *Let $\mathcal{S}$ be an abelian Rickart $*$-ring and $\Gamma^*(\mathcal{S}))$ be the zero-divisor graph of $\mathcal{S}$. If $\Gamma^*(\mathcal{S}))$ does not manifest as a complete bipartite graph, yet a complete bipartite subgraph can be derived by removing certain edges from $\Gamma^*(\mathcal{S})$, then $V(\Gamma^*(\mathcal{S}))$ is a subring of $\mathcal{S}$.*

*Proof.* Let $x, y \in V(\Gamma^*(\mathcal{S}))$. Clearly $xy \in V(\Gamma^*(\mathcal{S}))$ and $-x \in V(\Gamma^*(\mathcal{S}))$. Now, we have to show that $x + y \in V(\Gamma^*(\mathcal{S}))$. As edges can be selectively removed from $\Gamma^*(A)$ to create a complete bipartite graph, we can infer the existence of nonempty sets $U$ and $U'$ satisfying the conditions $U \cup U' = V(\Gamma^*(\mathcal{S}))$, $U \cap U' = \emptyset$, and $uv^* = 0$ for all $u \in U, v \in U'$. If $x \in U$ and $y \in U$, then take $v \in U'$. We have $(x + y)v^* = xv^* + yv^* = 0$. Thus, $x + y \in V(\Gamma^*(\mathcal{S}))$. Likewise, for $x \in U'$ and $y \in U'$. Therefore, we may assume, without loss of generality, that $x \in U$ and $y \in U'$. Since $\Gamma^*(\mathcal{S})$ is not complete bipartite, there exists an edge that does not connect a vertex of $U$ to a vertex of $U'$. Let it lie between $u_1, u_2 \in U$. Then $u_1(y + u_2)^* = u_1y^* + u_1u_2^* = 0$, so either $y + u_2 = 0, y + u_2 \in U'$, or $y + u_2 \in U$. If $y + u_2 = 0$, then $0 = y0 = y(y+u_2)^* = yy^* + yu_2^* = yy^*$. So $(x+y)y^* = xy^* + yy^* = 0+0 = 0$ and hence $x + y \in V(\Gamma^*(\mathcal{S}))$. If $y + u_2 \in U'$, then $0 = x(y + u_2)^* = xy^* + xu_2^* = xu_2^*$, and hence $(x + y)u_2^* = xu_2^* + yu_2^* = 0 + 0 = 0$, that is, $x + y \in V(\Gamma^*(\mathcal{S}))$. If $y + u_2 \in U$, then for any $v \in U'$, we have $0 = (y + u_2)v^* = yv^* + u_2v^* = yv^*$.

Thus, $(x+y)v^* = xv^* + yv^* = 0 + 0 = 0$. Therefore, $x + y \in V(\Gamma^*(\mathcal{S}))$. This completes the proof of the lemma. □

*Utilizing the lemmas outlined above, we establish the following theorem, which closely resembles Theorem 2.8 presented in [3] for zero-divisor graphs with a diameter of two. This theorem serves as an analogue in the context of diameter-two zero-divisor graphs.*

**Theorem 2.8.** *Let $\mathcal{S}$ be a left-Artinian abelian Rickart $*$-ring. If $\Gamma^*(\mathcal{S})$ is not complete bipartite but has a complete bipartite subgraph induced by removing only edges from $\Gamma^*(\mathcal{S})$, then for all $a, b \in V(\Gamma^*(\mathcal{S}))$ there exists nonzero element $c$ in $\mathcal{S}$ such that $ac = bc = 0$.*

*Proof.* By Lemma 2.7, $V(\Gamma^*(\mathcal{S}))$ is a subring of $\mathcal{S}$. We have to show that $V(\Gamma^*(\mathcal{S})) = V(V(\Gamma^*(\mathcal{S})))$. To prove this, let $x \in V(\Gamma^*(\mathcal{S}))$. Then there exists a nonzero $y \in A$ such that $xy = 0$, that is, $y^*x^* = 0$, which shows that $y^* \in V(\Gamma^*(\mathcal{S}))$. Using Theorem 2.2, we have $y \in V(\Gamma^*(\mathcal{S}))$ and hence $V(\Gamma^*(\mathcal{S})) = V(V(\Gamma^*(\mathcal{S})))$. If $diam(\Gamma^*(V(\Gamma^*(\mathcal{S})))) = 1$ then by Lemma 2.5, $V(\Gamma^*(\mathcal{S}))^2 = 0$ and the result is trivial. If $diam(\Gamma^*(V(\Gamma^*(\mathcal{S})))) = 2$ then Lemma 2.6, yields the required result. □

## 3. *Direct Product of $*$-Rings*

*Now, we are in a position to classifying the diameters of zero-divisor graph of direct product of rings with involution.*

**Theorem 3.1.** *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two left Artinian abelian Rickart $*$-rings such that $diam(\Gamma^*(\mathcal{S}_1)) = diam(\Gamma^*(\mathcal{S}_2)) = 1$ and let $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$ with componentwise involution. Then:*

- *(i) $diam(\Gamma^*(\mathcal{S})) = 1$ if and only if $(\mathcal{S}_1)^2 = (\mathcal{S}_2)^2 = 0$.*
- *(ii) $diam(\Gamma^*(\mathcal{S})) = 2$ if and only if $(\mathcal{S}_1)^2 = 0$ and $(\mathcal{S}_2)^2 \neq 0$ or $(\mathcal{S}_1)^2 \neq 0$ and $(\mathcal{S}_2)^2 = 0$.*
- *(iii) $diam(\Gamma^*(\mathcal{S})) = 3$ if and only if $(\mathcal{S}_1)^2 \neq 0$ and $(\mathcal{S}_2)^2 \neq 0$.*

*Proof.* *(i)* ($\Leftarrow$) Suppose that $(\mathcal{S}_1)^2 = (\mathcal{S}_2)^2 = 0$. Then $x_1x_2 = 0$ for all $x_1, x_2 \in \mathcal{S}_1$. Likewise, $y_1y_2 = 0$ for all $y_1, y_2 \in \mathcal{S}_2$. Thus, we get $(x_1, y_1) \cdot (x_2, y_2)^* = (x_1x_2^*, y_1y_2^*) = (0, 0)$ for all $(x_1, y_1), (x_2, y_2) \in A$, and hence $diam(\Gamma^*(\mathcal{S})) = 1$.

($\Rightarrow$) Let $diam(\Gamma^*(\mathcal{S})) = 1$ and $(\mathcal{S}_1)^2 \neq 0$. Then there exist nonzero elements $x_1, x_2 \in \mathcal{S}_1$ such that $x_1x_2 \neq 0$. This shows that $(x_1, 0) \cdot (x_2^*, 0)^* = (x_1x_2, 0) \neq (0, 0)$ for $(x_1, 0), (x_2^*, 0) \in V^*(\Gamma^*(\mathcal{S}))$, and hence $diam(\Gamma^*(\mathcal{S})) > 1$, which is a contradiction.

*(ii)* ($\Leftarrow$) Suppose that $(\mathcal{S}_1)^2 = 0$ and $(\mathcal{S}_2)^2 \neq 0$ then by *(i)*, $diam(\Gamma^*(\mathcal{S})) > 1$. However, since $(\mathcal{S}_1)^2 = 0$, there must exists a nonzero $x \in \mathcal{S}_1$ such that $xy = 0$ for all $y \in \mathcal{S}_1$. Let $(x_1, y_1)$ and $(x_2, y_2)$ be any two distinct vertices of $\Gamma^*(\mathcal{S})$ which are non-adjacent. Observe that, $(x, 0) \neq (x_1, y_1)$ and $(x, 0) \neq (x_2, y_2)$ otherwise, $(x_1, y_1) \cdot (x_2, y_2)^* = (0, 0)$. Thus, we have $(x_1, y_1) - (x, 0) - (x_2, y_2)$ is a path in $\Gamma^*(\mathcal{S})$ and so $diam(\Gamma^*(\mathcal{S})) \leq 2$. Hence, $diam(\Gamma^*(\mathcal{S})) = 2$.

($\Rightarrow$) Let $(\mathcal{S}_1)^2 = (\mathcal{S}_2)^2 = 0$. Then, by $(i)$, we have $diam(\Gamma^*(\mathcal{S})) = 1$. Now, suppose that $(\mathcal{S}_1)^2 \neq 0$ and $(\mathcal{S}_2)^2 \neq 0$, but $diam(\Gamma^*(\mathcal{S})) = 2$. Therefore by Lemma 2.5, since $(\mathcal{S}_1)^2 \neq 0$, there must exists $u \in \mathcal{S}_1 \setminus V(\Gamma^*(\mathcal{S}_1))$. Likewise, since $(\mathcal{S}_2)^2 \neq 0$ there must exist $v \in \mathcal{S}_2 \setminus V(\Gamma^*(\mathcal{S}_2))$. Let $x \in V(\Gamma^*(\mathcal{S}_1))$, $y \in V(\Gamma^*(\mathcal{S}_2))$. Observe that, $(u, y^*)$ and $(x^*, v)$ are vertices of $\Gamma^*(\mathcal{S})$ such that $(u, y^*) \cdot (x^*, v)^* = (ux, v^*y^*) \neq (0, 0)$. Since $diam(\Gamma^*(\mathcal{S})) = 2$, there must be some $(a, b) \in V^*(\Gamma^*(\mathcal{S}))$ such that $(u, y^*) - (a, b) - (x^*, v)$ is a path in $\Gamma^*(\mathcal{S})$. Thus, $ua^* = x^*a^* = 0$, which gives $a = 0$, and $y^*b^* = vb^* = 0$, we have $b = 0$. Consequently, $(a, b) = (0, 0)$, leads to a contradiction. Hence, it must be the case that either $(\mathcal{S}_1)^2 = 0$ and $(\mathcal{S}_2)^2 \neq 0$ or $(\mathcal{S}_1)^2 \neq 0$ and $(\mathcal{S}_2)^2 = 0$.

$(iii)$ Result follows from $(i)$ and $(ii)$. □

**Theorem 3.2.** *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be left Artinian abelian Rickart $*$-rings such that $diam(\Gamma^*(\mathcal{S}_1)) = 1$, $diam(\Gamma^*(\mathcal{S}_2)) = 2$ and let $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$ with componentwise involution. Then:*

$(i)$ *$diam(\Gamma^*(\mathcal{S})) \neq 1$.*

$(ii)$ *$diam(\Gamma^*(\mathcal{S})) = 2$ if and only if $\mathcal{S}_1 = V(\Gamma^*(\mathcal{S}_1))$ or $\mathcal{S}_2 = V(\Gamma^*(\mathcal{S}_2))$.*

$(iii)$ *$diam(\Gamma^*(\mathcal{S})) = 3$ if and only if $\mathcal{S}_1 \neq V(\Gamma^*(\mathcal{S}_1))$ and $\mathcal{S}_2 \neq V(\Gamma^*(\mathcal{S}_2))$.*

*Proof.* $(i)$ Since $diam(\Gamma^*(\mathcal{S}_2)) = 2$, there exist $y_1, y_2 \in V^*(\Gamma^*(\mathcal{S}_2))$, $y_1 \neq y_2$, such that $y_1y_2^* \neq (0, 0)$. This means that $(0, y_1) \cdot (0, y_2)^* = (0, y_1y_2^*) \neq (0, 0)$ for $(0, y_1), (0, y_2) \in V^*(\Gamma^*(\mathcal{S}))$. Therefore, $diam(\Gamma^*(\mathcal{S})) > 1$.

$(ii)$ ($\Leftarrow$) Assuming $\mathcal{S}_1 = V(\Gamma^*(\mathcal{S}_1))$, let $x_1 \in V^*(\Gamma^*(\mathcal{S}_1))$. According to Lemma 2.5, $(x_1^*, 0)$ annihilates any elements of $V(\Gamma^*(\mathcal{S}))$, implying $diam(\Gamma^*(\mathcal{S})) \leq 2$. However, from $(i)$, we know $diam(\Gamma^*(\mathcal{S})) > 1$, necessitating $diam(\Gamma^*(\mathcal{S})) = 2$. Now, suppose $\mathcal{S}_2 = V(\Gamma^*(\mathcal{S}_2))$. If $(x_1, y_1)$ and $(x_2, y_2)$ are distinct elements of $V^*(\Gamma^*(\mathcal{S}))$ and $(x_1, y_1) \cdot (x_2, y_2)^* \neq (0, 0)$. According to Lemma 2.6, there exists a nonzero $y_3$ such that $y_1y_3 = y_2y_3 = 0$. Notably, $(0, y_3^*)$ cannot be equal to $(x_1, y_1)$ or $(x_2, y_2)$; otherwise, $(x_1, y_1) \cdot (x_2, y_2)^* = (0, 0)$. Thus, $(x_1, y_1) - (0, y_3^*) - (x_2, y_2)$ forms a path in $\Gamma^*(\mathcal{S})$, leading to $diam(\Gamma^*(\mathcal{S})) \leq 2$. Consequently, as per $(i)$, $diam(\Gamma^*(\mathcal{S})) = 2$.

($\Rightarrow$) Assume that $\mathcal{S}_1 \neq V(\Gamma^*(\mathcal{S}_1))$ and $\mathcal{S}_2 \neq V(\Gamma^*(\mathcal{S}_2))$ but $diam(\Gamma^*(\mathcal{S})) = 2$. Let $x \in V(\Gamma^*(\mathcal{S}_1))$, $y \in V(\Gamma^*(\mathcal{S}_2))$, $u \in \mathcal{S}_1 \setminus V(\Gamma^*(\mathcal{S}_1))$ and $v \in \mathcal{S}_2 \setminus V(\Gamma^*(\mathcal{S}_2))$. Since $diam(\Gamma^*(\mathcal{S})) = 2$, there must exist $(a, b) \in V^*(\Gamma^*(\mathcal{S}))$ such that $(x, v)(a, b)^* = (u, y) \cdot (a, b)^* = 0$. Then $xa^* = ua^* = 0$ so $a = 0$, and $vb^* = yb^* = 0$, so $b = 0$. Therefore, $(a, b) = (0, 0)$, which is not an element of $V^*(\Gamma^*(\mathcal{S}))$, leads to a contradiction.

$(iii)$ Proof follows from $(i)$ and $(ii)$. □

**Theorem 3.3.** *Let $\mathcal{S}_1$, $\mathcal{S}_2$ be left Artinian abelian Rickart $*$-rings such that $diam(\Gamma^*(\mathcal{S}_1)) = 1$, $diam(\Gamma^*(\mathcal{S}_2)) = 3$ and let $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$ with componentwise involution. Then:*

$(i)$ *$diam(\Gamma^*(\mathcal{S})) \neq 1$.*

$(ii)$ *$diam(\Gamma^*(\mathcal{S})) = 2$ if and only if $\mathcal{S}_1 = V(\Gamma^*(\mathcal{S}_1))$.*

$(iii)$ *$diam(\Gamma^*(\mathcal{S})) = 3$ if and only if $\mathcal{S}_1 \neq V(\Gamma^*(\mathcal{S}_1))$.*

*Proof.* $(i)$ Since $diam(\Gamma^*(\mathcal{S}_2)) = 3$, there must exist $y_1, y_2 \in V^*(\Gamma^*(\mathcal{S}_2))$, $y_1 \neq y_2$ with $y_1 y_2^* \neq 0$. This means that $(0, y_1) \cdot (0, y_2)^* = (0, y_1 y_2^*) \neq (0, 0)$, for $(0, y_1), (0, y_2) \in V^*(\Gamma^*(\mathcal{S}))$, it follows that $diam(\Gamma^*(\mathcal{S})) > 1$.

$(ii)$ $(\Leftarrow)$ Let $\mathcal{S}_1 = V(\Gamma^*(\mathcal{S}_1))$. Then by Lemma 2.5, $(\mathcal{S}_1)^2 = 0$ that is $x_1 x_2 = 0$ for all $x_1, x_2 \in \mathcal{S}_1$. Let $(x_1, y_2), (x_2, y_2) \in V^*(\Gamma^*(\mathcal{S}))$ such that $(x_1, y_1) \cdot (x_2, y_2)^* \neq (0, 0)$. Thus $y_1 y_2^* \neq 0$ and so, $y_1 \neq 0$ and $y_2 \neq 0$. It is clear that $(x_1, y_1) - (x_1, 0) - (x_2, y_2)$ is a path in $\Gamma^*(\mathcal{S})$, and hence by $(i)$, $diam(\Gamma^*(\mathcal{S})) = 2$.

$(\Rightarrow)$ Assume that $\mathcal{S}_1 \neq V(\Gamma^*(\mathcal{S}_1))$ but $diam(\Gamma^*(\mathcal{S})) = 2$. Let $u \in \mathcal{S}_1 \setminus V(\Gamma^*(\mathcal{S}_1))$. Since $diam(\Gamma^*(\mathcal{S}_2)) = 3$, there are distinct $y_1, y_2 \in V^*(\Gamma^*(\mathcal{S}_2))$, with $y_1 y_2^* \neq 0$ and there is no $y_3 \in V^*(\Gamma^*(\mathcal{S}_2))$, such that $y_1 y_3^* = y_2 y_3^* = 0$. Since $(u, y_1) \cdot (u, y_2)^* = (uu^*, y_1 y_2^*) \neq (0, 0)$, and $diam(\Gamma^*(\mathcal{S})) = 2$, there must be some $(a, b) \in V^*(\Gamma^*(\mathcal{S}))$ such that $(u, y_1) \cdot (a, b)^* = (u, y_2) \cdot (a, b)^* = 0$. Thus, $ua^* = 0$, so, it gives that $a = 0$. Hence, we must have $b \in V^*(\Gamma^*(\mathcal{S}_2))$ such that $y_1 b^* = y_2 b^* = 0$, giving $b = 0$, a contradiction.

$(iii)$ Proof follows from $(i)$ and $(ii)$.

□

**Theorem 3.4.** *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be left Artinian abelian Rickart $*$-rings such that $diam(\Gamma^*(\mathcal{S}_1)) = diam(\Gamma^*(\mathcal{S}_2)) = 2$ and let $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$ with componentwise involution. Then:*

- $(i)$ *$diam(\Gamma^*(\mathcal{S})) \neq 1$.*
- $(ii)$ *$diam(\Gamma^*(\mathcal{S})) = 2$ if and only if $\mathcal{S}_1 = V(\Gamma^*(\mathcal{S}_1))$ or $\mathcal{S}_2 = V(\Gamma^*(\mathcal{S}_2))$.*
- $(iii)$ *$diam(\Gamma^*(\mathcal{S})) = 3$ if and only if $\mathcal{S}_1 \neq V(\Gamma^*(\mathcal{S}_1))$ and $\mathcal{S}_2 \neq V(\Gamma^*(\mathcal{S}_2))$.*

*Proof.* $(i)$ Since $diam(\Gamma^*(\mathcal{S}_1)) = 2$, so there must exist $x_1, x_2 \in V^*(\Gamma^*(\mathcal{S}_1))$, $x_1 \neq x_2$ with $x_1 x_2^* \neq 0$. This means that $(x_1, 0) \cdot (x_2, 0)^* = (x_1 x_2^*, 0) \neq (0, 0)$, for $(x_1, 0), (x_2, 0) \in V^*(\Gamma^*(\mathcal{S}))$, and hence $diam(\Gamma^*(\mathcal{S})) > 1$.

$(ii)$ $(\Leftarrow)$ Without loss of generality, let $\mathcal{S}_1 = V(\Gamma^*(\mathcal{S}_1))$. Then by Lemma 2.6, for all $x_1, x_2 \in V(\Gamma^*(\mathcal{S}_1))$, there exists a nonzero $x_3 \in \mathcal{S}_1$ such that $x_1 x_3 = x_2 x_3 = 0$. So, for any $(x_1, y_1), (x_2, y_2) \in V^*(\Gamma^*(\mathcal{S}))$, there exists $(x_3^*, 0) \in V^*(\Gamma^*(\mathcal{S}))$ such that $(x_1, y_1) \cdot (x_3^*, 0)^* = (x_2, y_2) \cdot (x_3^*, 0)^* = (0, 0)$. If, without loss of generality, $(x_2, y_2) = (x_3^*, 0)$, then we have $(x_1, y_1) \cdot (x_2, y_2)^* = (0, 0)$. Thus, $diam(\Gamma^*(\mathcal{S})) \leq 2$ and hence by $(i)$, it must be that $diam(\Gamma^*(\mathcal{S})) = 2$. Similarly, if $\mathcal{S}_2 = V(\Gamma^*(\mathcal{S}_2))$, then $diam(\Gamma^*(\mathcal{S})) \leq 2$ and hence again by $(i)$ $diam(\Gamma^*(\mathcal{S})) = 2$.

$(\Leftarrow)$ Assume that $\mathcal{S}_1 \neq V(\Gamma^*(\mathcal{S}_1))$, $\mathcal{S}_2 \neq V(\Gamma^*(\mathcal{S}_2))$ and $diam(\Gamma^*(\mathcal{S})) = 2$. Let $x \in V(\Gamma^*(\mathcal{S}_1))$, $y \in V(\Gamma^*(\mathcal{S}_2))$. Then, there must exist $m \in \mathcal{S}_1 \setminus V(\Gamma^*(\mathcal{S}_1))$ and $n \in \mathcal{S}_2 \setminus V(\Gamma^*(\mathcal{S}_2))$. Consider the vertices $(x, n)$, $(m, y)$ of $\Gamma^*(\mathcal{S})$. Since $(x, n) \cdot (m, y)^* = (xm^*, ny^*) \neq (0, 0)$ and $diam(\Gamma^*(\mathcal{S})) = 2$, there exists $(a, b) \in V^*(\Gamma^*(\mathcal{S}))$ such that $(x, n) \cdot (a, b)^* = (m, y) \cdot (a, b)^* = (0, 0)$. Then, $ma^* = nb^* = 0$, this implies that $(a, b) = (0, 0)$, a contradiction.

$(iii)$ The result follows from $(i)$ and $(ii)$. □

**Theorem 3.5.** *Let $\mathcal{S}_1, \mathcal{S}_2$ be left Artinian abelian Rickart $*$-rings such that $diam(\Gamma^*(\mathcal{S}_1)) = 2$, $diam(\Gamma^*(\mathcal{S}_2)) = 3$ and let $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$ with componentwise involution. Then:*

($i$) $diam(\Gamma^*(\mathcal{S})) \neq 1$.
($ii$) $diam(\Gamma^*(\mathcal{S})) = 2$ *if and only if* $\mathcal{S}_1 = V(\Gamma^*(\mathcal{S}_1))$.
($iii$) $diam(\Gamma^*(\mathcal{S})) = 3$ *if and only if* $\mathcal{S}_1 \neq V(\Gamma^*(\mathcal{S}_1))$.

*Proof.* ($i$) The proof follows a similar structure to that of Theorem 3.4 ($i$)

($ii$) ($\Leftarrow$) Equivalent to the proof of Theorem 3.4 ($ii$)

($\Rightarrow$) Assume $\mathcal{S}_1 \neq V(\Gamma^*(\mathcal{S}))$, but $diam(\Gamma^*(\mathcal{S})) = 2$. Let $u \in \mathcal{S}_1 \setminus V(\Gamma^*(\mathcal{S}_1))$. Since $diam(\Gamma^*(\mathcal{S}_2)) = 3$, there must exist $y_1, y_2 \in V^*(\Gamma^*(\mathcal{S}_2))$, $y_1 \neq y_2$, $y_1 y_2^* \neq 0$ such that there is no $y_3 \in V^*(\Gamma^*(\mathcal{S}_2))$ with $y_1 y_3^* = y_2 y_3^* = 0$. Consider elements $(u, y_1)$ and $(u, y_2)$ of $V^*(\Gamma^*(\mathcal{S}))$. Since $(u, y_1) \cdot (u, y_2)^* = (uu^*, y_1 y_2^*) \neq (0, 0)$ and $diam(\Gamma^*(\mathcal{S})) = 2$, there must exist some $(a, b) \in V^*(\Gamma^*(\mathcal{S}))$ such that $(u, y_1) \cdot (a, b)^* = (u, y_2) \cdot (a, b)^* = (0, 0)$. Then $ua^* = 0$, so it must be that $a = 0$. Also, we must have $b \in V^*(\Gamma^*(\mathcal{S}_2))$ such that $y_1 b^* = y_2 b^* = 0$. Which is a contradiction, since we assume that no such $b$ exists.

($iii$) By ($i$) and ($ii$). □

**Theorem 3.6.** *Let $\mathcal{S}_1, \mathcal{S}_2$ be left Artinian abelian Rickart $*$-rings such that $diam(\Gamma^*(\mathcal{S}_1)) = diam(\Gamma^*(\mathcal{S}_2)) = 3$ and let $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$ with componentwise involution. Then $diam(\Gamma^*(\mathcal{S})) = 3$.*

*Proof.* Since $diam(\Gamma^*(\mathcal{S}_1)) = 3$, there must exist $x_1, x_2 \in V^*(\Gamma^*(\mathcal{S}_1))$, $x_1 \neq x_2$ with $x_1 x_2^* \neq 0$ and there is no $x_3 \in V^*(\Gamma^*(\mathcal{S}_1))$ such that $x_1 x_3^* = x_2 x_3^* = 0$. Likewise, there must exist $y_1, y_2 \in V^*(\Gamma^*(\mathcal{S}_2))$, $y_1 \neq y_2$ with $y_1 y_2^* \neq 0$ and there is no $y_3 \in V^*(\Gamma^*(\mathcal{S}_2))$ such that $y_1 y_3^* = y_2 y_3^* = 0$. Consider elements $(x_1, y_1)$ and $(x_2, y_2)$ of $V^*(\Gamma^*(\mathcal{S}))$. Since $(x_1, y_1) \cdot (x_2, y_2)^* = (x_1 x_2^*, y_1 y_2^*) \neq (0, 0)$, that is, $diam(\Gamma^*(\mathcal{S})) > 1$. If $diam(\Gamma^*(\mathcal{S})) = 2$, then there must exist some $(a, b) \in V^*(\Gamma^*(\mathcal{S}))$ such that $(x_1, y_1) \cdot (a, b)^* = (x_2, y_2) \cdot (a, b)^* = (0, 0)$. Then $x_1 a^* = x_2 a^* = 0$, follows that $a = 0$. Also, it must be that $b \in V^*(\Gamma^*(\mathcal{S}_2))$, such that $y_1 b^* = y_2 b^* = 0$, giving $b = 0$, a contradiction. Hence, we must have $diam(\Gamma^*(\mathcal{S})) = 3$. □

## 4. ***Data Availability Statement***

*No data were used to support this study.*

## 5. ***Aknowledgement***

*The authors are deeply grateful to the referee for careful reading of the paper and helpful suggestions.*

## 6. ***Conflict of interest***

*The authors declare that they have no conflicts of interest.*

## *References*


[1] Alanazi, A. M., Nazim, M., & Rehman, N. (2021). Planar, outerplanar, and toroidal graphs of the generalized zero-divisor graph of commutative rings. *J. Math.*, **2021**, 4828579, 7 pages. https://doi.org/10.1155/2021/4828579

[2] Akbari,S. & Mohammadian, A. (2004). On the zero-divisor graph of a commutative ring. *J. Algebra*, **274**, 847-855. https://doi.org/10.1016/S0021-8693(03)00435-6

[3] Anderson, D. F. & Livingston, P. S. (1999). The zero-divisor graph of a commutative ring. *J. Algebra*, **217**, 434-447. https://doi.org/10.1006/jabr.1998.7840

[4] Anderson, D. D. & M. Naseer, Beck's coloring of a commutative ring, *J. Algebra* **159** (1993) 500-514. https://doi.org/10.1006/jabr.1993.1171

[5] Patil, A. & Waphare, B. N. (2018). The zeto-divisor graph of a ring with involution. *J. Algebra and its Appl.*, **16 (11)**, 1-17. https://doi.org/10.1142/S0219498818500500

[6] Beck, I. (1988). Coloring of commutative rings. *J. Algebra*, **116**, 208-226. https://doi.org/10.1016/0021-8693(88)90202-5

[7] Berberian, S. K. (1972). Bear $* - ring$. Springer - Verlag, Berlin, New York.

[8] DeMeyer, F., McKenzia, T. & Schneider, K. (2002). The zero-divisor graph of a commutative semigroup. *Semigroup Fourm*, **65**(2) 206-214. https://doi.org/10.1007/s002330010128

[9] Nazim, M., Nisar, J. & Rehman, N. (2021). On domination in zero-divisor graphs of rings with involution. *Bull. Korean Math. Soc.*, **58**(6), 1409-1418. https://doi.org/10.4134/BKMS.b200968

[10] Nazim, M. & Rehman, N. (2022). On the essential annihilating-ideal graph of commutative rings. *ARS Math. Contemp.*, **22**(3), 16 pages. https://doi.org/10.26493/1855-3974.2645.8fc

[11] Nazim, M., Rehman, N., & Mir, S. A. (2023). Some properties of the essential annihilating-ideal graph of commutative rings. *Commu. Comb. Opt.*, **8**(4), 715-724. https://doi.org/10.22049/CCO.2022.27827.1365

[12] Rehman, N., Nazim, M. & Selvakumar, K. (2022). On the planarity, genus and crosscap of new extension of zero-divisor graph of commutative rings. *AKCE Int. J. Graphs Comb.*, **19**(1), 61-68. https://doi.org/10.1080/09728600.2022.2058437

[13] Rehman, N., Nazim, M., & Selvakumar, K. (2024). On the genus of annihilator intersection graph of commutative rings. *Algebraic Structures and their Applications*, **11**(1) 25-36. https://doi.org/10.22034/as.2023.18830.1573

[14] Redmond, S. P. (2002). The zero divisor graph of a non-commutative ring. *Internet. J. Commutative Rings*, **1**(4) 203-211.

Mohd Nazim
Orcid number: 0000-0001-8817-4336
School of Computational Sciences
Faculty of Science and Technology
JSPM University
Pune, India
*Email address:* mnazim1882@gmail.com

Nadeem Ur Rehman
Orcid number: 0000-0003-3955-7941
Department of Mathematics
Aligarh Muslim University
Aligarh, India
*Email address:* nu.rehman.mm@amu.ac.in

Shabir Ahmad Mir
Orcid number: 0000-0003-4743-6859
School of Computaional Sciences
Faculty of Science and Technology
JSPM University
Pune, India
*Email address:* mirshabir967@gmail.com