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VIETA-LUCAS OPERATIONAL MATRIX TECHNIQUE FOR FRACTIONAL VARIABLE-ORDER INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. The aim of this article is to find an effective method for solving variable-order fractional integro-differential equations. This method transforms the problem into a system of algebraic equations. For this purpose, we first express Vieta-Lucas orthogonal polynomials, then, we express the operational matrices of these polynomials. At this stage, all components of the equation will be expressed in terms of the new shifted Vieta-Lucas operational matrices. After that, by placing these operational matrices in the main equation and using the spectral collocation method, the variable-order fractional integro-differential equation will become an algebraic system. By solving this algebraic system, we will find an approximate solution to the original equation. In the following, an analysis of the error is also presented by preparing some theorems. In the end, in order to express the efficiency and capability of the method, some numerical examples are given. Additionally, for the numerical examples, the condition number, numerical convergence order, and the computed CPU time are evaluated. Based on the obtained results, it was concluded that the proposed method is relatively stable, highly accurate and efficient, and has an appropriate convergence rate.

Keywords: Vieta-Lucas operational matrix, Fractional variable-order integro-differential equations, spectral collocation method, error analysis.
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1. Introduction

In recent decades, fractional calculus has gained significant attention across various disciplines, including mathematics, physics, engineering, and chemistry. This is due to its ability to describe systems with memory effects and hereditary properties, which cannot be adequately modeled by traditional integer-order differential equations. Several definitions of fractional derivatives and integrals exist in the literature, with the Caputo derivative and the Riemann-Liouville fractional integral operators are among the most widely used [27].

A significant extension of fractional calculus is the concept of variable-order derivatives, where the fractional order of differentiation is allowed to depend

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on the independent variable, such as time. This approach, known as fractional variable-order calculus, was introduced by Samko in [32] and has since been further developed. It has found numerous applications, particularly in fields such as nonlinear viscoelasticity, mechanics, and systems where the order of differentiation changes over time [5, 23, 28].

Given the complexity of solving equations that involve fractional derivatives with a variable-order, there has been substantial interest in developing efficient numerical methods to approximate solutions [2, 9, 26]. Various numerical methods have been proposed for solving these challenging equations, such as the use of Legendre polynomials [15], wavelet-based methods [7], and Chebyshev polynomials [22]. Additionally, significant work has been done in applying the Vieta-Lucas polynomials to solve a wide range of fractional and fractional-order differential equations, such as those related to heat conduction [21] and delay differential equations [20]. Other approaches, such as the fractional integration operators introduced by Zaeri et al. [37], have also contributed to the solution of integro-differential equations with fractional derivatives.

The focus of this paper is on the numerical solution of fractional variable-order integro-differential equations (FVOIDEs) of the form

$$(1)\quad {}_{0}^{LC}\mathcal{D}_z^{\eta(z)}\phi(z)+\sum_{k=0}^{m}G_k(z)\phi^{(k)}(z)+\lambda_1\mathcal{K}_f\phi+\lambda_2\mathcal{K}_v\phi=h(z),\quad 0<\eta(z)<1,$$

subject to initial conditions

$$(2)\qquad \phi^{(k)}(\delta)=\theta_k,\quad \delta\in[0,1],\quad k=0,1,\dots,m-1,$$

where

$$(3)\qquad \mathcal{K}_f\phi=\int_0^1\mathcal{K}_f(z,\xi)\phi(\xi)d\xi,\quad \mathcal{K}_v\phi=\int_0^z\mathcal{K}_v(z,\xi)\phi(\xi)d\xi.$$

Here, $\mathcal{D}_z^{\eta(z)}$ denotes the left Riemann-Liouville fractional derivative of variable-order, and $\mathcal{K}_f$ and $\mathcal{K}_v$ represent the Fredholm and Volterra kernels, respectively. The function $G_k(z)$ is a smooth function, and λ_1 and λ_2 are constants. The solution to this equation, $\phi(z)$, is the unknown function we aim to determine.

FVOIDEs like the one described in equation (1) have significant applications in various domains, including noise reduction, signal processing [13], geographic data processing [8], and more [25,31,34]. To solve such equations, various methods have been proposed. For example, the shifted Jacobi-Gauss collocation method and the shifted Legendre Gauss-Lobatto collocation method have been used to solve these types of equations [11, 12]. Additionally, Haar collocation methods have been employed [4], as well as effective numerical schemes [14].

In this work, we propose a new method for solving FVOIDEs based on shifted Vieta-Lucas polynomials (SVLOMT). This method utilizes the orthogonal Vieta-Lucas basis and transforms the fractional variable-order integro-differential equation into a system of algebraic equations. By constructing operational matrices from the fractional derivatives and the Vieta-Lucas basis vectors, the problem is efficiently solved. The SVLOMT method is advantageous because of its simplicity, accuracy, and ability to handle polynomial exact solutions.

The key advantages of the SVLOMT method include

- Simple implementation and ease of use, making it accessible for both practitioners and researchers.
- High accuracy in computing solutions, particularly when the exact solution is polynomial in nature.
- Efficient computational performance, as the matrices involved contain many zeros, reducing the computational load.
- Exact solutions for problems whose exact solutions are polynomial functions.

This paper proceeds as follows: we first define the variable-order fractional derivatives and discuss their essential properties. Then, we introduce the shifted Vieta-Lucas polynomials and explore their properties. Next, we detail the construction of operational matrices for fractional derivatives and present the SVLOMT method. The paper also includes a theoretical analysis of convergence, existence, uniqueness, and error bounds. To verify the method's efficacy, we present several numerical examples, compare the results with those obtained using other numerical methods, and assess the accuracy and stability of the proposed technique. The paper concludes by summarizing the findings and discussing future directions.

2. Some materials and mathematical tools

2.1. **Definitions of variable-order fractional derivatives.** In this Subsection, we describe the definitions of variable-order fractional derivatives and some of their important properties discussed in various literature.

Definition 2.1. [3,32] *The left Riemann-Liouville fractional integral of order $\eta(z)$ for the function $\phi(z)$ is defined as*

$$ {}_0^{RL}\mathcal{I}_z^{\eta(z)}\phi(z) = \frac{1}{\Gamma(\eta(z))}\int_0^z (z-\xi)^{\eta(z)-1}\phi(\xi)d\xi, \quad z>0, \tag{4}$$

where $\Gamma(.)$ is the Euler gamma function.

Remark 2.2. [3] *Let ϕ be the power function $\phi(z)=z^{\beta}$. Then, for $\beta>-1$, we have*

$$ {}_0^{RL}\mathcal{I}_z^{\eta(z)}z^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\eta(z))}z^{\beta+\eta(z)}. \tag{5}$$

Definition 2.3. [3,32] *The left Liouville-Caputo fractional derivative of order $\eta(z)$ for the function $\phi(z)$ is defined as*

$$ {}_0^{LC}\mathcal{D}_z^{\eta(z)}\phi(z) = \frac{1}{\Gamma(1-\eta(z))}\int_0^z (z-\xi)^{-\eta(z)}\phi'(\xi)d\xi, \quad 0<\eta(z)<1. \tag{6}$$

Remark 2.4. [1] *Let* ϕ *be the power function* $\phi(z) = z^m$. *Then, for* $m = 0$, ${}_0^{LC}\mathcal{D}_z^{\eta(z)} z^0 = 0$ *and for* $m \in \mathbb{Z}^+$ *we have*

$$ {}_0^{LC}\mathcal{D}_z^{\eta(z)} z^m = \frac{\Gamma(m+1)}{\Gamma(m+1-\eta(z))} z^{m-\eta(z)}. \tag{7}$$

Lemma 2.5. *[32,33] Consider the variable-order fractional integro-differential equation*

$$ {}_0^{LC}\mathcal{D}z^{\eta(z)}\phi(z) = \mathcal{H}(z,\phi). \tag{8}$$

Then the solution of this equation is

$$\phi(z) = \phi(0) + \frac{1}{\Gamma(\eta(z))}\int_0^z (z-\xi)^{\eta(z)-1}\mathcal{H}(\xi,\phi(\xi))d\xi. \tag{9}$$

Theorem 2.6. *[33] Assume that*

(1) $\mathcal{H} : [0,\infty] \times \mathbb{R} \to \mathbb{R}$ *is a Lipschitz continuous function, i.e.,* $|\mathcal{H}(z,\xi_1) - \mathcal{H}(z,\xi_2)| \leq \mathcal{L}|\xi_1 - \xi_2|$.

(2) $\mathcal{H}$ *has a weak singularity with respect to* z, *then there exists a constant* $\alpha \in (0,1]$ *such that* $(\Omega\phi)(z) = z^\alpha\mathcal{H}(z,\phi)$ *is a continuous bounded map defined on* $[0,1] \times [0,1]$.

Then the equation (1) *(or equivalently* (8)*) has a unique solution* $\phi \in C[0,\hbar^*]$, *where* $\hbar^* \in [0,1]$.

Definition 2.7. *The incomplete gamma function is defined as*

$$\Gamma(s,z) = \int_z^\infty \xi^{s-1}e^{-\xi}\,d\xi, \quad \text{for}\, s > 0 \,\text{and}\, z > 0.$$

Here, $\Gamma(s,z)$ *is referred to as the upper incomplete gamma function, and it generalizes the gamma function* $\Gamma(s)$ *as*

$$\Gamma(s) = \int_0^\infty \xi^{s-1}e^{-\xi}\,d\xi.$$

Definition 2.8. The space $L^2([a,b],\omega(z))$ consists of all measurable functions $f(z)$ defined on the interval $[a,b]$ such that the following condition holds

$$\|f\|_{L^2([a,b],\omega(z))} = \left(\int_a^b |f(z)|^2\omega(z)\,dz\right)^{1/2} < \infty,$$

where $\omega(z)$ is a weight function, which is a positive function defined on the interval $[a,b]$. In other words, $f(z) \in L^2([a,b],\omega(z))$ if and only if the weighted L^2-norm of f is finite

$$\int_a^b |f(z)|^2\omega(z)\,dz < \infty.$$

Definition 2.9. The *Gram matrix* associated with a set of functions

$$\{f_1, f_2, \ldots, f_n\},$$

defined on a domain D with respect to an inner product $\langle \cdot, \cdot \rangle$ is the square matrix $G \in \mathbb{R}^{n\times n}$ where the (i,j)-th entry is given by the inner product of the functions f_i and f_j

$$G_{ij} = \langle f_i, f_j \rangle = \int_D f_i(x) f_j(x)\, dx$$

for all $1 \le i, j \le n$, assuming the inner product is defined as an integral over the domain D. The Gram matrix represents the pairwise inner products between the functions, reflecting their similarities and orthogonality properties.

Corollary 2.10. *The Gram matrix G corresponding to the functions*

$$\{VL_0^*, VL_1^*, \ldots, VL_N^*\},$$

is invertible.

2.2. **Vieta-Lucas polynomials.** In this Subsection, we introduce Vieta-Lucas orthogonal polynomials [19, 24]. We also present some important relationships involving these polynomials.

Definition 2.11. *The Vieta-Lucas polynomials are orthogonal and are defined for $\xi \in [-2, 2]$ as follows*

$$VL_n(\xi) = 2\cos(n\gamma), \quad \gamma = \arccos\left(\frac{\xi}{2}\right), \quad \gamma \in [0, \pi]. \tag{10}$$

The recurrence relationship of Vieta-Lucas polynomials $VL_n(\xi)$ is

$$VL_n(\xi) = \xi VL_{n-1}(\xi) - VL_{n-2}(\xi),$$

and the first two Vieta-Lucas polynomials are $VL_0(\xi) = 2$ and $VL_1(\xi) = \xi$. Also, $VL_n(\xi)$ can be obtained through the following power formula

$$VL_n(\xi) = \sum_{k=0}^{\lceil \frac{n}{2} \rceil} (-1)^k \frac{n\Gamma(n-k)}{\Gamma(k+1)\Gamma(n+1-2k)} \xi^{n-2k}, \;\; n = 2, 3, \ldots.$$

These polynomials are orthogonal on the interval $[-2, 2]$ with respect to the weight function $\omega(\xi) = \frac{1}{\sqrt{4-\xi^2}}$, therefore, the following relationship holds

$$\langle VL_m(\xi), VL_n(\xi) \rangle = \int_{-2}^{2} \frac{VL_m(\xi) VL_n(\xi)}{\sqrt{4-\xi^2}} d\xi = \begin{cases} 0 & m \neq n,\, m \neq 0,\, n \neq 0, \\ 4\pi & m = n = 0, \\ 2\pi & m = n \neq 0. \end{cases}$$

2.3. **Shifted Vieta-Lucas polynomials.** A new class of orthogonal polynomials on the interval $[0, 1]$ is created from $VL_n(z)$ using $\xi = 4z - 2$. This new class of polynomials is referred to by the symbol $VL_n^*(z)$.

Definition 2.12. *The shifted Vieta-Lucas polynomials of degree n on $[0, 1]$ can be obtained as*

$$VL_n^*(z) = VL_n\left(4z - 2\right).$$

The well-known shifted Vieta-Lucas polynomials $VL_n^*(z)$ satisfy the following recurrence formulae

(11) $$VL_n^*(z) = (4z-2)VL_{n-1}^*(z) - VL_{n-2}^*(z),\ n = 2, 3, \dots,$$

where $VL_0^*(z) = 2$, $VL_1^*(z) = 4z - 2$. The shifted Vieta-Lucas polynomials are the orthogonal polynomials on the interval $[0, 1]$ with respect to the weight function $\omega(z) = \frac{1}{\sqrt{z(1-z)}}$, and so we have the following orthogonality property

(12)
$$\langle VL_m^*(z), VL_n^*(z)\rangle = \int_0^1 \frac{VL_m^*(z)VL_n^*(z)}{\sqrt{z(1-z)}}dz = \begin{cases} 0 & m \neq n,\ m \neq 0,\ n \neq 0, \\ 4\pi & m = n = 0, \\ 2\pi & m = n \neq 0. \end{cases}$$

The explicit formula for the shifted Vieta-Lucas polynomials $VL_n^*(z)$ is given by

(13) $$VL_n^*(z) = \sum_{j=0}^{n} (-1)^j \frac{n(2)^{2j+1}\Gamma(n+j)}{\Gamma(n-j+1)\Gamma(2j+1)} z^j, \quad n \geq 1.$$

2.4. **Fundamental matrix relations.** In this Subsection, we will make a matrix representation of all components of the equation (1). This representation consists of the shifted Vieta-Lucas polynomials, the function $\phi(z)$ and its derivatives, the fractional variable-order derivative of $\phi(z)$, the function $G_k(z)$, the terms $\mathcal{K}f\phi$ and $\mathcal{K}v\phi$, and the conditions given in (2), which are also necessary for the method.

2.4.1. *Matrix representation of* $VL_n^*(z)$. Using Eq. (13), the vector of shifted Vieta-Lucas polynomials

(14) $$\mathbf{\Psi}(z) = \begin{bmatrix} VL_0^*(z) & VL_1^*(z) & \cdots & VL_N^*(z) \end{bmatrix}^T,$$

can be written in the form

(15) $$\mathbf{\Psi}(z) = \mathbf{A}\mathbf{\Lambda}(z), \quad \mathbf{\Lambda}(z) = \begin{bmatrix} 1 & z & \cdots & z^N \end{bmatrix}^T,$$

where $\mathbf{A}$ is an $(N+1) \times (N+1)$ lower triangular matrix with elements defined as

$$\mathbf{A} = \begin{bmatrix} a_{00} & 0 & 0 & 0 \\ a_{10} & a_{11} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{N0} & a_{N1} & \cdots & a_{NN}, \end{bmatrix},$$

where the elements of the matrix $\mathbf{A}$ are given by

(16) $$a_{ij} = \begin{cases} 2, & i = j = 0, \\ \dfrac{i(-1)^{i-j}(2)^{2j+1}\Gamma(i+j)}{\Gamma(i-j+1)\Gamma(2j+1)}, & i \geq j, \\ 0, & \text{Otherwise}. \end{cases}$$

If $n = 1$, 2, and 3, then the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -2 & 4 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 4 & 0 \\ 2 & -16 & 16 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 2 & -16 & 16 & 0 \\ -2 & 36 & -96 & 64 \end{bmatrix}.$$

The matrix $\mathbf{A}$ has $N(N+1)/2$ elements equal to zero and $(N+1)(N+2)/2$ non-zero elements. Note that $\mathbf{A}$ is a lower triangular matrix with non-zero elements on the main diagonal, so its determinant is non-zero ($|\mathbf{A}| = 2^{(N+1)^2-N}$) therefore, it is invertible. Thus, from equation (15), we obtain

$$\mathbf{\Lambda}(z) = \mathbf{A}^{-1}\mathbf{\Psi}(z). \tag{17}$$

2.4.2. *Matrix representation of* $\frac{d}{dz}VL_n^*(z)$. The derivative of the vector $\mathbf{\Psi}(z)$ in Eq. (14) is given by

$$\frac{d}{dz}\mathbf{\Psi}(z) = \mathbb{D}\mathbf{\Psi}(z), \tag{18}$$

where $\mathbb{D}$ is the $(N+1) \times (N+1)$ operational lower triangular matrix with all elements on the main diagonal is zero. Also, if N is odd, we have

$$\mathbb{D} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 8 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2(N-2) & 0 & 4(N-2) & \cdots & 0 & 0 & 0 \\ 0 & 4(N-1) & 0 & \cdots & 4(N-1) & 0 & 0 \\ 2N & 0 & 4N & \cdots & 0 & 4N & 0 \end{bmatrix}, \tag{19}$$

and for the even N,

$$\mathbb{D} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 8 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 4(N-2) & 0 & \cdots & 0 & 0 & 0 \\ 2(N-1) & 0 & 4(N-1) & \cdots & 4(N-1) & 0 & 0 \\ 0 & 4N & 0 & \cdots & 0 & 4N & 0 \end{bmatrix}. \tag{20}$$

The number of zero and non-zero elements in the matrix $\mathbb{D}$ is shown in Table 1. Substituting (15) into (18), results in

$$\frac{d}{dz}\mathbf{\Psi}(z) = \mathbb{D}\mathbf{A}\mathbf{\Lambda}(z) = \mathbb{D}\mathbf{\Psi}(z). \tag{21}$$

For the k-th order derivative of $\mathbf{\Psi}(z)$, we have

$$\frac{d^k}{dz^k}\mathbf{\Psi}(z) = \mathbb{D}^k\mathbf{A}\mathbf{\Lambda}(z) = \mathbb{D}^k\mathbf{\Psi}(z). \tag{22}$$

Table 1. The number of zero and non-zero elements of matrix $\mathbb{D}$

N	Number of zero elements	Number of non-zero elements
even	$\frac{1}{4}N(N+2)$	$\frac{1}{4}\left(3(N+1)^2+1\right)$
odd	$\frac{1}{4}(N+1)^2$	$\frac{3}{4}(N+1)^2$

2.4.3. *Matrix representation of* ${}_0^{LC}\mathcal{D}_z^{\eta(z)}\mathbf{\Psi}(z)$. By applying the operator ${}_0^{LC}\mathcal{D}_z^{\eta(z)}$ on both sides of equation (15), the following relationship will be obtained

$$(23)\qquad {}_0^{LC}\mathcal{D}_z^{\eta(z)}\mathbf{\Psi}(z) = {}_0^{LC}\mathcal{D}_z^{\eta(z)}\mathbf{A}\mathbf{\Lambda}(z) = \mathbf{A}{}_0^{LC}\mathcal{D}_z^{\eta(z)}\mathbf{\Lambda}(z) = \mathbf{A}\mathbf{S}_{\eta(z)}\mathbf{\Lambda}(z),$$

where $\mathbf{S}_{\eta(z)}$ is a $(N+1)(N+1)$ diagonal matrix as follows

$$(24)\qquad \mathbf{S}_{\eta(z)} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & s_1(z) & 0 & \cdots & 0 \\ 0 & 0 & s_2(z) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s_N(z) \end{bmatrix}.$$

The elements of the matrix $\mathbf{S}_{\eta(z)}$ can be easily calculated from the following relation

$$s_k(z) = \frac{\Gamma(k+1)}{\Gamma(k+1-\eta(z))} z^{-\eta(z)}, \quad k = 1, 2, \ldots, N.$$

Here, via Eqs. (17) and (23) we claim

$$(25)\qquad {}_0^{LC}\mathcal{D}_z^{\eta(z)}\mathbf{\Psi}(z) = \mathbf{A}\mathbf{S}_{\eta(z)}\mathbf{A}^{-1}\mathbf{\Psi}(z),$$

and it will result that for the operator ${}_0^{LC}\mathcal{D}_z^{\eta(z)}$

$$(26)\qquad {}_0^{LC}\mathcal{D}_z^{\eta(z)} = \mathbf{A}\mathbf{S}_{\eta(z)}\mathbf{A}^{-1}.$$

The matrix $\mathbf{AS}\eta(z)\mathbf{A}^{-1}$ is the operational matrix for the fractional variable-order operator ${}_0^{LC}\mathcal{D}_z^{\eta(z)}$.

2.4.4. *Matrix representation of* $\phi(z)$, $\phi^{(k)}(z)$ *and* ${}_0^{LC}\mathcal{D}_z^{\eta(z)}\phi(z)$. Suppose that $\phi(z) \in L^2[0,1]$, then it can be approximated as

$$(27)\qquad \phi(z) \simeq \phi_N(z) = \sum_{j=0}^{N} c_j VL_j^*(z) = \mathbf{C}^T\mathbf{\Psi}(z),$$

where $\mathbf{C}$ is the Vieta-Lucas coefficients vector as follows

$$\mathbf{C} = \begin{bmatrix} c_0 & c_1 & \cdots & c_N \end{bmatrix}^T.$$

This coefficients, can be calculated by the following relation

$$(28)\qquad c_j = \frac{1}{\pi\delta_j}\int_0^1 \frac{\phi(z)VL_j^*(z)}{\sqrt{z(1-z)}}dz,\ \delta_j = \begin{cases} 4 & j = 0, \\ 2 & j = 1, 2, ..., N. \end{cases}$$

According to Eq. (15), $\mathbf{\Psi}(z) = \mathbf{A\Lambda}(z)$, so equation (27) is equivalent to

$$\phi(z) \simeq \phi_N(z) = \mathbf{C}^T \mathbf{A\Lambda}(z), \tag{29}$$

For the k-th order derivative of $\phi(z)$, from Eqs. (15), (22), (27) and (29) results in

$$\frac{d^k}{dz^k}\phi(z) \simeq \frac{d^k}{dz^k}\phi_N(z) = \mathbf{C}^T \mathbb{D}^k \mathbf{A\Lambda}(z) = \mathbf{C}^T \mathbb{D}^k \mathbf{\Psi}(z). \tag{30}$$

Using equations (27) and (15), one obtains

$$\begin{aligned} {}_0^{LC}\mathcal{D}_z^{\eta(z)}\phi(z) &= {}_0^{LC}\mathcal{D}_z^{\eta(z)}\mathbf{C}^T\mathbf{\Psi}(z) = \mathbf{C}^T\,{}_0^{LC}\mathcal{D}_z^{\eta(z)}\mathbf{\Psi}(z) \\ &= \mathbf{C}^T\,{}_0^{LC}\mathcal{D}_z^{\eta(z)}\mathbf{A\Lambda}(z) = \mathbf{C}^T\mathbf{A}\,{}_0^{LC}\mathcal{D}_z^{\eta(z)}\mathbf{\Lambda}(z). \end{aligned}$$

Now, using equation (23) results in

$${}_0^{LC}\mathcal{D}_z^{\eta(z)}\phi(z) = \mathbf{C}^T\mathbf{A}\mathbf{S}_{\eta(z)}\mathbf{\Lambda}(z). \tag{31}$$

2.4.5. *Function approximation $G_k(z)$, $\mathcal{K}_f(z,\xi)$ and $\mathcal{K}_v(z,\xi)$.* Any function of one dimensional $G_k(z) \in L^2[0,1]$ can be approximated in terms of the shifted Vieta-Lucas polynomials as

$$G_k(z) \simeq G_{k,N}(z) = \sum_{j=0}^{N} g_{k,j} VL_j^*(z) = \mathbf{G}_k^T \mathbf{\Psi}(z), \tag{32}$$

where $\mathbf{G}_k^T = \begin{bmatrix} g_{k,0} & g_{k,1} & \cdots & g_{k,N} \end{bmatrix}$ and $g_{k,j}$'s $(j = 0, 1 \ldots, N)$ can be calculated by the following relation

$$g_{k,j} = \frac{1}{\pi\delta_j}\int_0^1 \frac{G_k(z)VL_j^*(z)}{\sqrt{z(1-z)}}dz, \;\; \delta_j = \begin{cases} 4 & j = 0, \\ 2 & j = 1, 2, ..., N. \end{cases} \tag{33}$$

The function $\mathcal{K}_f(z,\xi) \in L^2([0,1]\times[0,1])$ can be approximated by the following series form

$$\mathcal{K}_f(z,\xi) \simeq \sum_{n=0}^{N}\sum_{m=0}^{M} k_{mn}^f VL_m^*(\xi)VL_n^*(z) = \mathbf{\Psi}^T(z)\mathbf{K}_f\mathbf{\Psi}(\xi), \tag{34}$$

where $\mathbf{K}_f$ is a $(N+1)\times(N+1)$ matrix, whose elements are

$$k_{mn}^f = \frac{1}{\pi^2\delta_n\delta_m}\int_0^1\int_0^1 \frac{\mathcal{K}_f(z,\xi)VL_n^*(z)VL_m^*(\xi)}{\sqrt{z(1-z)}\sqrt{\xi(1-\xi)}}dzd\xi. \tag{35}$$

Similarly,

$$\mathcal{K}_v(z,\xi) \simeq \sum_{n=0}^{N}\sum_{m=0}^{M} k_{mn}^v VL_m^*(\xi)VL_n^*(z) = \mathbf{\Psi}^T(z)\mathbf{K}_v\mathbf{\Psi}(\xi), \tag{36}$$

and components k_{mn}^v is similar to k_{mn}^f in relation (35).

2.4.6. *Matrix representation of $\mathcal{K}_f\phi$ and $\mathcal{K}_v\phi$.* By putting relations (34) and (27) in (3) and using Eq. (15) and some simplifications, the following relation is gained

$$\mathcal{K}_f\phi = \mathbf{\Psi}^T(z)\mathbf{K}_f\left(\mathbf{A}\int_0^1 \mathbf{\Lambda}(\xi)\mathbf{\Lambda}^T(\xi)d\xi\right)\mathbf{A}^T\mathbf{C} = \mathbf{\Psi}^T(z)\mathbf{K}_f\mathbf{A}\mathbf{Q}^f\mathbf{A}^T\mathbf{C},$$

in which the matrix $\mathbf{Q}^f = \int_0^1 \mathbf{\Lambda}(\xi)\mathbf{\Lambda}^T(\xi)d\xi = [q^f_{ij}]$ is named the dual matrix of $\mathbf{\Psi}(\xi)$. The components of this matrix can be computed as follows

$$q^f_{ij} = \frac{1}{i+j+1}, \quad i,j = 0,1,\ldots,N. \tag{37}$$

If we define $\mathbf{E}^f = \mathbf{K}_f\mathbf{A}\mathbf{Q}^f\mathbf{A}^T$, then we have

$$\mathcal{K}_f\phi = \mathbf{\Psi}^T(z)\mathbf{E}^f\mathbf{C} = \mathbf{\Lambda}^T(z)\mathbf{A}^T\mathbf{E}^f\mathbf{C}. \tag{38}$$

Similarly, for part $\mathcal{K}_v\phi$

$$\mathcal{K}_v\phi = \mathbf{\Psi}^T(z)\mathbf{E}^v(z)\mathbf{C} = \mathbf{\Lambda}^T(z)\mathbf{A}^T\mathbf{E}^v(z)\mathbf{C}, \tag{39}$$

where $\mathbf{E}^v(z) = \mathbf{K}_v\mathbf{A}\mathbf{Q}^v(z)\mathbf{A}^T$, $\mathbf{Q}^v(z) = \int_0^z \mathbf{\Lambda}(\xi)\mathbf{\Lambda}^T(\xi)d\xi = [q^v_{ij}(z)]$ and the elements of matrix $\mathbf{Q}^v$ are calculated as follows

$$q^v_{ij}(z) = \frac{1}{i+j+1}z^{i+j+1}, \quad i,j = 0,1,\ldots,N. \tag{40}$$

2.4.7. *Matrix representation of initial conditions.* According to relation (30), the matrix form of initial conditions (2) are as follows

$$\mathbf{C}^T\mathbb{D}^k\mathbf{\Psi}(\delta) = \theta_k. \tag{41}$$

3. Description of SVLOMT

In this section, we propose SVLOMT for solving problem (1)-(3). To do this, we first substitute Eqs. (30)-(32), (38) and (39) in (1), yielding

$$\mathbf{C}^T\mathbf{A}\mathbf{S}_{\eta(z)}A^{-1}\mathbf{\Psi}(z) + \sum_{k=0}^{m}\left(\mathbf{G}_k^T\mathbf{\Psi}(z)\mathbf{C}^T\mathbb{D}^k\mathbf{\Psi}(z)\right) + \lambda_1\mathbf{\Psi}^T(z)\mathbf{E}^f\mathbf{C} + \lambda_2\mathbf{\Psi}^T(z)\mathbf{E}^v(z)\mathbf{C} = h(z).$$

Let us define

$$\mathcal{R}_N(z) = \left(\mathbf{C}^T\mathbf{A}\mathbf{S}_{\eta(z)}A^{-1} + \sum_{k=0}^{m}\mathbf{G}_k^T\mathbf{\Psi}(z)\mathbf{C}^T\mathbb{D}^k\right)\mathbf{\Psi}(z) + \mathbf{\Psi}^T(z)\left(\lambda_1\mathbf{E}^f + \lambda_2\mathbf{E}^v(z)\right)\mathbf{C} - h(z) = 0.$$

To find the solution $\phi(z)$ according to collocation method, we utilize the following equations

$$\mathcal{R}_N(z_j) = \left(\mathbf{C}^T \mathbf{A} \mathbf{S}_{\eta(z_j)} A^{-1} + \sum_{k=0}^{m} \mathbf{G}_k^T \mathbf{\Psi}(z_j) \mathbf{C}^T \mathbb{D}^k\right) \mathbf{\Psi}(z_j) + \mathbf{\Psi}^T(z_j) \left(\lambda_1 \mathbf{E}^f + \lambda_2 \mathbf{E}^v(z_j)\right) \mathbf{C} - h(z_j) = 0, \tag{42}$$

to build $(N+1)$ algebraic system equations, where

$$z_j = \frac{2j+1}{2N+2}, \quad j = 0, 1, \ldots, N.$$

Eq. (42) can be written as $\mathbf{WC} = \mathbf{G}$. To impose the initial conditions (2), we replace m rows of the matrix $\mathbf{W}$ to obtain $\hat{\mathbf{W}}\mathbf{C} = \hat{\mathbf{G}}$. For simplicity, if the last m rows of matrix $\mathbf{W}$ are replaced, the matrix system $\hat{\mathbf{W}}\mathbf{C} = \hat{\mathbf{G}}$ will be obtained. However, it is not necessary to replace only the last m rows. For example, if the matrix $\hat{\mathbf{W}}$ is singular, the rows that are linearly dependent or zero will be replaced. If rank$(\hat{\mathbf{W}}) = N+1$, the matrix $\mathbf{C}$ will be determined as $\mathbf{C} = \hat{\mathbf{W}}^{-1}\hat{\mathbf{G}}$. Therefore, system (1) with the initial conditions (2) has a unique solution. It should be noted that if the matrix $\hat{\mathbf{W}}$ is singular and rank$(\hat{\mathbf{W}}) < N+1$, then a specific solution may exist. Consequently, by solving for the unknown parameters of the vector $\mathbf{C}$, one can obtain the solution as $\phi(z) = \mathbf{C}^T \mathbf{\Psi}(z)$.

4. **Error analysis**

The aim of this section is to provide error bounds for the numerical solution obtained using the method proposed in Section Sec. 3.

Theorem 4.1. *Assume that $\phi(z) \in L^2([0,1];\omega)$, where $\omega(z) = \frac{1}{\sqrt{z(1-z)}}$, and that $|\phi''(z)| \leq \mathcal{M}$. Then, $\phi(z)$ can be expressed as an infinite linear combination of the shifted Vieta-Lucas polynomials, specifically in terms of $VL_j^*(z)$, and $\phi_N(z)$ contains only $N+1$ terms from this series. Furthermore, this series converges uniformly to the function $\phi(z)$ as $N \to \infty$. Moreover, the coefficients given in* (28) *are bounded, i.e.,*

$$|c_j| \leq \frac{\mathcal{M}}{4j(j^2-1)}. \tag{43}$$

Proof. Any function $\phi(z) \in L^2([0,1];\omega)$ can be expressed by Vieta-Lucas polynomials as Eq. (27) via coefficients c_j in (28). To compute the integrals, apply the substitution $4z - 2 = \cos(2\gamma)$ in Eq. (28), and then the following is gained

$$c_j = \frac{2}{\pi \delta_j} \int_0^{\pi} \phi\left(\frac{1}{2}\cos(\gamma) + \frac{1}{2}\right) \cos(j\gamma) d\gamma.$$

Integrating by parts gives

$$c_j = \frac{2}{\pi\delta_j}\left[\frac{1}{j}\sin(j\gamma)\phi\left(\frac{1}{2}\cos(\gamma)+\frac{1}{2}\right)\Big|_0^{\pi} + \frac{1}{2j}\int_0^{\pi}\sin(j\gamma)\sin(\gamma)\phi'\left(\frac{1}{2}\cos(\gamma)+\frac{1}{2}\right)d\gamma\right].$$

With the help of trigonometric relation

$$\sin(j\gamma)\sin(\gamma) = \frac{1}{2}\left(\cos((j-1)\gamma) - \cos((j+1)\gamma)\right),$$

and reuse of integration by part one obtains

$$\begin{aligned} c_j &= \frac{2}{\pi\delta_j}\frac{1}{2j}\left[\frac{1}{2}\left[\frac{\sin((j-1)\gamma)}{(j-1)} - \frac{\sin((j+1)\gamma)}{(j+1)}\right]\phi'\left(\frac{1}{2}\cos(\gamma)+\frac{1}{2}\right)\Big|_0^{\pi} + \frac{1}{4}\int_0^{\pi}\sin(\gamma)\phi''\left(\frac{1}{2}\cos(\gamma)+\frac{1}{2}\right)\Upsilon_j(\gamma)d\gamma\right] \\ &= \frac{1}{4\pi j\delta_j}\int_0^{\pi}\sin(\gamma)\phi''\left(\frac{1}{2}\cos(\gamma)+\frac{1}{2}\right)\Upsilon_j(\gamma)d\gamma, \end{aligned}$$

where $\Upsilon_j(\gamma) = \frac{\sin((j-1)\gamma)}{(j-1)} - \frac{\sin((j+1)\gamma)}{(j+1)}$. On the other hand $|\Upsilon_j(\gamma)| \le \frac{2}{j^2-1}$, $j \ge 2$. This relationship along with the property of function $\phi(z)$ implies that

$$|c_j| \le \frac{1}{4\pi j\delta_j}\frac{2\pi\mathcal{M}}{j^2-1} = \frac{\mathcal{M}}{4j(j^2-1)}.$$

□

Remark 4.2. In Theorem 4.1, for the two special cases of c_0 and c_1, the upper bounds of the error are as follows

$$|c_0| \le \frac{\tau}{2}, \quad |c_1| \le \frac{2\tau}{\pi},$$

where $|\phi(z)| \le \tau$ for all $z \in (0,1)$.

Lemma 4.3. *Let $\phi(z) \in L^2([0,1];\omega)$, where $\omega(z) = \frac{1}{\sqrt{z(1-z)}}$, and let $\vartheta = span\{VL_0^*, VL_1^*, \ldots, VL_N^*\}$. If the Gram matrix G corresponding to ϑ is invertible (Matrix G in Corrolary 2.10), the best approximation of $\phi(z)$ in the subspace ϑ exists and is unique.*

Proof. **Existence:** The subspace ϑ is finite-dimensional, spanned by the functions $\{VL_0^*, VL_1^*, \ldots, VL_N^*\}$. Since ϑ is finite-dimensional, the projection of $\phi(z)$ onto ϑ exists. In a finite-dimensional Hilbert space, the projection of any element onto a subspace always exists, thus the best approximation of $\phi(z)$ in ϑ exists.

Uniqueness: Since the Gram matrix G is invertible, we have that the set of functions $\{VL_0^*, VL_1^*, \ldots, VL_N^*\}$ is linearly independent. From the properties of the Gram matrix, the invertibility ensures that the projection of $\phi(z)$

onto the subspace ϑ is unique. In fact, the solution to the least-squares problem, which provides the best approximation of $\phi(z)$ in ϑ, is unique due to the invertibility of the Gram matrix. □

For more details on the existence and uniqueness of the best approximation and related topics, refer to [29].

Theorem 4.4. *Let $\phi(z) \in L^2([0,1];\omega)$ and*

$$\vartheta = span\{VL_0^*, VL_1^*, \dots, VL_N^*\}.$$

If $\phi_N(z)$ is the best approximation to $\phi(z)$ from ϑ, then the error bound is as follows

$$\|\phi(z) - \phi_N(z)\|_2 \le \frac{\Delta\sqrt{\pi}}{(N+1)!},$$

where $\Delta = \max\{\phi^{(N+1)}(z),\ z \in [0,1]\}$, and $\phi(z)$ is assumed to be sufficiently smooth, specifically that it has a derivative of order $(N+1)$.

Proof. For $z_0 \in [0,1]$, the function $\phi(z)$ expands as

$$\phi(z) = \phi(z_0) + (z - z_0)\phi'(z_0) + \frac{(z-z_0)^2}{2!}\phi''(z_0) + \cdots + \frac{(z-z_0)^N}{N!}\phi^{(N)}(z_0) + \frac{(z-z_0)^{N+1}}{(N+1)!}\phi^{(N+1)}(\eta),$$

where $\eta \in [z_0, z]$. Suppose $\tilde{\phi}_N(z) \in \vartheta$, and assume that

$$\tilde{\phi}_N(z) = \phi(z_0) + (z - z_0)\phi'(z_0) + \frac{(z-z_0)^2}{2!}\phi''(z_0) + \cdots + \frac{(z-z_0)^N}{N!}\phi^{(N)}(z_0),$$

then

$$\phi(z) - \tilde{\phi}_N(z) = \frac{(z-z_0)^{N+1}}{(N+1)!}\phi^{(N+1)}(\eta).$$

Since $\phi_N(z)$ is the best approximation to $\phi(z)$ out of ϑ, one obtains

$$\begin{aligned}
\|\phi(z) - \phi_N(z)\|_2 \le \left\|\phi(z) - \tilde{\phi}_N(z)\right\|_2 &= \sqrt{\int_0^1 \left|\phi(z) - \tilde{\phi}_N(z)\right|^2 \omega(z)dz} \\
&= \sqrt{\int_0^1 \left|\frac{(z-z_0)^{N+1}}{(N+1)!}\phi^{(N+1)}(\eta)\right|^2 \omega(z)dz} \\
&\le \sqrt{\frac{\Delta^2}{[(N+1)!]^2}\int_0^1 \left|(z-z_0)^{N+1}\right|^2 \omega(z)dz} \\
&\le \sqrt{\frac{\Delta^2\left[(1)^{N+1}\right]^2}{[(N+1)!]^2}\int_0^1 \omega(z)dz} = \frac{\Delta\sqrt{\pi}}{(N+1)!},
\end{aligned}$$

in which $\omega(z) = \frac{1}{\sqrt{z(1-z)}}$ and $\int_0^1 \omega(z)dz = \pi$. These complete the proof. $\square$

Remark 4.5. If $\tilde{\phi}_N(z) \in \vartheta$, where $\vartheta = \text{span}\{VL_0^*, VL_1^*, \ldots, VL_N^*\}$, and $\phi(z) \in L^2([0,1];\omega)$, then by considering the Taylor expansions of the functions $\phi(z)$ and $\tilde{\phi}_N^{(k)}(z)$, it follows that for any $z_0 \in [0,1]$, the order of accuracy of the derivative approximation in equation (30) is $O((z-z_0)^{N+1-k})$.

5. Consistency and Stability [6, 18]

In this section, we analyze the consistency and stability of the approximation method presented in this paper, specifically focusing on the best approximation of the function $\phi(z)$ in the subspace $\vartheta = \text{span}\{VL_0^*, VL_1^*, \ldots, VL_N^*\}$.

5.1. **Consistency.** Consistency refers to the property of the approximation method where the approximation converges to the true function $\phi(z)$ as the number of basis functions increases. Specifically, we show that the error of the best approximation $\phi_N(z)$ decreases as N increases and converges to zero in the limit.

From Theorem 4.4, we obtain a clear bound on the error of the best approximation

$$\|\phi(z) - \phi_N(z)\|_2 \leq \frac{\Delta\sqrt{\pi}}{(N+1)!},$$

where $\Delta = \max\{\phi^{(N+1)}(z), z \in [0,1]\}$. This error bound shows that as N increases, the error in the approximation diminishes. The rapid decay of the factor $(N+1)!$ ensures that as more basis functions are added, the approximation $\phi_N(z)$ becomes very close to $\phi(z)$. Therefore, the method is consistent as the number of basis functions increases, the approximation improves.

5.2. **Stability.** Stability refers to the robustness of the approximation method with respect to small perturbations in the input data or the choice of basis functions. In other words, small changes in $\phi(z)$ or in the set of basis functions $\{VL_0^*, VL_1^*, \ldots, VL_N^*\}$ should not lead to large deviations in the approximation $\phi_N(z)$.

The stability of this approximation is ensured by the invertibility of the Gram matrix G, which is discussed in Corrolary 2.10. Since the Gram matrix is invertible, the system of equations for the best approximation has a unique and stable solution. This means that small changes in $\phi(z)$ or in the coefficients of the basis functions result in only small changes in the approximation $\phi_N(z)$. Therefore, the method is stable. Furthermore, since the functions $\{VL_0^*, VL_1^*, \ldots, VL_N^*\}$ are linearly independent and the Gram matrix is positive definite, the approximation error does not grow disproportionately in response to small changes in the data. This ensures that the method is stable.

6. Test Problems

In this section, we illustrate the accuracy and efficiency of the proposed technique by solving some numerical examples. The method is implemented using the Maple 16 software. The error function is given by $e_N(z) = |\phi(z) - \phi_N(z)|$. To numerically calculate the convergence order ord_N^∞, we use the following formula [21]

$$\text{ord}_N^\infty = \frac{\ln\left(\max\limits_{0<z<1} e_N(z)\right) - \ln\left(\max\limits_{0<z<1} e_{2N}(z)\right)}{\ln 2}. \tag{44}$$

Example 6.1. *[1, 22] As a first example, we consider the following fractional variable-order integro-differential equation*

$${}_0^{LC}\mathcal{D}_z^{\eta(z)}\phi(z) + 6\int_0^z \phi(\xi)d\xi + 2z\phi'(z) + \phi(z) = h(z), \tag{45}$$

where $\eta(z) = \frac{3}{5}(\sin(z) + \cos(z))$, *and*

$$h(z) = \frac{5\Gamma(3)}{\Gamma(3-\eta(z))} z^{2-\eta(z)} + \frac{15\Gamma(2)}{\Gamma(2-\eta(z))} z^{1-\eta(z)} + 5z(2z^2 + 14z + 9).$$

Equation (45)*, with the initial condition* $\phi(0) = 0$*, has the exact solution* $5z^2 + 15z$*. We apply the presented method to find the approximate solution using the truncated Vieta-Lucas series for* $N = 2$*. The essential matrix equation for this problem is given by*

$$\begin{aligned} &\left[\mathbf{C}^T\mathbf{A}\mathbf{S}_{\eta(z)}A^{-1} + \mathbf{G}_0^T\mathbf{\Psi}(z)\mathbf{C}^T + \mathbf{G}_1^T\mathbf{\Psi}(z)\mathbf{C}^T\mathbb{D}\right]\mathbf{\Psi}(z) \\ &\qquad\qquad\qquad\qquad +6\mathbf{\Psi}^T(z)\mathbf{E}^v\mathbf{C} = h(z), \end{aligned} \tag{46}$$

where

$$\mathbf{C}^T = \begin{bmatrix} c_0 & c_1 & c_2 \end{bmatrix}, \quad \mathbf{G}_0^T = \begin{bmatrix} \frac{1}{2} & 0 & 0 \end{bmatrix}, \quad \mathbf{G}_1^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix},$$

and

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 4 & 0 \\ 2 & -16 & 19 \end{bmatrix}, \quad \mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{3}{16} & \frac{1}{4} & \frac{1}{16} \end{bmatrix}, \quad \mathbb{D} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 8 & 0 \end{bmatrix},$$

$$\mathbf{S}_{\eta(z)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\Gamma(2)z^{-\eta(z)}}{\Gamma(2-\eta(z))} & 0 \\ 0 & 0 & \frac{\Gamma(3)z^{-\eta(z)}}{\Gamma(3-\eta(z))} \end{bmatrix}, \quad \mathbf{K}_v = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{\Psi}(z) = \begin{bmatrix} 2 \\ 4z-2 \\ 16z^2 - 16z + 2 \end{bmatrix}, \quad \mathbf{Q}^v(z) = \begin{bmatrix} z & \frac{z^2}{2} & \frac{z^3}{3} \\ \frac{z^2}{2} & \frac{z^3}{3} & \frac{z^4}{4} \\ \frac{z^3}{3} & \frac{z^4}{4} & \frac{z^5}{5} \end{bmatrix}.$$

From the relation $\mathbf{E}^v(z) = \mathbf{K}_v \mathbf{A} \mathbf{Q}^v(z) \mathbf{A}^T$, *we get*

$$\mathbf{E}^v(z) = \begin{bmatrix} z & z^2 - z & \frac{8z^3}{3} - 4z^2 + z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix equation for the initial condition $\phi(0) = 0$, *according to* (41), *is given by*

$$\mathbf{C}^T \mathbf{\Psi}(0) = 0, \quad or \quad 2c_0 - 2c_1 + 2c_2 = 0.$$

By solving the system at collocation points $z_0 = \frac{1}{6}$, $z_1 = \frac{3}{6}$, *and* $z_2 = \frac{5}{6}$, *we obtain the following system*

$$\begin{cases} 4c_0 + 0.900126952c_1 - 10.61518232c_2 = 19.93339028, \\ 8c_0 + 4.813894046c_1 - 6.390304879c_2 = 59.57249995, \\ 12c_0 + 10.50647492c_1 + 20.18894180c_2 = 115.0914189. \end{cases}$$

Replacing the third equation with the initial condition $2c_0 - 2c_1 + 2c_2 = 0$ *and solving the system, we obtain the following result for the unknown vector* $\mathbf{C}$

$$\mathbf{C}^T = \begin{bmatrix} \frac{75}{16} & 5 & \frac{5}{16} \end{bmatrix}.$$

Thus, we obtain the solution $\phi(z) = \mathbf{C}^T \mathbf{\Psi}(z) = 5z^2 + 15z$, *which is the exact solution of the equation.*

In [1] and [22], numerical methods for solving variable-order fractional integro-differential equations (FVOIDEs) are proposed. The method in [1] uses Vieta-Fibonacci polynomials to construct operational matrices and convert the problem into an algebraic system with fractional differentiation and integration operators. The method in [22] applies the second kind of Chebyshev polynomials to derive operational matrices for fractional differentiation and integration, also transforming the equation into an algebraic system. These methods provide approximate solutions, whereas the proposed method using Vieta-Lucas polynomials yields the exact solution to the problem.

Example 6.2. *[10, 35] As a second example, we consider the following fractional variable-order integro-differential equation (FVOIDE)*

$$(47) \qquad {}_0^{LC}\mathcal{D}_z^{\eta(z)} \phi(z) - {}_0^{C}\mathcal{D}_z^{\beta(z)} \phi(z) - 56 \int_0^1 (z + \xi) \phi^3(\xi) \, d\xi = h(z),$$

where

$$h(z) = 9 + 6z + \frac{2\Gamma(3) z^{2-\eta(z)}}{\Gamma(3 - \eta(z))} - \frac{6\Gamma(2) z^{1-\eta(z)}}{\Gamma(2 - \eta(z))} - \frac{\Gamma(3) z^{2-\beta(z)}}{\Gamma(3 - \beta(z))} + \frac{3\Gamma(2) z^{1-\beta(z)}}{\Gamma(2 - \beta(z))},$$

with initial conditions

$$\phi(0) = 1, \quad \phi'(0) = -3,$$

and the exact solution $\phi(z) = z^2 - 3z + 1$. *The variable-order fractional derivatives are given by* $\eta(z) = 1 + \frac{z}{3}$ *and* $\beta(z) = \frac{z}{3}$.

Now, we aim to find the approximate solution of this problem in the form

$$\phi_N(z) = \sum_{j=0}^{N} c_j VL_j^*(z), \tag{48}$$

where $N = 2$ *as described in Section 3, with collocation points* $z_0 = \frac{1}{6}, z_1 = \frac{3}{6}, z_2 = \frac{5}{6}$. *By applying the method outlined in Section 3, we obtain the following system of equations*

$$\begin{cases} -\dfrac{3136}{5}c_0c_2^2 - \dfrac{448}{5}c_2c_2^2 - \dfrac{352}{5}c_1c_3^2 - 448c_0c_2^2 + 448c_0^2c_2 - 224c_1^2c_1 \\ \quad + \dfrac{98609}{17104}c_1 + \dfrac{51744}{4903}c_2 - 448c_0^3 - \dfrac{224}{5}c_1^3 + \dfrac{576}{5}c_2^3 \\ \quad - \dfrac{448}{5}c_0c_1c_2 = \dfrac{64264}{6573}, \\ 2c_0 - 2c_1 + 2c_2 = 1, \\ 4c_1 - 16c_2 = -3. \end{cases}$$

Solving this system, we obtain the following values for the coefficients

$$c_0 = -\frac{1}{16}, \quad c_1 = -\frac{1}{2}, \quad c_2 = \frac{1}{16}.$$

Finally, by substituting the obtained coefficient vector $\mathbf{C}$ *into the expression* (48), *we get the approximate solution*

$$\phi_2(z) = 1 - 3z + z^2,$$

which coincides with the exact solution of the problem.

From the obtained results, it is evident that our method provides the exact solution for $N = 2$*, unlike the approximate solutions obtained by the methods in [10] and [35]. The Vieta-Lucas polynomial-based approach, combined with the collocation method, achieves the exact solution with a low-order expansion, whereas the operational matrix method based on Bernstein polynomials [35] and the Galerkin method with Chebyshev polynomials [10] require higher values of* N *to reach similar accuracy.*

The key advantage of our method lies in its ability to capture the solution's behavior accurately using fewer terms, which allows it to outperform other methods that depend on higher-order expansions. While methods like Bernstein and Chebyshev polynomials are effective, they struggle to provide exact solutions for variable-order fractional integro-differential equations without significant computational effort and a higher N*. Therefore, the Vieta-Lucas polynomial approach is more efficient, providing the exact solution with fewer computational resources.*

Example 6.3. *[35] Consider the following fractional variable-order integro-differential equation (FVOIDE)*

$${}_0^{LC}\mathcal{D}_z^{\eta(z)}\phi(z) - \int_0^1 z\xi\phi(\xi)\,d\xi - \int_0^z (z\xi)^2\phi(\xi)\,d\xi = h(z), \quad \phi(0) = 1, \tag{49}$$

where

$$h(z) = -z - z^2\left(-2 + e^z\left(2 + (z-2)z\right)\right) + \frac{e^z\left(\Gamma(1-\eta(z)) - \Gamma(1-\eta(z), z)\right)}{\Gamma(1-\eta(z))},$$

and $\Gamma(.,.)$ *denotes the incomplete gamma function. The analytic solution to this equation is* $\phi(z) = e^z$*, and in this case, we set* $\eta(z) = \sin(z)$*.*

By applying the SVLOMT for $N = 4$ *and* $N = 6$*, we obtain the following approximate solutions*

$$\phi_4(z) = 1 + 0.999288127z + 0.506280082z^2 + 0.147504731z^3 + 0.064870034z^4,$$
$$\phi_6(z) = 1 + 0.999997164z + 0.500043250z^2 + 0.166412246z^3 + 0.500043250z^4 + 0.007225671z^5 + 0.002199893z^6.$$

A comparison of the L_2 *errors obtained by the proposed method for* $N = 2, 4, 6, 8$*, along with the results from [35] using Bernstein polynomials, is presented in Table 2. Additionally, a comparison of the absolute errors at selected points, obtained using different numbers of basis functions (*$N = 2, 3, 4, 5, 6$*), is shown in Table 3. The graphs of the absolute error functions for different values of* N *are provided in Figs. 1 and 2. Figure 3 compares the approximate solutions* $\phi_N(z)$ *with the exact solution for various values of* N*. From these comparisons, it is evident that the errors decrease as* N *increases. Moreover, the method in Section 3 proves to be quite effective, and the calculations performed in Maple demonstrate its reliability. In this example, the convergence order has been computed and is presented in Table 4, along with the CPU time and condition number. The results indicate that the SVLOMT yields high-order accurate results. Since the condition number is less than or equal to 1, the problem is fully stable, meaning the output is not sensitive to changes in the input data.*

TABLE 2. L_2-norm errors (Example 6.3)

N	*2*	*4*	*6*	*8*
SVLOMT	*2.66e−03*	*1.14e−05*	*2.34e−08*	*5.64e−10*
Method [35]	*1.75e−02*	*4.94e−05*	*1.23e−07*	*2.11e−08*

Example 6.4. *[30] In this example, we consider the following variable-order fractional integro-differential equation*

$$ {}_{0}^{LC}\mathcal{D}_z^{\eta(z)}\phi(z) + \int_0^z e^z\phi(\xi)\,d\xi + \phi(z) = h(z), \tag{50}$$

where $h(z) = (2z+3)(z+6)$ *and* $\eta(z) = \frac{1}{3}\left(\sin(z) + \cos(z)\right)$*. The exact solution of this problem is not known.*

TABLE 3. Absolute error (Example 6.3)

z	N				
	2	*3*	*4*	*5*	*6*
0.1	*3.79e−04*	*2.53e−05*	*1.00e−06*	*4.21e−08*	*9.41e−10*
0.3	*3.16e−05*	*9.13e−07*	*3.38e−07*	*1.31e−08*	*4.56e−11*
0.5	*1.72e−04*	*1.47e−05*	*3.85e−08*	*1.67e−08*	*5.11e−10*
0.7	*2.20e−04*	*4.36e−06*	*4.08e−07*	*3.90e−08*	*9.42e−10*
0.9	*1.25e−03*	*6.47e−05*	*1.93e−06*	*6.09e−08*	*6.72e−10*

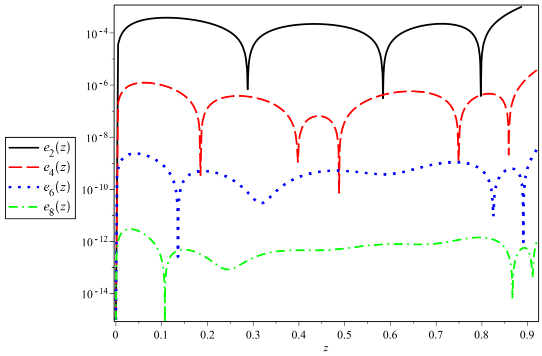

FIGURE 1. Comparison of the absolute error for $\phi(z)$ in problem 6.3

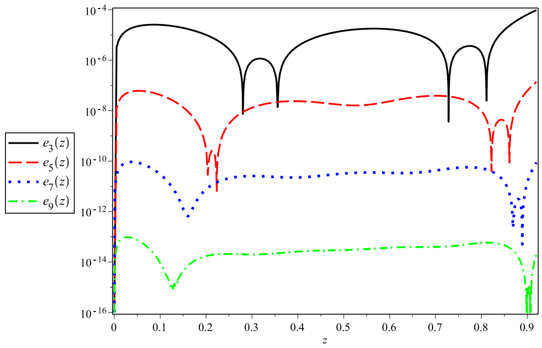

FIGURE 2. Comparison of the absolute error for $\phi(z)$ in problem 6.3

Since the exact solution is unavailable for comparison, the approximate error is estimated using the following expression

$$e_N(z) = \left| {}_0^{LC}\mathcal{D}_z^{\eta(z)}\phi_N(z) + \int_0^z e^z \phi_N(\xi)\, d\xi + \phi_N(z) - h(z) \right| \simeq 0.$$

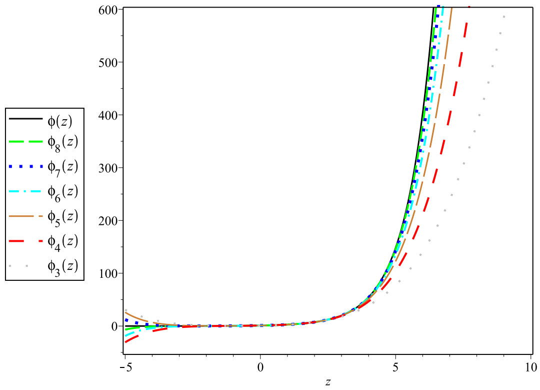

FIGURE 3. Comparison of the exact and approximate solutions of the problem (49)

TABLE 4. The numerical calculation of the condition number (C. N), convergence order (ord_N^∞), and time (S) for Example 6.3

N	2	3	4	5	6	7
C. N	0.90451	0.99849	0.99987	0.99999	0.99999	1.00000
ord_N^∞	10.9658	12.5507	17.9316	19.60964046	24.2535	29.8973
Time	0.96	1.15	1.45	1.64	1.89	2.09

Table 5 presents a comparison of the absolute error at selected points for various values of N (i.e., $N = 5, 6, 7, 8, 9$) obtained using the current method. Figures 4-5 show the graphs of the absolute error functions for different values of N. In Figure 6, the approximate solutions $\phi_N(z)$ for the problem in (50) are compared for different values of N. The results from these comparisons clearly demonstrate that the errors decrease as the value of N increases. The convergence order for this example has been calculated and is presented in Table 6, along with the CPU time and the condition number. Given that the condition numbers are small, it can be concluded that the problem is stable and well-conditioned.

Example 6.5. *Consider the following variable-order fractional integro-differential equation (FVOIDE)*

$$(51)\quad {}_{0}^{LC}\mathcal{D}_z^{\eta(z)}\phi(z) - \int_0^1 \xi \sin(z)\phi(\xi)\, d\xi - \int_0^z (z-\xi)\phi(\xi)\, d\xi = h(z), \quad \phi(0) = 0,$$

TABLE 5. Absolute error (Example 6.4)

z	N 5	6	7	8	9
0.1	7.60e−04	2.25e−04	5.80e−05	1.80e−06	1.43e−06
0.3	3.93e−04	3.74e−06	5.61e−07	1.70e−08	1.32e−09
0.5	2.93e−04	4.38e−06	0	3.24e−09	0
0.7	3.70e−04	2.17e−06	2.13e−07	6.46e−09	5.00e−10
0.9	7.04e−04	4.98e−05	5.54e−06	1.65e−07	1.27e−07

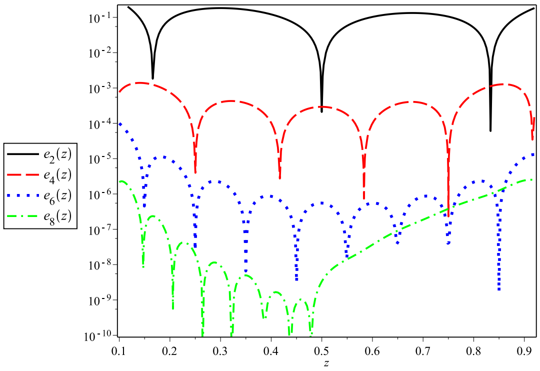

FIGURE 4. Comparison of the absolute error for $\phi(z)$ in problem 6.4

TABLE 6. The numerical calculation of the condition number (C. N), convergence order (ord_N^∞), and time for Example 6.4

N	2	3	4	5	6	7
C. N	0.97870	1.03985	1.04089	1.04066	1.04055	1.04038
ord_N^∞	4.21864	7.92964	11.4750	16.7095	19.6843	22.6842
Time	0.54	0.70	0.89	1.00	1.07	1.15

where

$$h(z) = \frac{\Gamma\left(\frac{23}{4}\right)}{\Gamma\left(\frac{23}{4}-\eta(z)\right)} z^{\frac{19}{4}-\eta(z)} + \frac{\Gamma\left(\frac{36}{5}\right)}{\Gamma\left(\frac{36}{5}-\eta(z)\right)} z^{\frac{31}{5}-\eta(z)} - \frac{16}{621} z^{\frac{27}{4}} - \frac{25}{1476} z^{\frac{41}{5}} - \frac{299}{1107}\sin(z).$$

In this problem, we take $\eta(z) = z$ *and the exact solution is given by* $\phi(z) = z^{\frac{19}{4}} + z^{\frac{31}{5}}$ *[16].*

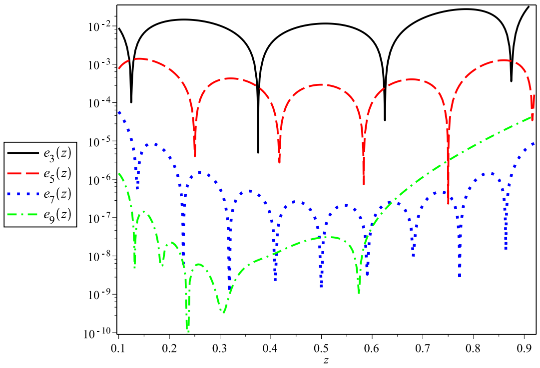

FIGURE 5. Comparison of the absolute error for $\phi(z)$ in problem 6.4

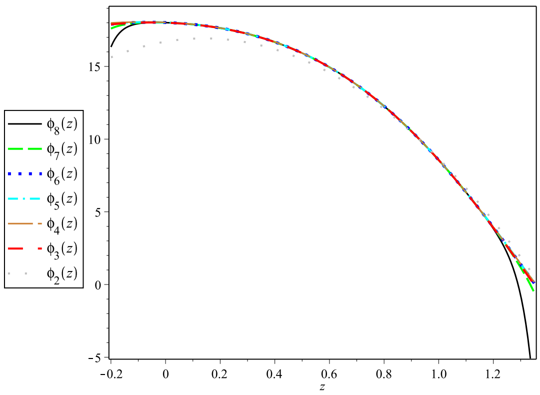

FIGURE 6. Comparison of the exact and approximate solutions of the problem (50)

Figures 7-8 present the graphs of the absolute error functions for various values of N. Figure 9 shows the approximate solutions $\phi_N(z)$ for different values of N. Furthermore, Table 7 compares the absolute error at selected points, obtained using the current method and the methods from [16, 17, 36], all with the same number of basis functions ($N = 7$).

Additionally, the condition number is computed and presented in Table 8 to assess the stability of the problem. This table also includes the CPU time and the convergence order ord_N^∞. Given that the condition number for this problem is less than 5, it indicates that the problem is relatively stable; however, caution should still be exercised regarding rounding errors and computational inaccuracies. This condition number suggests that the problem's output is not *highly sensitive to input variations, and standard computational precision can provide fairly reliable results. Moreover, as seen from the data in the table, the condition number decreases gradually as N increases, indicating that increasing N leads to greater stability of the problem.*

In conclusion, the results demonstrate that the SVLOMT provides high-order accurate solutions, with stable performance as indicated by the computed condition number, CPU time, and convergence order. The comparison of absolute errors across different methods further supports the effectiveness of the proposed approach.

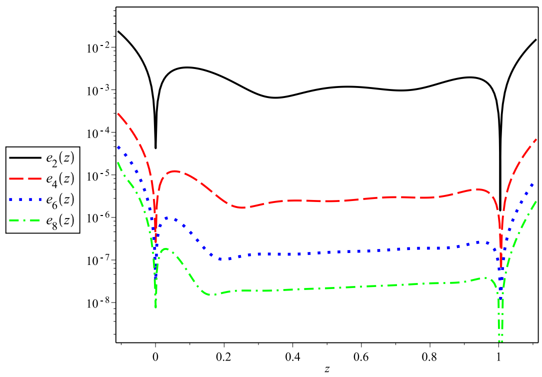

FIGURE 7. Comparison of the absolute error for $\phi(z)$ in problem 6.5

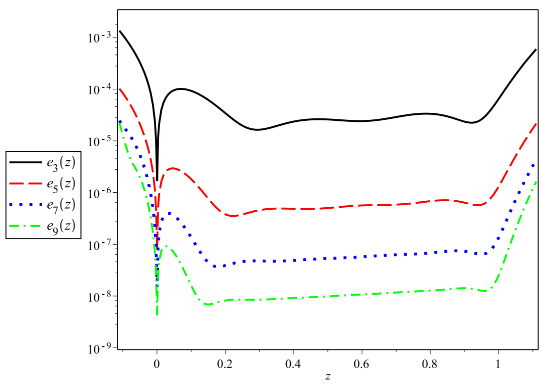

FIGURE 8. Comparison of the absolute error for $\phi(z)$ in problem 6.5

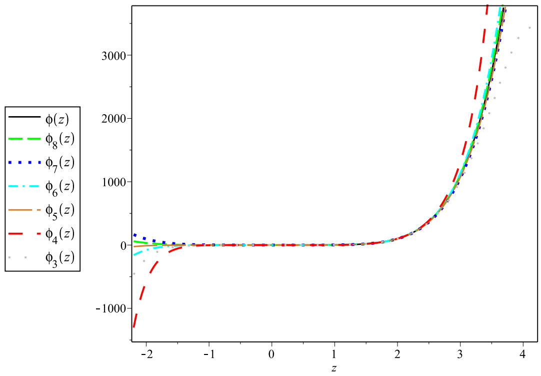

FIGURE 9. Comparison of the exact and approximate solutions of the problem (51)

TABLE 7. The comparison of the absolute error with the methods presented in [17], [16], [36], and the SVLOMT method for $N = 7$, Example 6.5.

z	SVLOMT	Method [17]	Method [16]	Method [36]
0.1	1.37e−07	4.39e−07	7.82e−07	8.25e−04
0.3	4.77e−08	5.54e−07	1.22e−06	9.11e−05
0.5	5.33e−08	3.41e−06	1.66e−07	1.26e−03
0.7	6.27e−08	4.78e−07	7.92e−07	9.58e−03
0.9	7.32e−08	2.72e−07	6.61e−07	8.66e−02

TABLE 8. The numerical calculation of the condition number (C. N), convergence order (ord_N^∞), and time for Example 6.5

N	2	3	4	5	6	7
C. N	5.44855	5.39532	4.91202	4.88266	4.87906	4.87844
ord_N^∞	5.38082	6.79586	9.38082	11.5162	13.0504	16.6540
Time	0.60	0.65	0.96	1.10	1.23	1.52

7. Concluding Remarks

In this article, an operational matrix method based on the shifted Vieta-Lucas polynomials (SVLOMT) and the collocation technique is presented for solving linear (or nonlinear) fractional-variable-order integro-differential equations (FVOIDEs). In the proposed method, all terms in the equation are replaced by corresponding operational matrices, and combined with the collocation technique, this transforms the problem into an algebraic system of equations. The method is particularly straightforward to implement, and its software implementation is relatively simple. Furthermore, with minor modifications, it can also be applied to solve nonlinear equations, as demonstrated in the examples provided.

One of the main advantages of this approach is that, when the exact solution of the equation is in polynomial form, the approximate solution obtained by the proposed method exactly matches the exact solution. Moreover, an error analysis of the SVLOMT was conducted, demonstrating that the method achieves high accuracy and efficiency. The condition number, CPU time, and numerical convergence rate were also calculated, with the results indicating that when the condition number is small, the method exhibits excellent stability. The convergence speed and accuracy of the method are both impressive, highlighting its effectiveness. Stability and consistency analyses further confirmed that the method performs robustly under varying conditions.

In future work, this method could be extended to higher-order equations and more complex nonlinear systems. Optimization for parallel computing may also enhance its efficiency for large-scale problems. Furthermore, its applicability in more intricate real-world systems could be explored, and hybrid approaches combining this method with other numerical techniques could lead to further improvements in both accuracy and stability. In conclusion, the SVLOMT proves to be an effective and powerful tool for solving fractional-variable-order integro-differential equations, making it highly suitable for large-scale and complex scientific and industrial applications.

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