Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

BIHARMONIC HYPERSURFACES IN THE STANDARD LORENTZ 5-PSEUDOSPHERE

G. Haghighatdoost, S. Hoseinpour, F. Pashaie ✉, and L. Shahbaz

Article type: Research Article
(Received: 04 November 2024, Received in revised form 29 January 2025)
(Accepted: 01 March 2025, Published Online: 02 March 2025)

Abstract. In this manuscript, we study the Lorentz hypersurfaces of the Lorentz 5-pseudosphere (i.e. the pseudo-Euclidean 5-sphere) $\mathbb{S}_1^5$ having three distinct principal curvatures. A well-known conjecture of Bang-Yen Chen on Euclidean spaces says that every submanifold is minimal. We consider an advanced version of the conjecture on Lorentz hypersurfaces of $\mathbb{S}_1^5$. We present an affirmative answer to the extended conjecture on Lorentz hypersurfaces with three distinct principal curvatures.

Keywords: C-biharmonic, isoparametric, de-Sitter space, 1-minimal.
2020 MSC: 53A30, 53B30, 53C40, 53C43.

1. Introduction

The category of biharmonic submnifolds plays an important role in the study of minimal submnifolds. The first steps in the field of biharmonic submanifolds have been taken by B.-Y. Chen and G. Y. Jiang ([4,9]). In 1987 Chen claimed (as a conjecture) that every biharmonic submanifold of an Euclidean space has to be minimal. In several cases, Chen conjecture has been proved (see [1,3–8,13,18]).

In this paper, we study an extended version of biharmonicity condition on Lorentz hypersurfaces of 5-dimensional Lorentz pseudo-sphere with some additional conditions. In Section 2, we present notations and preliminary concepts. In section 3, we illustrate some C-biharmonic examples of Lorentz hypersurfaces in Lorentz 5-pseudosphere. In section 4, we have several propositions and theorems on Lorentz hypersurfaces in 5-dimensional Lorentz pseudo-sphere satisfying the C-biharmonicity condition. The case of diagonal shape operator is discussed in Section 4. The non-diagonal shape operator case will be explained in Section 5.

2. Preliminaries

First, we recall necessary notations and concepts from [2,10–12,14,17]. In general, we remember the definition of pseudo-Euclidean k-space $\mathbb{E}_t^k$ of index

✉ f_pashaie@maragheh.ac.ir, ORCID: 0000-0002-3020-7649
https://doi.org/10.22103/jmmr.2025.24319.1716
Publisher: Shahid Bahonar University of Kerman
How to cite: G. Haghighatdoost, S. Hoseinpour, F. Pashaie, L. Shahbaz, *Biharmonic hypersurfaces in the standard Lorentz 5-pseudosphere*, J. Mahani Math. Res. 2025; 14(2): 203-219.

$t \geq 0$, obtained from Euclidean k-space $\mathbb{E}^k$ by considering the non-degenerate scalar product $\langle \mathbf{a}, \mathbf{b} \rangle := -\sum_{i=1}^{t} a_i b_i + \sum_{i=t+1}^{k} a_i b_i$, for every $\mathbf{a}, \mathbf{b} \in \mathbb{E}^k$. We deal with the Lorentz k-space $\mathbb{L}^k := \mathbb{E}_1^k$ and specially with the standard 5-dimensional Lorentz pseudo-sphere $\mathbb{S}_1^5 := \mathbb{S}_1^5(1)$ which are obtained from the following general definition, for positive real number r:

$$\mathbb{M}_1^5(c) = \begin{cases} \mathbb{S}_1^5(r) = \{\mathbf{a} \in \mathbb{L}^6 | \langle \mathbf{a}, \mathbf{a} \rangle = r^2\} & \text{(where } c = 1/r^2) \\ \mathbb{H}_1^5(-r) = \{\mathbf{a} \in \mathbb{E}_2^6 | \langle \mathbf{a}, \mathbf{a} \rangle = -r^2, a_1 > 0\} & \text{(where } c = -1/r^2) \\ \mathbb{L}^5 & \text{(where } c = 0) \end{cases}$$

where $\mathbb{S}_1^5(r)$ denotes the 5-dimensional pseudo-sphere of radius r and curvature $1/r^2$, and $\mathbb{H}_1^5(-r)$ stands for the pseudo-hyperbolic 5-space of radius $-r$ and curvature $-1/r^2$. In the canonical cases $c = \pm 1$, we get the standard 5-dimensional Lorentz pseudo-sphere (i.e. the de Sitter 5-space) $d\mathbb{S}^5 := \mathbb{S}_1^5$ and the standard pseudo-hyperbolic 5-space (i.e. the anti-de-Sitter 5-space) $Ad\mathbb{S}^5 = \mathbb{H}_1^5 := \mathbb{H}_1^5(-1)$.

Clearly, the metric induced from $\mathbb{S}_1^5$ on its each Lorentzian hypersurface M_1^4 (i.e. induced by means of isometric immersion $\mathbf{x} : M_1^4 \to \mathbb{S}_1^5$) is Lorentzian. A given basis $\Omega := \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ of the tangent space of M_1^4 has two possible cases as follows.

Definition 2.1. Let $\Omega := \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ be a basis for the tangent space of Lorentz hypersurface $\mathbf{x} : M_1^4 \to \mathbb{S}_1^5$ in de Sitter 5-space.

(1) Ω is said to be *orthonormal* if it satisfies equalities $\langle \mathbf{w}_1, \mathbf{w}_1 \rangle = -1$, $\langle \mathbf{w}_2, \mathbf{w}_2 \rangle = \langle \mathbf{w}_3, \mathbf{w}_3 \rangle = \langle \mathbf{w}_4, \mathbf{w}_4 \rangle = 1$ and $\langle \mathbf{w}_i, \mathbf{w}_j \rangle = 0$ for each $i \neq j$.

(2) Ω is called *pseudo-orthonormal* if it satisfies $\langle \mathbf{w}_1, \mathbf{w}_1 \rangle = \langle \mathbf{w}_2, \mathbf{w}_2 \rangle = 0$, $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = -1$ and $\langle \mathbf{w}_i, \mathbf{w}_j \rangle = \delta_i^j$ for $i = 1, 2, 3, 4$ and $j = 3, 4$. As well-known, δ is the Kronecker delta.

The induced metric on M_1^4 with respect to an orthonormal basis has the matrix form $\mathcal{M}_1 := \mathrm{diag}[-1, 1, 1, 1]$ and with respect to an pseudo-orthonormal basis it is of form $\mathcal{M}_2 = \mathrm{diag}[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, 1, 1]$.

According to the orthonormal basis, the shape operator of M_1^4 has two possible matrix form $\mathcal{S}_1 = \mathrm{diag}[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$ and $\mathcal{S}_2 = \mathrm{diag}[\begin{bmatrix} \kappa & \lambda \\ -\lambda & \kappa \end{bmatrix}, \eta_1, \eta_2]$, where $\lambda \neq 0$. We note that, when the matrix form of shape operator is $\mathcal{S}_2$, it has two eigenvalues $\kappa \pm i\lambda$ which are complex conjugate.

Now, consider the case that $\mathcal{M}_2$ is the matrix form of metric tensor with respect to a pseudo-orthonormal basis. So, the matrix form of shape operator has to be of forms $\mathcal{S}_3 = \mathrm{diag}[\begin{bmatrix} \kappa & 0 \\ 1 & \kappa \end{bmatrix}, \lambda_1, \lambda_2]$ or $\mathcal{S}_4 = \mathrm{diag}[\begin{bmatrix} \kappa & 0 & 0 \\ 0 & \kappa & 1 \\ -1 & 0 & \kappa \end{bmatrix}, \lambda]$.

Remark 2.2. In the case $\mathcal{M}_2$, we substitute the pseudo-orthonormal basis by a new orthonormal one $\Omega := \{\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2, \mathbf{w}_3, \mathbf{w}_4\}$, where $\tilde{\mathbf{w}}_1 := \frac{1}{2}(\mathbf{w}_1 + \mathbf{w}_2)$ and $\tilde{\mathbf{w}}_2 := \frac{1}{2}(\mathbf{w}_1 - \mathbf{w}_2)$.

Then, we obtain new matrix forms $\mathcal{S}_3 = \mathrm{diag}[\begin{bmatrix} \kappa+\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \kappa-\frac{1}{2} \end{bmatrix}, \lambda_1, \lambda_2]$ and $\mathcal{S}_4 = \mathrm{diag}[\begin{bmatrix} \kappa & 0 & \frac{\sqrt{2}}{2} \\ 0 & \kappa & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \kappa \end{bmatrix}, \lambda]$

As usual, the principal curvatures of M_1^4, denoted by $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ in four possible cases of shape operator are defined as follows. When $S = \mathcal{S}_1$, we take $\kappa_i := \lambda_i$, for $i = 1, \dots, 4$, such that λ_i's are the eigenvalues of $\mathcal{S}_1$. In the case $S = \mathcal{S}_2$, we put $\kappa_1 = \kappa + i\lambda$, $\kappa_2 = \kappa - i\lambda$, and $\kappa_i := \eta_{i-2}$, for $i = 3, 4$. In the case $S = \mathcal{S}_3$, we take $\kappa_i := \kappa$ for $i = 1, 2$, and $\kappa_i := \lambda_{i-2}$, for $i = 3, 4$. Finally, when $S = \mathcal{S}_4$, we take $\kappa_i := \kappa$ for $i = 1, 2, 3$, and $\kappa_4 := \lambda$.

The characteristic polynomial of S on M_1^4 is of the form

$$Q(t) = \prod_{i=1}^{4}(t - \kappa_i) = \sum_{j=0}^{4}(-1)^j s_j t^{4-j},$$

where, $s_0 := 1$, $s_i := \sum_{1\le j_1<\dots<j_i\le 4} \kappa_{j_1} \dots \kappa_{j_i}$ for $i = 1, \dots, 4$.

For $j = 1, \dots, 4$, the *jth mean curvature* H_j of M_1^4 is defined by $H_j = \frac{1}{\binom{4}{j}} s_j$. When H_j is zero, M_1^4 is said to be $(j-1)$-*minimal*. The following function on M_1^4 will be used frequently:

$$\mu_{i;k} = \sum_{1\le j_1<\dots<j_k\le 4; j_l\ne i} \kappa_{j_1} \dots \kappa_{j_k}, \ (i = 1, \dots, 4;\ 1 \le k \le 3).$$

Here, we give the definition of isoparametric hypersurface in two different cases.

Definition 2.3. Consider $\mathbf{x} : M_1^4 \to \mathbb{S}_1^5$ as a timelike (Lorentz) hypersurface in the 5-pseudosphere. Let S be its shape operator.

(1) If S has a diagonal matrix form with constant eigenvalues, then M_1^4 is said to be *isoparametric*.
(2) If S has a non-diagonal matrix form and the minimal polynomial of S is constant, then M_1^4 is said to be *isoparametric*.

Remark 2.4. We note that, M_1^4 cannot be isoparametric in the case that it has complex principal curvatures (see Theorem 4.10 from [12]).

The Newton operator on M_1^4 is defined by

$$\mathcal{N}_0 = I,\ \mathcal{N}_j = s_j I - S \circ \mathcal{N}_{j-1},\ (j = 1, \dots, 4), \tag{1}$$

where I is the identity map. Also, its explicit formula is $\mathcal{N}_j = \sum_{i=0}^{j}(-1)^i s_{j-i} S^i$ (where $S^0 = I$) (see [2,15]).

When $S = \mathcal{S}_1$, we have $\mathcal{N}_j = diag[\mu_{1;j}, \dots, \mu_{4;j}]$, for $j = 1, 2, 3$.

In the case $S = \mathcal{S}_2$, we have $\mathcal{N}_1 = \mathrm{diag}\left[\begin{bmatrix} \kappa+\eta_1+\eta_2 & -\lambda \\ \lambda & \kappa+\eta_1+\eta_2 \end{bmatrix}, 2\kappa+\eta_2, 2\kappa+\eta_1\right]$ and

$$\mathcal{N}_2 = \mathrm{diag}\left[\begin{bmatrix} \kappa(\eta_1+\eta_2)+\eta_1\eta_2 & -\lambda(\eta_1+\eta_2) \\ \lambda(\eta_1+\eta_2) & \kappa(\eta_1+\eta_2)+\eta_1\eta_2 \end{bmatrix}, \kappa^2+\lambda^2+2\kappa\eta_2, \kappa^2+\lambda^2+2\kappa\eta_1\right].$$

When $S=\mathcal{S}_3$, we have

$$\mathcal{N}_1=\mathrm{diag}[\begin{bmatrix}\lambda_1+\lambda_2+\kappa-\frac{1}{2} & -\frac{1}{2}\\ \frac{1}{2} & \lambda_1+\lambda_2+\kappa+\frac{1}{2}\end{bmatrix},2\kappa+\lambda_2,2\kappa+\lambda_1]$$

and

$$\mathcal{N}_2=\mathrm{diag}[\begin{bmatrix}\lambda_1\lambda_2+(\kappa-\frac{1}{2})(\lambda_1+\lambda_2) & -\frac{1}{2}(\lambda_1+\lambda_2)\\ \frac{1}{2}(\lambda_1+\lambda_2) & \lambda_1\lambda_2+(\kappa+\frac{1}{2})(\lambda_1+\lambda_2)\end{bmatrix},\kappa(\kappa+2\lambda_2),\kappa(\kappa+2\lambda_1)].$$

If $S=\mathcal{S}_4$, we have $\mathcal{N}_1=\mathrm{diag}[\begin{bmatrix}2\kappa+\lambda & 0 & -\frac{\sqrt{2}}{2}\\ 0 & 2\kappa+\lambda & \frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2\kappa+\lambda\end{bmatrix},3\kappa]$ and

$$\mathcal{N}_2=\mathrm{diag}[\begin{bmatrix}2\kappa\lambda+\kappa^2-\frac{1}{2} & -\frac{1}{2} & -\frac{\sqrt{2}}{2}(\kappa+\lambda)\\ \frac{1}{2} & 2\kappa\lambda+\kappa^2+\frac{1}{2} & \frac{\sqrt{2}}{2}(\kappa+\lambda)\\ \frac{\sqrt{2}}{2}(\kappa+\lambda) & \frac{\sqrt{2}}{2}(\kappa+\lambda) & 2\kappa\lambda+\kappa^2\end{bmatrix},3\kappa^2].$$

The following identities are recalled from [2, 15].

(2) $\mu_{i,1}=4H_1-\lambda_i,\ \mu_{i,2}=6H_2-\lambda_i\mu_{i,1}=6H_2-4\lambda_iH_1+\lambda_i^2,\ (1\le i\le 4),$

(3) $\mathrm{tr}(\mathcal{N}_1)=12H_1,\ \mathrm{tr}(\mathcal{N}_2)=12H_2,\ \mathrm{tr}(\mathcal{N}_1\circ S)=12H_2,\ \mathrm{tr}(\mathcal{N}_2\circ S)=12H_3,$

(4)
$\mathrm{tr}S^2=4(4H_1^2-3H_2),\ \mathrm{tr}(\mathcal{N}_1\circ S^2)=12(2H_1H_2-H_3),\ \mathrm{tr}(\mathcal{N}_2\circ S^2)=4(4H_1H_3-H_4).$

We consider the Cheng-Yau operator $\mathrm{C}:\mathcal{C}^\infty(M_1^4)\to\mathcal{C}^\infty(M_1^4)$ given by $\mathrm{C}(f)=tr(\mathcal{N}_1\circ\nabla^2 f)$, where, $\nabla^2 f:\chi(M)\to\chi(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f which is defined by $(\nabla^2 f)X=\nabla_X(\nabla f)$ for every smooth vector field X on M_1^3, where $\nabla f=\sharp df$. In other words, $\mathrm{C}(f)$ is given by

(5) $$\mathrm{C}(f)=\sum_{i=1}^{4}\epsilon_i\mu_{i,1}(e_ie_if-\nabla_{e_i}e_if).$$

For a Lorentzian hypersurface $\mathbf{x}:M_1^4\to\mathbb{S}_1^5$ we have

(6) $$\mathrm{C}\mathbf{x}=12H_2\mathbf{n}-12H_1\mathbf{x},$$

$$\begin{aligned}\mathrm{C}^2\mathbf{x}&=24\left(\mathcal{N}_2\nabla H_2-\mathcal{N}_1\nabla H_1-9H_2\nabla H_2\right)\\&+12[\mathrm{C}H_2-12H_2(2H_1H_2-H_3)-12H_1H_2]\mathbf{n}\\&-12c[\mathrm{C}H_1-12(H_2^2+H_1^2)]\mathbf{x}.\end{aligned}$$

Definition 2.5. A hypersurface $\mathbf{x}:M_1^4\to\mathbb{S}_1^5$ is said to be *strongly* C-*biharmonic* if it satisfies the condition $\mathrm{C}^2\mathbf{x}=0$. It is said to be C-*biharmonic* if it satisfies the following conditions

(7) $$\begin{aligned}&\text{(i) } \mathcal{N}_2\nabla H_2-\mathcal{N}_1\nabla H_1=9H_2\nabla H_2\\&\text{(ii) } \mathrm{C}H_2=12H_2(2H_1H_2-H_3)+12H_1H_2.\end{aligned}$$

3. Some examples

Here, we give some examples of C-biharmonic Lorentz hypersurfaces in $\mathbb{S}^5_1$, with constant first and second mean curvatures (see [15, 16]).

Example 3.1. Let $0 < r < 1$ and $\Theta_1 = \mathbb{S}^4_1(r) \subset \mathbb{S}^5_1$ defined as

$$\Theta_1 = \{(y_1, \ldots, y_6) \in \mathbb{L}^6 | - y_1^2 + \sum_{i=2}^{5} y_i^2 = r^2, y_6 = \sqrt{1-r^2}\},$$

having the normal vector field

$$\mathbf{n}(y) = \frac{\sqrt{1-r^2}}{r}(y_1, \ldots, y_5, 0) + \frac{-r}{\sqrt{1-r^2}}(0, \ldots, 0, y_6)$$

as the Gauss map. Clearly, it has only one principal curvature of multiplicity 4 as $\kappa_1 = \ldots = \kappa_4 = \frac{-\sqrt{1-r^2}}{r}$,. One can see that Θ_1 is C-biharmonic and all of its mean curvatures are constant.

Example 3.2. Let $0 < r < 1$ and $\Theta_2 = \mathbb{S}^3_1(r) \times \mathbb{S}^1(\sqrt{1-r^2}) \subset \mathbb{S}^5_1$ defined as

$$\Theta_2 = \{(y_1, \ldots, y_6) \in \mathbb{L}^6 | - y_1^2 + y_2^2 + y_3^2 + y_4^2 = r^2, y_5^2 + y_6^2 = 1 - r^2\},$$

having the normal vector field

$$\mathbf{n}(y) = \frac{\sqrt{1-r^2}}{r}(y_1, y_2, y_3, y_4, 0, 0) + \frac{-r}{\sqrt{1-r^2}}(0, 0, 0, 0, y_5, y_6)$$

as the Gauss map. Clearly, it has two distinct principal curvatures $\kappa_1 = \kappa_2 = \kappa_3 = \frac{-\sqrt{1-r^2}}{r}$ and $\kappa_4 = \frac{r}{\sqrt{1-r^2}}$. One can see that Θ_2 is C-biharmonic and all of its mean curvatures are constant.

Example 3.3. Let $0 < r < 1$ and $\Theta_3 = \mathbb{S}^2_1(r) \times \mathbb{S}^2(\sqrt{1-r^2}) \subset \mathbb{S}^5_1$ defined as

$$\Theta_3 = \{(y_1, \ldots, y_6) \in \mathbb{L}^6 | - y_1^2 + y_2^2 + y_3^2 = r^2, y_4^2 + y_5^2 + y_6^2 = 1 - r^2\},$$

having the normal vector field

$$\mathbf{n}(y) = \frac{\sqrt{1-r^2}}{r}(y_1, y_2, y_3, 0, 0, 0) + \frac{-r}{\sqrt{1-r^2}}(0, 0, 0, y_4, y_5, y_6)$$

as the Gauss map. Clearly, it has two distinct principal curvatures $\kappa_1 = \kappa_2 = \frac{-\sqrt{1-r^2}}{r}$ and $\kappa_3 = \kappa_4 = \frac{r}{\sqrt{1-r^2}}$. One can see that Θ_3 is C-biharmonic and all of its mean curvatures are constant.

Example 3.4. Let $0 < r, t < 1$, $r^2 + s^2 < 1$ and $\Theta_4 = \mathbb{S}^1_1(r) \times \mathbb{S}^1(t) \times \mathbb{S}^1(\sqrt{1-r^2-t^2}) \subset \mathbb{S}^5_1$ defined as

$$\Theta_4 = \{(y_1, \ldots, y_6) \in \mathbb{L}^6 | - y_1^2 + y_2^2 = r^2, y_3^2 + y_4^2 = t^2, y_5^2 + y_6^2 = 1 - r^2 - t^2\},$$

having three distinct principal curvatures $\kappa_1 = \frac{-\sqrt{1-r^2-t^2}}{r}$, $\kappa_2 = \frac{\sqrt{1-r^2-t^2}}{t}$ and $\kappa_3 = \kappa_4 = \frac{\sqrt{r^2+t^2}}{\sqrt{1-r^2-t^2}}$. Clearly, Θ_4 is C-biharmonic and all of its mean curvatures are constant.

Example 3.5. Let $\mathbf{v} \in \mathbb{L}^6$ be a spacelike unit constant vector. For each $\sigma \in (-1, 1)$, we consider the subset

$$\Phi_\sigma := \{\mathbf{p} \in \mathbb{S}^5_1 : \langle \mathbf{p}, \mathbf{v} \rangle = \sigma\}.$$

Φ_σ is a totally umbilical hypersurface in $\mathbb{S}^5_1$ with Gauss map $\mathbf{n}(\mathbf{p}) = \frac{1}{\sqrt{1-\sigma^2}}(\mathbf{v} - \sigma\mathbf{p})$ and shape operator $S = \frac{\sigma}{\sqrt{1-\sigma^2}} I$. Φ_σ is isometric to $\mathbb{S}^4_1(\sqrt{1-\sigma^2})$. So, it is C-biharmonic.

4. C-biharmonic hypersurfaces with diagonal shape operator

The main focus in this section is on Lorentz hypersurfaces of $\mathbb{S}^5_1$, whose shape operator is assumed to be diagonal. In this case, we try to confirm a modified version of conjecture.

Proposition 4.1. We consider a Lorentz hypersurface $\mathbf{x} : M^4_1 \to \mathbb{S}^5_1$ satisfying the following conditions:
(a) The shape operator of M^4_1 is a diagonal matrix of form $\mathcal{S}_1$,
(b) The ordinary mean curvature of M^4_1 is constant,
(c) The second mean curvature of M^4_1 is non-constant.
If M^4_1 is C-biharmonic, then it has a nonconstant principal curvature of multiplicity one.

Proof. We take the open subset $\mathcal{K} \subset M^4_1$ consisting of points in M^4_1, at which ∇H_2 is non-zero. By conditions (7)(i), taking $\mathbf{w}_1 := \frac{\nabla H_2}{||\nabla H_2||}$, we get $\mathcal{N}_2\mathbf{w}_1 = 9H_2\mathbf{w}_1$ on $\mathcal{K}$. Clearly, one can choose a suitable orthonormal local basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ of principal directions on M^4_1. So, we have $S\mathbf{w}_i = \kappa_i\mathbf{w}_i$ and $\mathcal{N}_2\mathbf{w}_i = \mu_{i,2}\mathbf{w}_i$, for $i = 1, 2, 3, 4$. Then, we get

$$\mu_{1,2} = 9H_2. \tag{8}$$

By canonic decomposition $\nabla H_2 = \sum_{i=1}^{4} \mathbf{w}_i(H_2)\mathbf{w}_i$, we get

$$\mathbf{w}_1(H_2) \neq 0,\ \mathbf{w}_2(H_2) = \mathbf{w}_3(H_2) = \mathbf{w}_4(H_2) = 0. \tag{9}$$

From (2) and (8) we get

$$H_2 = \frac{1}{3}\kappa_1(\kappa_1 - 4H_1). \tag{10}$$

So, since H_1 is assumed to be constant, from (9) we get

$$\mathbf{w}_1(\kappa_1) \neq 0,\ \mathbf{w}_2(\kappa_1) = \mathbf{w}_3(\kappa_1) = \mathbf{w}_4(\kappa_1) = 0, \tag{11}$$

which gives that κ_1 is non-constant. Now, using $\nabla_{\mathbf{w}_i}\mathbf{w}_j = \sum_{k=1}^4 \omega^k_{ij}\mathbf{w}_k$ for $j, i = 1, 2, 3, 4$, the identity $\mathbf{w}_k < \mathbf{w}_i, \mathbf{w}_j >= 0$ gives $\epsilon_j\omega^j_{ki} = -\epsilon_i\omega^i_{kj}$ for $k, j, i = 1, 2, 3, 4$. On the other hand, for distinct $k, j, i = 1, 2, 3, 4$, the Codazzi equation gives

$$\mathbf{w}_i(\kappa_j) = (\kappa_i - \kappa_j)\omega^j_{ji},\ (\kappa_i - \kappa_j)\omega^j_{ki} = (\kappa_k - \kappa_j)\omega^j_{ik}. \tag{12}$$

Since from (11) we get $\mathbf{w}_1(\kappa_1) \neq 0$, so we can claim $\kappa_j \neq \kappa_1$ for $j = 2, 3, 4$. Assuming $\kappa_j = \kappa_1$ for some integer $j \neq 1$, we have $\mathbf{w}_1(\kappa_j) = \mathbf{w}_1(\kappa_1) \neq 0$. On the other hand, from (12) we obtain $0 = (\kappa_1 - \kappa_j)\omega_{j1}^j = \mathbf{w}_1(\kappa_j) = \mathbf{w}_1(\kappa_1)$. Hence, we have gotten a contradiction. □

The ordinary versions of Proposition 4.1 may be seen in [7, 18].

Proposition 4.2. Suppose that M_1^4 be an orientable timelike hypersurface of $\mathbb{S}_1^5$ with shape operator of form $\mathcal{S}_1$, constant mean curvature, non-constant second mean curvature and exactly three distinct principal curvatures. If M_1^4 is C-biharmonic, then the following equalities occur according to the orthonormal tangent frame $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ of principal directions of M_1^4 with associated principal curvatures $\kappa_1, \kappa_2 = \kappa_3, \kappa_4$, which satisfy the following equalities:

(13)

$$(i)\nabla_{\mathbf{w}_1}\mathbf{w}_1 = 0,\ \nabla_{\mathbf{w}_2}\mathbf{w}_1 = \alpha\mathbf{w}_2,\ \nabla_{\mathbf{w}_3}\mathbf{w}_1 = \alpha\mathbf{w}_3,\ \nabla_{\mathbf{w}_4}\mathbf{w}_1 = -\beta\mathbf{w}_4,$$
$$(ii)\nabla_{\mathbf{w}_2}\mathbf{w}_2 = -\alpha\mathbf{w}_1 + \omega_{22}^3\mathbf{w}_3 + \gamma\mathbf{w}_4,\ \nabla_{\mathbf{w}_i}\mathbf{w}_2 = \omega_{i2}^3\mathbf{w}_3 \ for\ i = 1, 3, 4\ ;$$
$$(iii)\nabla_{\mathbf{w}_3}\mathbf{w}_3 = -\alpha\mathbf{w}_1 - \omega_{32}^3\mathbf{w}_3 + \gamma\mathbf{w}_4,\ \nabla_{\mathbf{w}_i}\mathbf{w}_3 = -\omega_{i2}^3\mathbf{w}_2 \ for\ i = 1, 2, 4\ ,$$
$$(iv)\nabla_{\mathbf{w}_1}\mathbf{w}_4 = 0,\ \nabla_{\mathbf{w}_2}\mathbf{w}_4 = -\gamma\mathbf{w}_2,\ \nabla_{\mathbf{w}_3}\mathbf{w}_4 = -\gamma\mathbf{w}_3,\ \nabla_{\mathbf{w}_4}\mathbf{w}_4 = \beta\mathbf{w}_1,$$

where $\alpha := \frac{\mathbf{w}_1(\kappa_2)}{\kappa_1 - \kappa_2}$, $\beta := \frac{\mathbf{w}_1(\kappa_1 + 2\kappa_2)}{\kappa_1 - \kappa_4}$, $\gamma := \frac{\mathbf{w}_4(\kappa_2)}{\kappa_2 - \kappa_4}$.

Proof. We consider the local orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ of TM as the previous proposition. So, the equations $(8) - (12)$ occur and κ_1 is of multiplicity one and we get

$$[\mathbf{w}_2, \mathbf{w}_3](\kappa_1) = [\mathbf{w}_3, \mathbf{w}_4](\kappa_1) = [\mathbf{w}_2, \mathbf{w}_4](\kappa_1) = 0,$$

which gives

$$\omega_{23}^1 = \omega_{32}^1,\ \omega_{34}^1 = \omega_{43}^1,\ \omega_{24}^1 = \omega_{42}^1. \quad (14)$$

Since, by assumption, there exist 3 distinct principal curvatures, we can take $\kappa_2 = \kappa_3$. So, we have $\kappa_4 = 4H_1 - \kappa_1 - 2\kappa_2$. Therefore, by (12) for distinct i, j and k less than 5, we have $\mathbf{w}_2(\kappa_2) = \mathbf{w}_3(\kappa_2) = 0$ and then,

(15)

$$(i)\ \omega_{11}^1 = \omega_{12}^1 = \omega_{13}^1 = \omega_{14}^1 = \omega_{31}^2 = \omega_{21}^3 = \omega_{34}^2 = \omega_{24}^3 = \omega_{42}^4 = \omega_{43}^4 = 0,$$
$$(ii)\ \omega_{21}^2 = \omega_{31}^3 = \frac{\mathbf{w}_1(\kappa_2)}{\kappa_1 - \kappa_2},\ \omega_{41}^4 = \frac{-\mathbf{w}_1(\kappa_1 + 2\kappa_2)}{\kappa_1 - \kappa_4},\ \omega_{24}^2 = \omega_{34}^3 = \frac{-\mathbf{w}_4(\kappa_2)}{\kappa_2 - \kappa_4},$$
$$(iii)\ (\kappa_1 - \kappa_4)\omega_{24}^1 = (\kappa_1 - \kappa_2)\omega_{42}^1,\ (\kappa_1 - \kappa_4)\omega_{34}^1 = (\kappa_1 - \kappa_2)\omega_{43}^1.$$

From (14) and (15) we get $\omega_{24}^1 = \omega_{42}^1, = \omega_{34}^1 = \omega_{43}^1 = \omega_{12}^4 = \omega_{13}^4 = 0$. All claimed equalities can be gotten from the last results. □

Proposition 4.3. Assume M_1^4 to be an orientable timelike hypersurface in $\mathbb{S}_1^5$ with shape operator of form $\mathcal{S}_1$, three distinct principal curvatures, non-constant second mean curvature and constant ordinary mean curvature. Let

M_1^4 be C-biharmonic. Then, the principal curvatures $\kappa_1, \kappa_2 = \kappa_3, \kappa_4$ associated to its orthonormal (local) tangent frame $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ of principal directions satisfy the equalities $\mathbf{w}_4(\kappa_2) = 0$ and

$$\mathbf{w}_1(\kappa_2)\mathbf{w}_1(\kappa_1 + 2\kappa_2) = \frac{1}{2}\kappa_2(\kappa_1 - \kappa_2)(\kappa_4 - \kappa_1)(2\kappa_1 + 4\kappa_2 + \kappa_4). \tag{16}$$

Proof. First, we recall the Gauss curvature tensor formula for every tangent vector fields V, W and Z as $R(V,W)Z = \nabla_V \nabla_W Z - \nabla_W \nabla_V Z - \nabla_{[V,W]} Z$.

By putting different choices of $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ and $\mathbf{w}_4$ instead of vector fields V, W and Z and applying Proposition 4.2, we get

$$\begin{aligned}
&(i)\ \mathbf{w}_1(\alpha) + \alpha^2 = -\kappa_1\kappa_2,\ \beta^2 - \mathbf{w}_1(\beta) = -\kappa_1\kappa_4;\\
&(ii)\ \mathbf{w}_1\left(\frac{\mathbf{w}_4(\kappa_2)}{\kappa_2 - \kappa_4}\right) + \alpha\frac{\mathbf{w}_4(\kappa_2)}{\kappa_2 - \kappa_4} = 0;\\
&(iii)\ \mathbf{w}_4(\alpha) - (\alpha + \beta)\frac{\mathbf{w}_4(\kappa_2)}{\kappa_2 - \kappa_4} = 0;\\
&(iv)\ \beta^2 - \mathbf{w}_1(\beta) = -\kappa_1\kappa_4;\\
&(v)\ \mathbf{w}_4\left(\frac{\mathbf{w}_4(\kappa_2)}{\kappa_2 - \kappa_4}\right) + \alpha\beta - \left(\frac{\mathbf{w}_4(\kappa_2)}{\kappa_2 - \kappa_4}\right)^2 = \kappa_2\kappa_4,
\end{aligned} \tag{17}$$

where $\alpha := \frac{\mathbf{w}_1(\kappa_2)}{\kappa_1 - \kappa_2}$ and $\beta := \frac{\mathbf{w}_1(\kappa_1 + 2\kappa_2)}{\kappa_1 - \kappa_4}$. Also, from (5) and (7), by using the result of Proposition (4.2) we get

$$\begin{aligned}
&(\kappa_1 - 4H_1)\mathbf{w}_1\mathbf{w}_1(H_2) - (2(\kappa_2 - 4H_1)\alpha + (\kappa_1 + 2\kappa_2)\beta)\mathbf{w}_1(H_2)\\
&= 12H_2(2H_1H_2 - H_3).
\end{aligned} \tag{18}$$

On the other hand, using (9) and (13), we have

$$\mathbf{w}_i\mathbf{w}_1(H_{k+1}) = 0, \tag{19}$$

for $i = 2, 3, 4$. Also, from derivation of α and β according to $\mathbf{w}_4$, we get

$$(\kappa_1 - \kappa_2)\mathbf{w}_4(\alpha) - \alpha\mathbf{w}_4(\kappa_2) = \mathbf{w}_4\mathbf{w}_1(\kappa_2) = \frac{1}{2}(\kappa_1 - \kappa_4)\mathbf{w}_4(\beta) + \beta\mathbf{w}_4(\kappa_2),$$

then

$$\frac{1}{2}(\kappa_1 - \kappa_4)\mathbf{w}_4(\beta) = (\kappa_1 - \kappa_2)\mathbf{w}_4(\alpha) - (\alpha + \beta)\mathbf{w}_4(\kappa_2),$$

which, using the value of $\mathbf{w}_4(\alpha)$ from (17), gives

$$\mathbf{w}_4(\beta) = \frac{-8\mathbf{w}_4(\kappa_2)(\alpha + \beta)(\kappa_2 - H_1)}{(\kappa_1 - \kappa_4)(\kappa_2 - \kappa_4)}.$$

Again, differentiating (18)along $\mathbf{w}_4$ and using (19), (17) and the last value of $\mathbf{w}_4(\beta)$, we get $\mathbf{w}_4(\kappa_2) = 0$ or

$$\frac{4(\alpha + \beta)[-H_1(8\kappa_1 + 12\kappa_2) + {\kappa_1}^2 + 3\kappa_1\kappa_2 + 16H_1^2]\mathbf{w}_1(H_2)}{\kappa_4 - \kappa_1} + 6H_2(\kappa_2 - \kappa_4)^2 = 0. \tag{20}$$

Finally, we claim that $\mathbf{w}_4(\kappa_2) = 0$.

Indeed, if the claim is not true, then we obtain

$$\frac{4(\alpha+\beta)\gamma \mathbf{w}_1(H_2)}{\kappa_1-\kappa_4} = 6H_2(\kappa_2-\kappa_4)^2, \tag{21}$$

where $\gamma = -8H_1\kappa_1 + {\kappa_1}^2 + 3\kappa_1\kappa_2 - 12H_1\kappa_2 + 16H_1^2$. Differentiating (21) along $\mathbf{w}_4$, we get

$$\frac{2(\alpha+\beta)\left[6\gamma(\kappa_2-H_1) + (3\kappa_1-12H_1)(\kappa_1+\kappa_2-2H_1)(\kappa_1+3\kappa_2-4H_1)\right]\mathbf{w}_1(H_2)}{(\kappa_1+\kappa_2-2H_1)^2} = 36H_2(4H_1+\kappa_1+3\kappa_2)^2. \tag{22}$$

Eliminating $\mathbf{w}_1(H_2)$ from (21) and (22), we obtain

$$\gamma(2\kappa_1-2H_1) = (\kappa_1-4H_1)(\kappa_1+\kappa_2-2H_1)(-4H_1+\kappa_1+3\kappa_2). \tag{23}$$

Also, we differentiate (23) along $\mathbf{w}_4$ which gives $4H_1 = \kappa_1$. This is impossible since κ_1 is nonconstant. So, $\mathbf{w}_4(\kappa_2) = 0$. The main result can be gotten from equation (17). □

Theorem 4.4. Suppose that M_1^4 is an orientable timelike hypersurface in $\mathbb{S}_1^5$ with shape operator of form $\mathcal{S}_1$, three distinct principal curvatures and constant ordinary mean curvature. If M_1^4 is C-biharmonic, then it has to be 1-minimal.

Proof. Assume that $\kappa_1, \kappa_2 = \kappa_3, \kappa_4$ are the distinct principal curvatures of M_1^4 according to its orthonormal (local) tangent frame $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ of principal directions. In the first step, we claim that H_2 is constant on M_1^4. Let H_2 be non-constant on an open subset $\mathcal{W}$ of M_1^4. We try to get a contradiction. By derivation of (10) in the direction of $\mathbf{w}_1$ and using the notation $\beta = \frac{\mathbf{w}_1(\kappa_1+2\kappa_2)}{\kappa_1-\kappa_4}$, we get

$$\mathbf{w}_1(H_2) = \frac{4}{3}(2H_1-\kappa_1)\mathbf{w}_1(\kappa_2) + \frac{4}{3}(\kappa_1+\kappa_2-2H_1)(\kappa_1-2H_1)\beta. \tag{24}$$

By Proposition 4.3 and equalities (17), from (24) we obtain

$$\begin{aligned} \mathbf{w}_1\mathbf{w}_1(H_2) &= \frac{4}{3}\kappa_1\kappa_2(\kappa_1-\kappa_2)(\kappa_1+2H_1) \\ &+ \frac{4}{3}(4H_1-\kappa_1-2\kappa_2)(\kappa_1-2H_1)(4\kappa_1\kappa_2+{\kappa_1}^2-4H_1\kappa_2-2H_1\kappa_1) \\ &+ \left[3\beta-4\alpha+2\frac{(\kappa_1+\kappa_2-2H_1)\beta-(\kappa_1-\kappa_2)\alpha}{\kappa_1-2H_1}\right]\mathbf{w}_1(H_2). \end{aligned} \tag{25}$$

Combining (18) and (25), we get

$$(P_{1,2}\alpha + P_{2,2}\beta)\mathbf{w}_1(H_2) = P_{3,6}, \tag{26}$$

where the polynomials $P_{3,6}$, $P_{2,2}$ and $P_{1,2}$ in terms κ_1 and κ_2 are of degrees 6, 2 and 2, respectively.

Derivation of (26) in direction $\mathbf{w}_1$ and equalities (16), (17)(i) and (26), give

$$P_{4,8}\alpha + P_{5,8}\beta = P_{6,5}\mathbf{w}_1(H_2). \tag{27}$$

The polynomials $P_{6,5}$, $P_{4,8}$ and $P_{5,8}$ in terms κ_1 and κ_2 are of degrees 5, 8 and 8, respectively.

By (24) and (27), we get

$$\begin{aligned}&\left(P_{4,8}+\frac{4}{3}P_{6,5}(\kappa_1-\kappa_2)(\kappa_1-2H_1)\right)\alpha\\&+\left(P_{5,8}-\frac{4}{3}P_{6,5}(\kappa_1+\kappa_2-2H_1)(\kappa_1-2H_1)\right)\beta=0.\end{aligned}\tag{28}$$

Also, by Proposition 4.3, from equalities (24) and (26), we obtain

$$P_{2,2}(\kappa_1+\kappa_2-2H_1)(\kappa_1-2H_1)\beta^2-P_{1,2}(\kappa_1-\kappa_2)(\kappa_1-2H_1)\alpha^2=\zeta,\tag{29}$$

where ζ is given by

$$\zeta=\kappa_2(4H_1-\kappa_1-2\kappa_2)(\kappa_1-2H_1)\left(\frac{\hat{a}}{\hat{a}}P_{2,2}(\kappa_1-\kappa_2)-P_{1,2}(\kappa_1+\kappa_2-2H_1)\right)+\frac{3}{4}P_{3,6}.$$

Using Proposition 4.3 and equality (28), we get

$$\begin{aligned}\alpha^2&=\frac{\frac{2}{3}P_{6,5}(\kappa_1-\kappa_4)(\kappa_1-2H_1)+P_{5,8}}{P_{4,8}+\frac{4}{3}P_{6,5}(\kappa_1-\kappa_2)(\kappa_1-2H_1)}\kappa_2\kappa_4,\\\beta^2&=\frac{\frac{4}{3}P_{6,5}(\kappa_1-\kappa_2)(\kappa_1-2H_1)-P_{4,8}}{P_{5,8}-\frac{2}{3}P_{6,5}(\kappa_1-\kappa_4)(\kappa_1-2H_1)}\kappa_2\kappa_4.\end{aligned}\tag{30}$$

From (29) and (32), we obtain a polynomial of degree 22 as follow.

$$\begin{aligned}&-\kappa_2\kappa_4(\kappa_1+2H_1)(\kappa_2-\kappa_1)P_{1,2}\left(P_{5,8}-\frac{2}{3}P_{6,5}(\kappa_1-\kappa_4)(\kappa_1-2H_1)\right)^2\\&-\frac{1}{2}\kappa_2\kappa_4(\kappa_1+2H_1)(\kappa_1-\kappa_4)P_{2,2}\left(P_{4,8}+\frac{4}{3}P_{6,5}(\kappa_1-\kappa_2)(\kappa_1-2H_1)\right)^2\\&=\zeta\left(P_{5,8}-\frac{2}{3}P_{6,5}(\kappa_1-\kappa_4)(\kappa_1-2H_1)\right)\left(P_{4,8}+\frac{4}{3}P_{6,5}(\kappa_1-\kappa_2)(\kappa_1-2H_1)\right),\end{aligned}\tag{31}$$

We choose an integral curve of $\mathbf{w}_1$ as $\gamma(t)$ (where $t\in I$) passing through $p=\gamma(t_0)$. So, $\mathbf{w}_1(\kappa_1)$ and $\mathbf{w}_1(\kappa_2)$ are nonzero and for $i=2,3,4$ we have $\mathbf{w}_i(\kappa_1)=\mathbf{w}_i(\kappa_2)=0$. We take $\kappa_2=\kappa_2(t)$ and $\kappa_1=\kappa_1(\kappa_2)$ in some neighborhood of $\kappa_0=\kappa_2(t_0)$. Using (28), we have

$$\begin{aligned}\frac{d\kappa_1}{d\kappa_2}&=\frac{d\kappa_1}{dt}\frac{dt}{d\kappa_2}=\frac{\mathbf{w}_1(\kappa_1)}{\mathbf{w}_1(\kappa_2)}\\&=2\frac{(\kappa_1+\kappa_2-2H_1)\beta-(\kappa_1-\kappa_2)\alpha}{(\kappa_1-\kappa_2)\alpha}\\&=\frac{2\left(P_{4,8}+\frac{4}{3}P_{6,5}(\kappa_1-\kappa_2)(\kappa_1-2H_1)\right)(\kappa_1+\kappa_2-2H_1)}{\left(\frac{4}{3}P_{6,5}(\kappa_1+\kappa_2-2H_1)(\kappa_1-2H_1)-P_{5,8}\right)(\kappa_1-\kappa_2)}-2\end{aligned}\tag{32}$$

Now, we differentiate (31) with respect to κ_2 and then substitute $\frac{d\kappa_1}{d\kappa_2}$ from (32), which gives

$$f(\kappa_1, \kappa_2) = 0, \tag{33}$$

where $f(\kappa_1, \kappa_2)$ is an algebraic polynomial of degree 30 in terms of κ_1 and κ_2.

By resorting polynomials (31) and (33) in terms of κ_2, we get two power series equations as:

$$\begin{aligned} &\text{(i)} \sum_{n=0}^{22} f_n(\kappa_1)\kappa_2^n = 0, \\ &\text{(ii)} \sum_{m=0}^{30} g_m(\kappa_1)\kappa_2^m = 0. \end{aligned} \tag{34}$$

By eliminating κ_2^{30} between (34)(i) and (34)(ii) we get a degree 29 polynomial equation in terms κ_2. By combining obtained equation with (34)(i), we get degree 28 polynomial equation in terms κ_2. By continuing this method, from (34)(i) and its consequences we can eliminate κ_2. In final, we obtain a non-trivial algebraic polynomial equation in κ_1 with constant coefficients which implies that κ_1 is constant and then by (10), κ_2 and H_2 are constants, which contradicts with the first assumption. So, H_2 is constant on M_1^4.

Now, we prove $H_2 = 0$. Assuming $H_2 \neq 0$, from condition (7)(ii), we get that H_3 is constant. So, M_1^4 is isoparametric. But, from Corollary 2.7 in [12], we know that every isoparametric timelike hypersurface of type $\mathcal{S}_1$ has at most one nonzero principal curvature. Then, we have a contradiction with the assumption of having three distinct principal curvatures. So $H_2 \equiv 0$. □

5. Three Cases of shape operator with non-diagonal matrix

A timelike hypersurface of $\mathbb{S}_1^5$, whose shape operator has non-diagonal matrix form, can satisfy the extended conjecture if it has at most three distinct principal curvatures and constant mean curvature. First, we consider the case that the matrix of shape operator of form $\mathcal{S}_2$. In this case, the modified conjecture will be confirmed.

Theorem 5.1. Suppose that an orientable timelike hypersurface $x : M_1^4 \to \mathbb{S}_1^5$ has shape operator of matrix form $\mathcal{S}_2$, also suppose that it has one real constant principal curvature and constant mean curvature. If M_1^4 is C-biharmonic, then it has a constant second mean curvature. Furthermore, either M_1^4 is 1-minimal or it is isoparametric and 3-minimal.

Proof. We have to show the constancy of H_2. In fact, we prove the emptiness of $\mathcal{U} = \{p \in M_1^4 : \nabla H_2^2(p) \neq 0\}$. By assumption $\mathcal{U} \neq \emptyset$, we try to get a contradiction. Using the matrix form $\mathcal{S}_2$ for S with respect to a suitable (local) orthonormal tangent frame $\{\mathbf{w}_1, \ldots, \mathbf{w}_4\}$, we have $S\mathbf{w}_1 = \kappa\mathbf{w}_1 - \lambda\mathbf{w}_2$, $S\mathbf{w}_2 = \lambda\mathbf{w}_1 + \kappa\mathbf{w}_2$, $S\mathbf{w}_3 = \eta_1\mathbf{w}_3$, $S\mathbf{w}_4 = \eta_2\mathbf{w}_4$ and then, the 2nd Newton

transformation satisfies:
$\mathcal{N}_2\mathbf{w}_1 = [\kappa(\eta_1+\eta_2)+\eta_1\eta_2]\mathbf{w}_1 + \lambda(\eta_1+\eta_2)\mathbf{w}_2$,
$\mathcal{N}_2\mathbf{w}_2 = -\lambda(\eta_1+\eta_2)\mathbf{w}_1 + [\kappa(\eta_1+\eta_2)+\eta_1\eta_2]\mathbf{w}_2$,
$\mathcal{N}_2\mathbf{w}_3 = (\kappa^2+\lambda^2+2\kappa\eta_2)\mathbf{w}_3$ and $\mathcal{N}_2\mathbf{w}_4 = (\kappa^2+\lambda^2+2\kappa\eta_1)\mathbf{w}_4$.

Using the canonic decomposition $\nabla H_2 = \sum_{i=1}^{4} \epsilon_i \mathbf{w}_i(H_2)\mathbf{w}_i$, by (7)(i) we get

$$(35)\quad \begin{aligned} &(i)\ (\kappa\eta_1+\kappa\eta_2+\eta_1\eta_2-9H_2)\epsilon_1\mathbf{w}_1(H_2) - \lambda(\eta_1+\eta_2)\epsilon_2\mathbf{w}_2(H_2) = 0,\\ &(ii)\ \lambda(\eta_1+\eta_2)\epsilon_1\mathbf{w}_1(H_2) + (\kappa\eta_1+\kappa\eta_2+\eta_1\eta_2-9H_2)\epsilon_2\mathbf{w}_2(H_2) = 0,\\ &(iii)\ (\kappa^2+\lambda^2+2\kappa\eta_2-9H_2)\epsilon_3\mathbf{w}_3(H_2) = 0,\\ &(iv)\ (\kappa^2+\lambda^2+2\kappa\eta_1-9H_2)\epsilon_4\mathbf{w}_4(H_2) = 0. \end{aligned}$$

Now, by assumption the constancy of H_1 and η_1, we prove a simple claim as $\mathbf{w}_i(H_2) = 0$ for $i = 1,2,3,4$.
If $\mathbf{w}_1(H_2) \neq 0$, dividing equalities (35)(i) and (35)(ii) by $\epsilon_1\mathbf{w}_1(H_2)$ and putting $u := \frac{\epsilon_2\mathbf{w}_2(H_2)}{\epsilon_1\mathbf{w}_1(H_2)}$ we get

$$(36)\quad \begin{aligned} &(i)\ \kappa(\eta_1+\eta_2)+\eta_1\eta_2-9H_2 = \lambda(\eta_1+\eta_2)u,\\ &(ii)\ (\kappa(\eta_1+\eta_2)+\eta_1\eta_2-9H_2)u = -\lambda(\eta_1+\eta_2), \end{aligned}$$

which gives $\lambda(\eta_1+\eta_2)(1+u^2) = 0$, then $\lambda(\eta_1+\eta_2) = 0$. Since $\lambda \neq 0$, we get $\eta_1+\eta_2 = 0$. So, using (36)(i), we obtain $\kappa^2+\lambda^2 = \frac{1}{3}\eta_1^2$. Since η_1 is assumed to be constant, so $9H_2 = -\eta_1^2 = -\eta_1^2$ is constant. Also, $H_1 = \frac{1}{2}\kappa$ is assumed constant, then $H_3 = \frac{-1}{2}\kappa\eta_1^2$ and $H_4 = \frac{-1}{3}\eta_1^4$ are constants. These results are in contradiction with the assumption $\mathbf{w}_1(H_2) \neq 0$. Hence, the claim is affirmed for $i = 1$.

By a similar manner for $i = 2$, we assume $\mathbf{w}_2(H_2) \neq 0$. Dividing (35)(i) and (35)(ii) by $\epsilon_2\mathbf{w}_2(H_2)$ and taking $v := \frac{\epsilon_1\mathbf{w}_1(H_2)}{\epsilon_2\mathbf{w}_2(H_2)}$, we get $\lambda(\eta_1+\eta_2)(1+v^2) = 0$. Hence, by a similar way we get the same results, which contradicts with assumption $\mathbf{w}_2(H_2) \neq 0$. Therefore, the claim is satisfied for $i = 2$.

Now, we start to prove the claim when $i = 3$. We assume $\mathbf{w}_3(H_2) \neq 0$. From equality (35)(iii) we have $\kappa^2+\lambda^2+2\kappa\eta_2 = 9H_2$, and by a straightforward computation we get

$$-3\kappa^2 + 2(4H_1-\eta_1)\kappa + 3\eta_1(4H_1-\eta_1) = -\lambda^2 < 0,$$

then,

$$-2[2\kappa^2 + (\eta_1-4H_1)\kappa + 2\eta_1(\eta_1-3H_1)] = -(\lambda^2+\kappa^2+\eta_1^2) < 0.$$

Clearly, the last inequality satisfies if and only if

$$\delta = (\eta_1-4H_1)^2 - 16\eta_1(\eta_1-3H_1) = -15\eta_1^2 + 40\eta_1 H_1 + 16H_1^2$$

is less than zero. On the other hand, $\delta < 0$ if and only if $\bar{\delta} < 0$ where

$$\bar{\delta} = (40H_1)^2 + (4\times 15\times 16)H_1^2 = 2560H_1^2.$$

This is a contradiction. So, the claim is affirmed in case $i = 3$.

For case $i = 4$ of the claim, we start with assumption $\mathbf{w}_4(H_2) \neq 0$. From equality (35)(iv) we have $\kappa^2 + \lambda^2 + 2\kappa\eta_1 = 9H_2$, and by a straightforward computation we get

$$-11\kappa^2 + (24H_1 - 10\eta_1)\kappa + 12\eta_1 H_1 - 3\eta_1^2 = -\lambda^2 < 0,$$

then,

$$-2[6\kappa^2 + (5\eta_1 - 12H_1)\kappa + 2\eta_1(\eta_1 - 3H_1)] = -(\lambda^2 + \kappa^2 + \eta_1^2) < 0.$$

It is straightforward to check that the last inequality occurs if and only if $\vartheta < 0$ where

$$\vartheta = (5\eta_1 - 12H_1)^2 - 48\eta_1(\eta_1 - 3H_1) = -23\eta_1^2 + 24\eta_1 H_1 + 144H_1^2.$$

Also, we have $\vartheta < 0$ if and only if $\bar{\vartheta} < 0$ where

$$\bar{\vartheta} = (24H_1)^2 + (4 \times 23 \times 144)H_1^2 = 13824H_1^2.$$

The last inequality is clearly impossible. Hence, the case $i = 4$ of claim is checked.

Therefore, H_2 is constant.

For the second part of the theorem, by constancy of H_2 we have $\mathrm{C}H_2 = 0$. Then, by (7)(ii), we have $9H_1H_2^2 - 3H_2H_3 = 0$. If $H_2 \neq 0$, the last equality gives the constancy of $H_3 = 3H_1H_2$. Also, we get

$$H_4 = (4H_3 - 6H_2\eta_1)\eta_1 + (4H_1 + \eta_1)\eta_1^3 - 2\eta_1^4,$$

which gives the constancy of H_4. Therefore, M_1^4 is isoparametric. Also, S can have at most one non-zero real eigenvalue ([12]). Hence, we have $\eta_1\eta_2 = 0$ which gives $H_4 = (\kappa^2 + \lambda^2)\eta_1\eta_2 = 0$. So, M_1^4 is 3-minimal. □

Theorem 5.2. Suppose that an orientable timelike hypersurface $x : M_1^4 \to \mathbb{S}_1^5$ has shape operator of matrix form $\mathcal{S}_3$, three distinct principal curvatures and constant mean curvature. If M_1^4 is C-biharmonic, then it is isoparametric and 1-minimal.

Proof. Our first step is to show the constancy of H_2. We prove the emptiness of $\mathcal{U} = \{p \in M_1^4 : \nabla H_2^2(p) \neq 0\}$. Assuming $\mathcal{U} \neq \emptyset$, we try to get a contradiction. Since the shape operator S of M_1^4 is of type $\mathcal{S}_3$, there exists a local orthonormal basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_4\}$ for $T(M_1^4)$ satisfying: $S\mathbf{w}_1 = (\kappa + \frac{1}{2})\mathbf{w}_1 - \frac{1}{2}\mathbf{w}_2$, $S\mathbf{w}_2 = \frac{1}{2}\mathbf{w}_1 + (\kappa - \frac{1}{2})\mathbf{w}_2$, $S\mathbf{w}_3 = \lambda_1\mathbf{w}_3$ and $S\mathbf{w}_4 = \lambda_2\mathbf{w}_4$. So, the second Newton transformation satisfies the equalities:
$\mathcal{N}_2\mathbf{w}_1 = [\mu_{1,2;2} + (\kappa - \frac{1}{2})\mu_{1,2;1}]\mathbf{w}_1 + \frac{1}{2}\mu_{1,2;1}\mathbf{w}_2$,
$\mathcal{N}_2\mathbf{w}_2 = -\frac{1}{2}\mu_{1,2;1}\mathbf{w}_1 + [\mu_{1,2;2} + (\kappa - \frac{1}{2})\mu_{1,2;1}]\mathbf{w}_2$,
$\mathcal{N}_2\mathbf{w}_3 = \mu_{3;2}\mathbf{w}_3$ and $\mathcal{N}_2\mathbf{w}_4 = \mu_{4;2}\mathbf{w}_4$.

From condition (7)(i), using the polar decomposition $\nabla H_2 = \sum_{i=1}^{4} \epsilon_i \mathbf{w}_i(H_2)\mathbf{w}_i$, we obtain

(37)

$$(i)\ [\lambda_1\lambda_2 + (\kappa - \frac{1}{2})(\lambda_1+\lambda_2) - 9H_2]\epsilon_1\mathbf{w}_1(H_2) = \frac{1}{2}(\lambda_1+\lambda_2)\epsilon_2\mathbf{w}_2(H_2),$$
$$(ii)\ [\lambda_1\lambda_2 + (\kappa + \frac{1}{2})(\lambda_1+\lambda_2) - 9H_2]\epsilon_2\mathbf{w}_2(H_2) = -\frac{1}{2}(\lambda_1+\lambda_2)\epsilon_1\mathbf{w}_1(H_2),$$
$$(iii)\ (\kappa^2 + 2\kappa\lambda_2 - 9H_2)\epsilon_3\mathbf{w}_3(H_2) = 0,$$
$$(iv)\ (\kappa^2 + 2\kappa\lambda_1 - 9H_2)\epsilon_3\mathbf{w}_4(H_2) = 0.$$

Now, we claim that $\mathbf{w}_i(H_2) = 0$ for $i = 1, 2, 3, 4$.
If $\mathbf{w}_1(H_2) \neq 0$, then dividing equalities (37)(i, ii) by $\epsilon_1\mathbf{w}_1(H_2)$ we get

$$(38)\quad \begin{aligned} &(i)\ \lambda_1\lambda_2 + (\kappa - \frac{1}{2})(\lambda_1+\lambda_2) - 9H_2 = \frac{1}{2}(\lambda_1+\lambda_2)u,\\ &(ii)\ [\lambda_1\lambda_2 + (\kappa + \frac{1}{2})(\lambda_1+\lambda_2) - 9H_2]u = -\frac{1}{2}(\lambda_1+\lambda_2),\end{aligned}$$

where $u := \frac{\epsilon_2\mathbf{w}_2(H_2)}{\epsilon_1\mathbf{w}_1(H_2)}$. From (38)$(i, ii)$, we obtain $(\lambda_1+\lambda_2)(u+1)^2 = 0$, then either $u = -1$ or $\lambda_1 + \lambda_2 = 0$.
If $\lambda_1+\lambda_2 = 0$, using (38)(i) we get $9H_2 = -\lambda_1^2$, which gives $3\kappa^2 = -\lambda_1^2$. Since H_1 is assumed constant on M_1^4, then $\kappa = 2H_1$, λ_1 and λ_2 are constant on M_1^4. Hence, M_1^4 is isoparametric having three real principal curvatures. This result contradicts with Corollary 2.7 in [12]. So, $\lambda_1+\lambda_2 \neq 0$ and $u = -1$. Hence, we have $\lambda_1\lambda_2 + \kappa(\lambda_1+\lambda_2) = 9H_2$ and then

$$3\kappa^2 + 4\kappa(\lambda_1+\lambda_2) + \lambda_1\lambda_2 = 0.$$

By constancy of $4H_1 = 2\kappa + \lambda_1 + \lambda_2$, from the last equality we get $\lambda^2 - H_1\lambda - 3H_1^2 = 0$, which means λ, κ and H_ks (for $k = 2, 3, 4$) are constant. This contradiction implies that the claim is true for $i = 1$.

Similarly, for $i = 2, 3, 4$, the assumptions $\mathbf{w}_i(H_2) \neq 0$ gives $\lambda^2 + 2\kappa\lambda = 9H_2$, which implies the affirmation of claim.

Now, we prove that $H_2 = 0$. Constancy of H_1 and H_2 and the condition (7)(ii) give that H_3 is constant. So, M_1^4 is isoparametric. Then, by Corollary 2.7 from [12], it has at most one nonzero principal curvature, so $\lambda = 0$ (for example). Then $H_1 = \frac{1}{2}\kappa$, $H_2 = \frac{1}{6}\kappa^2$ and $H_3 = 0$, and consequently, by (7)(ii), we get $\kappa = 0$ and then $H_2 = 0$. □

Theorem 5.3. Suppose that an orientable timelike hypersurface $x : M_1^4 \to \mathbb{S}_1^5$ has shape operator of matrix form $\mathcal{S}_4$ and constant mean curvature. If M_1^4 is C-biharmonic, then it is isoparametric and 1-minimal.

Proof. First we prove that H_2 is constant. In fact, we show that the open subset $\mathcal{U} = \{p \in M_1^4 : \nabla H_2^2(p) \neq 0\}$ has no member. Assuming $\mathcal{U} \neq \emptyset$ we try to get a contradiction. Since M_1^4 is of type $\mathcal{S}_4$, there exists an orthonormal tangent

frame $\{\mathbf{w}_1, \ldots, \mathbf{w}_4\}$ on M_1^4 such that the shape operator is of form $\mathcal{S}_4$. So, we have $S\mathbf{w}_1 = \kappa\mathbf{w}_1 - \frac{\sqrt{2}}{2}\mathbf{w}_3$, $S\mathbf{w}_2 = \kappa\mathbf{w}_2 - \frac{\sqrt{2}}{2}\mathbf{w}_3$, $S\mathbf{w}_3 = \frac{\sqrt{2}}{2}\mathbf{w}_1 - \frac{\sqrt{2}}{2}\mathbf{w}_2 + \kappa\mathbf{w}_3$ and $S\mathbf{w}_4 = \lambda\mathbf{w}_4$. and then, we have
$\mathcal{N}_2\mathbf{w}_1 = (\kappa^2 + 2\kappa\lambda - \frac{1}{2})\mathbf{w}_1 + \frac{1}{2}\mathbf{w}_2 + \frac{\sqrt{2}}{2}(\kappa+\lambda)\mathbf{w}_3$,
$\mathcal{N}_2\mathbf{w}_2 = \frac{-1}{2}\mathbf{w}_1 + (\kappa^2 + 2\kappa\lambda + \frac{1}{2})\mathbf{w}_2 + \frac{\sqrt{2}}{2}(\kappa+\lambda)\mathbf{w}_3$,
$\mathcal{N}_2\mathbf{w}_3 = \frac{-\sqrt{2}}{2}(\kappa+\lambda)\mathbf{w}_1 + \frac{\sqrt{2}}{2}(\kappa+\lambda)\mathbf{w}_2 + (\kappa^2 + 2\kappa\lambda)\mathbf{w}_3$ and $\mathcal{N}_2\mathbf{w}_4 = 3\kappa^2\mathbf{w}_4$.

From (7)(i) and polar decomposition $\nabla H_2 = \sum_{i=1}^{4} \epsilon_i \mathbf{w}_i(H_2)\mathbf{w}_i$, we get
(i) $(\kappa^2 + 2\kappa\lambda - \frac{1}{2} - 9H_2)\epsilon_1\mathbf{w}_1(H_2) - \frac{1}{2}\epsilon_2\mathbf{w}_2(H_2) - \frac{\sqrt{2}}{2}(\kappa+\lambda)\epsilon_3\mathbf{w}_3(H_2) = 0$,
(ii) $\frac{1}{2}\epsilon_1\mathbf{w}_1(H_2) + (\kappa^2 + 2\kappa\lambda + \frac{1}{2} - 9H_2)\epsilon_2\mathbf{w}_2(H_2) + \frac{\sqrt{2}}{2}(\kappa+\lambda)\epsilon_3\mathbf{w}_3(H_2) = 0$,
(iii) $\frac{\sqrt{2}}{2}(\kappa+\lambda)(\epsilon_1\mathbf{w}_1(H_2) + \epsilon_2\mathbf{w}_2(H_2)) + (\kappa^2 + 2\kappa\lambda - 9H_2)\epsilon_3\mathbf{w}_3(H_2) = 0$,
(iv) $(3\kappa^2 - 9H_2)\epsilon_4\mathbf{w}_4(H_2) = 0$.

Our first claim is $\mathbf{w}_i(H_2) = 0$ for $i = 1, 2, 3, 4$.
If $\mathbf{w}_1(H_2) \neq 0$, then by dividing both sides of equalities (i), (ii) and (iii) by $\epsilon_1\mathbf{w}_1(H_2)$, and using the identity $2H_2 = \kappa^2 + \kappa\lambda$ (in the case $\mathcal{S}_4$),we get

$$(39)\qquad \begin{aligned} &(a)\ -\frac{1}{2} - \frac{7}{2}\kappa^2 - \frac{5}{2}\kappa\lambda - \frac{1}{2}\nu_1 - \frac{\sqrt{2}}{2}(\kappa+\lambda)\nu_2 = 0\\ &(b)\ \frac{1}{2} + (\frac{1}{2} - \frac{7}{2}\kappa^2 - \frac{5}{2}\kappa\lambda)\nu_1 + \frac{\sqrt{2}}{2}(\kappa+\lambda)\nu_2 = 0\\ &(c)\ \frac{-\sqrt{2}}{2}(\kappa+\lambda)(1+\nu_1) - (\frac{7}{2}\kappa^2 + \frac{5}{2}\kappa\lambda)\nu_2 = 0, \end{aligned}$$

where, $\nu_1 := \frac{\epsilon_2\mathbf{w}_2(H_2)}{\epsilon_1\mathbf{w}_1(H_2)}$ and $\nu_2 := \frac{\epsilon_3\mathbf{w}_3(H_2)}{\epsilon_1\mathbf{w}_1(H_2)}$.

By comparing (39)(a) and (39)(b), we get $\frac{-1}{2}\kappa(7\kappa+5\lambda)(1+u_1) = 0$. If $\kappa = 0$, then $H_2 = 0$. Assuming $\kappa \neq 0$, we get $u_1 = -1$ or $\lambda = -\frac{7}{5}\kappa$. If $u_1 \neq -1$ then $\lambda = -\frac{7}{5}\kappa$. So, by (39)($c$) we obtain $u_1 = -1$, which is a contradiction. Hence we have $u_1 = -1$, which by (39)(a, c) gives $u_2 = 0$.

Now we discuss on two cases $\lambda = -\frac{7}{5}\kappa$ and $\lambda \neq -\frac{7}{5}\kappa$. If $\lambda = -\frac{7}{5}\kappa$, then, $\kappa = \frac{5}{2}H_1$, $H_2 = \frac{-1}{5}\kappa^2$, $H_3 = \frac{-4}{5}\kappa^3$ and $H_4 = \frac{-7}{5}\kappa^4$ are all constants on $\mathcal{U}$. Also, the case $\lambda \neq -\frac{7}{5}\kappa$ is in contradiction with (39)(b).

Hence, the claim in the case $i = 1$ is affirmed. The second case of claim (i.e. $\mathbf{w}_2(H_2) = 0$) can be proved by a similar way.

By applying the results $\mathbf{w}_1(H_2) = \mathbf{w}_2(H_2) = 0$, from (39)($b$) and (39)($c$) we get $\mathbf{w}_3(H_2) = 0$.

The final case of claim (i.e. $\mathbf{w}_4(H_2) = 0$), can be proved using (iv), in a straightforward manner.

In the second step, we prove that $H_2 = 0$. By (7)(ii), we have $\mathrm{C}H_2 = 9H_1H_2^2 - 3H_2H_3 = 0$. If $H_2 = 0$, it remains nothing to prove. By assumption $H_2 \neq 0$, we get $3H_1H_2 = H_3$, which gives $\kappa(\kappa^2 - 3H_1\kappa + 3H_1^2) = 0$, where $\kappa^2 - 3H_1\kappa + 3H_1^2 > 0$, Hence, $\kappa = 0$. Therefore, $H_2 = H_3 = H_4 = 0$. So, M_1^4 is isoparametric and 1-minimal □

6. Aknowledgement

We would like to thank the reviewers for their thoughtful comments and efforts towards improving our manuscript.

References

[1] Akutagawa, K., Maeta, S. (2013), Biharmonic properly immersed submanifolds in Euclidean spaces, Geom. Dedicata, 164, 351-355. http://doi:10.1007/s10711-012-9778-1.

[2] Alias, L.J., Gürbüz, N. (2006), An extension of Takahashi theorem for the linearized operators of the higher order mean curvatures, Geom. Dedicata, 121, 113-127. http://doi:10.1007/s10711-006-9093-9.

[3] Arvanitoyeorgos, A., Defever, F., Kaimakamis, G., Papantoniou, B. J. (2007), Biharmonic Lorentz hypersurfaces in E^4_1, Pacific J. Math., 229, 293-306. http://doi:10.2140/pjm.2007.229.293

[4] Chen, B. Y. (2014), Some open problems and conjetures on submanifolds of finite type: Recent development, Tamkang J. Math., 56, 87-108. http://doi:10.5556/tkjm.45.2014.1564.

[5] Defever, F. (1998), Hypersurfaces of $\mathbb{E}^4$ with harmonic mean curvature vector, Math. Nachr., 196, 61-69. https://doi.org/10.1002/mana.19981960104.

[6] Dimitrić, I. (1992) Submanifolds of $\mathbb{E}^n$ with harmonic mean curvature vector, Bull. Inst. Math. Acad. Sin., 20, 53-65. https://www.researchgate.net/publication/266138884.

[7] Gupta, R. S. (2016), Biharmonic hypersurfaces in $\mathbb{E}^5_s$, An. Stiint. Univ. Al. I. Cuza Iasi Mat. (N.S.), 112:2, 585-593.

[8] Hasanis, T. and Vlachos, T. (1995) Hypersurfaces in $\mathbb{E}^4$ with harmonic mean curvature vector field, Math. Nachr., 172, 145-169. doi:10.1002/mana.19951720112.

[9] Jiang, G. Y. (1987), The conservation law for 2-harmonic maps between Riemannian manifolds, Acta Math. Sin., 30, 220–225.

[10] Kashani, S. M. B. (2009), On some L_1-finite type (hyper)surfaces in $\mathbb{R}^{n+1}$, Bull. Korean Math. Soc., 46:1, 35-43. http://doi:10.4134/BKMS.2009.46.1.035.

[11] Lucas, P. and Ramirez-Ospina, H. F. (2011), Hypersurfaces in the Lorentz-Minkowski space satisfying $L_k\psi = A\psi + b$, Geom. Dedicata, 153, 151-175. http://doi:10.1007/s10711-010-9562-z.

[12] Magid, M. A. (1985), Lorentzian isoparametric hypersurfaces, Pacific J. of Math., 118:1 , 165-197. http://doi:10.2140/PJM.1985.118.165.

[13] Pashaie, F. and Mohammadpouri, A. (2017), L_k-biharmonic spacelike hypersurfaces in Minkowski 4-space $\mathbb{E}^4_1$, Sahand Comm. Math. Anal., 5:1 , 21-30. http://doi:10.22130/scma.2017.20589.

[14] O'Neill, B. (1983), Semi-Riemannian Geometry with Applicatins to Relativity, 2nd edition, Academic Press Inc..

[15] Pashaie, F. and Kashani, S. M. B. (2013), Spacelike hypersurfaces in Riemannian or Lorentzian space forms satisfying $L_kx = Ax + b$, Bull. Iran. Math. Soc., 39:1, 195–213. http://bims.iranjournals.ir/article_338.html.

[16] Pashaie, F. and Kashani, S. M. B. (2014), Timelike hypersurfaces in the Lorentzian standard space forms satisfying $L_kx = Ax + b$, Mediterr. J. Math., 11:2, 755-773. https://link.springer.com/article/10.1007/s00009-013-0336-3.

[17] Petrov, A. Z. (1969), Einstein Spaces, Oxford and New York: Pergamon Press, Hungary.

[18] Turgay, N. C. (2014), Some classifications of biharmonic Lorentzian hypersurfaces in Minkowski 5-space $\mathbb{E}^5_1$, Mediterr. J. Math., 13:1, 401-412. http://doi: 10.1007/s00009-014-0491-1.

Ghorbanali Haghighatdoost
Orcid number: 0000-0002-7979-6122
Department of Mathematics, Faculty of Sciences
Azarbaijan Shahid Madani University
Tabriz, Iran
Email address: gorbanali@azaruniv.ac.ir

Sara Hoseinpour
Orcid number: 0009-0002-8696-4398
Department of Mathematics, Faculty of Sciences
Azarbaijan Shahid Madani University
Tabriz, Iran
Email address: s.h.13971402@gmail.com

Firooz Pashaie
Orcid number: 0000-0002-3020-7649
Department of Mathematics, Faculty of Basic Sciences
University of Maragheh, P.O.Box 55181-83111
Maragheh, Iran
Email address: f_pashaie@maragheh.ac.ir

Leila Shahbaz
Orcid number: 0000-0001-6312-6231
Department of Mathematics, Faculty of Basic Sciences
University of Maragheh, P.O.Box 55181-83111
Maragheh, Iran
Email address: l_shahbaz@maragheh.ac.ir