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NEW GENERALIZED CLOSURE OPERATORS INDUCED BY LOCAL FUNCTIONS VIA IDEALS

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Abstract. This work aims to introduce and examine four new operators based on the topological structure “ideal” and the notion of “generalized” generating two generalized ideal topological spaces. The proposed structures are discussed in detail in terms of topological properties and some basic theories. Moreover, we obtain bases for the generated generalized ideal topological spaces. Further, we define the concept of topology suitable for an ideal. In addition, we provide several essential findings pertaining to these novel frameworks. We also provide several counterexamples that are related to our findings.

Keywords: ideal, $(\cdot)^{\bullet}$-operator, $\coprod^{\bullet}$-operator, generalized topology.
2020 MSC: 54A05, 54B99, 54C60, 54C05

1. Introduction

In several branches of mathematics, topology plays a crucial role. Researchers from a wide range of scientific and social disciplines have been drawn to the applicability of different topological concepts to several natural problems. Numerous novel concepts have been introduced in topology, enhancing it with a variety of newly developed areas of study. Topologists have created various novel structures in an effort to find legitimate answers to a number of these topological problems, including closure space, proximity, filters [12], ideals [10], grills [4] and primals [2] are a few examples of these structures. Kuratowski [12] developed and examined the concept of ideals. Furthermore, several topologists have examined this idea from various angles [13, 14]. The dual structure of filters appears as the idea of ideals. A nonempty family of sets with finite additivity and hereditary attributes closed is called an ideal. By reducing the border region and increasing the set's accuracy values, it is a whole new method of representing vagueness and uncertainty, and it has helped academics handle numerous real-world difficulties.

Kuratovski [12] in 1933 proposed the notion of ideal and then studied from very different aspects by many mathematicians. An ideal topological space (briefly, ideal-TS), is a topological space $(\beth, \Delta)$ with an ideal $\mathcal{L}$ on $\beth$ and is denoted by $(\beth, \Delta, \mathcal{L})$. For a subset $\Gamma \subseteq \beth$, the local function with respect

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to $\mathcal{L}$ and Δ is denoted by Γ^*. The local function Γ^* of $\Gamma \subseteq \beth$ is defined as $\Gamma^*(\mathcal{L}, \Delta) = \{\daleth \in \beth | (\forall \Upsilon \in \Delta(\daleth))(\Upsilon \cap \Gamma \notin \mathcal{L})\}$, where $\Delta(\daleth)$ is the collection of all open subsets containing $\daleth \in \Gamma$ (see [12]). In particular, in 1945, Vaidyanathaswamy [19] looked into more specific features of the local function. The notion of the local function gave the literature a new topology known as $*$-topology, which was later researched by Samuel [11] in 1975 along with several other researchers, including Hayashi [9] in 1964 and Njastad [16] in 1966. Following a 15-year break, Jankovic and Hamlett [10] returned to this subject in 1990. Not only have they compiled all of the available data on the subject in one piece, but they have also included some novel findings. The generalized topological spaces (briefly, GTS) were introduced around 2002 [5], which differ from topological spaces in the condition of intersection. In more specific terms, they are closed under arbitrary unions. In [5, 6, 18], a lot of its characteristics are studied in detail, such as the generalized interior of a set Γ, the generalized closure of a set Γ, the generalized neighbourhood, separation axioms, and so on. The notion of continuity gets a lot of attention under the new appellations "generalized continuity", "θ-continuity", "γ-continuity", etc. (see [5, 6]). Various kinds of generalized topology appeared and were studied, such as "extremely disconnected generalized topology", "δ- and θ-modifications generalized topology", etc. (see [5, 6, 18]).

Many articles have been written on the topological operators subject. The authors [1, 3], for example, examined the idealization of a few weak separation axioms, and Navaneethakrishnan [15] (2008) focused on the study of g-closed sets in ideal-TSs. Hatır [8] and Ekici [7], among others, have examined the decompositions of continuity in I-Alexandroff topological spaces and ideal-TSs, respectively.

The majority of these approaches to the topic don't really differ from one another. In one method, the specification of the local function is changed to produce a new topology with new attributes; in another method, topologies resulting from several ideals that are essentially known are taken into consideration collectively.

In this work, we generate two generalized topological spaces using the notion of ideal by applying two new local functions, and we study some basic characteristics of these new structures as well as these two operators. By using the concept of ideal, we construct and examine the two new local functions given by $(\cdot)^\bullet$ and $(\cdot)^\bullet_R$ to define two new generalized closure operators given by $\coprod^\bullet$ and $\coprod^\bullet_R(\cdot)$. Thus, we are able to derive two new, finer generalized topologies than Δ_δ: $\Delta^\bullet$ and $\Delta^\bullet_R$. Conversely, we demonstrate the independence of the concepts of $\Delta^\bullet_R$-open and Δ-open. Also, the authors study the fundamental characteristics of the produced operators and give an answer to the important question, "do they satisfy Kuratowskis closure axioms?". Furthermore, we establish a number of basic findings about the operators $\coprod^\bullet(\cdot)$, $(\cdot)^\bullet_R$, $(\cdot)^\bullet$, and $\coprod^\bullet_R(\cdot)$. Additionally, we provide several illustrative examples in addition to a few connections. Finally, we provide the idea of topology compatible for an ideal and derive its characterizations and other properties.

2. Preliminaries

In this paper, $(\beth, \Delta)$ (briefly, $\beth$) is a TS unless otherwise indicated. We represent, respectively, the closure and interior of a subset Γ of a space $\beth$ by $\coprod(\Gamma)$ and $\prod(\Gamma)$. The power set of a set $\beth$ will be denoted by $2^{\beth}$. The family of all open neighbourhoods of a point $\gimel$ of $\beth$ will be represented by $O(\beth, \gimel)$.

The operator $\coprod^{*} : 2^{\beth} \rightarrow 2^{\beth}$ defined by $\coprod^{*}(\Gamma) = \Gamma \cup \Gamma^{*}$ is a Kuratowski closure operator. The topology proposed by the closure operator $\coprod^{*}$ is $\Delta^{*}(\mathcal{L}, \Delta) = \{\Gamma \subseteq \beth | \coprod^{*}(\beth \backslash \Gamma) = \beth \backslash \Gamma\}$ and denoted by $*$-topology which is finer than the topology Δ. Any subset Γ of a space $\beth$ is said to be regular open [17] if $\Gamma = \prod(\coprod(\Gamma))$. A regular closed set [17] is the complement of a regular open set. The union of all regular open subsets of $\beth$ in Γ is known as the δ-interior of Γ and defined by δ-$\prod(\Gamma)$. On the other hand, the intersection of all regular closed subsets of $\beth$ containing Γ of a space $\beth$ is said to be δ-closure of Γ and given by δ-$\coprod(\Gamma)$. A subset Γ of a space $\beth$ is said to be δ-open [20] if $\Gamma = \delta$-$\prod(\Gamma)$. The complement of a δ-open set in a space $\beth$ is said to be δ-closed [20]. All regular open [17] (resp. regular closed [17], δ-open [20], δ-closed [20]) subsets of a space $\beth$ will be denoted by $RO(\beth)$ (resp. $RC(\beth)$, $\delta O(\beth)$, $\delta C(\beth)$). All regular open (resp. regular closed, δ-open, δ-closed) sets of $\beth$ containing a point $\gimel$ of $\beth$ are denoted by $RO(\beth, \gimel)$ (resp. $RC(\beth, \gimel)$, $\delta O(\beth, \gimel)$, $\delta C(\beth, \gimel)$).

Now, we recall the following results concerning the δ-interior and the δ-closure of a set Γ in a TS $\beth$.

Theorem 2.1. *[20] Let $\beth$ be a TS and $\Gamma \subseteq \beth$. Then the following hold.*

(a) δ-$\prod(\Gamma) = \{\gimel | (\exists \Upsilon \in RO(\beth, \gimel))(\Upsilon \subseteq \Gamma)\} = \{\gimel | (\exists \Upsilon \in O(\beth, \gimel))(\prod(\coprod(\Upsilon)) \subseteq \Gamma)\}$,

(b) δ-$\coprod(\Gamma) = \{\gimel | (\forall \Upsilon \in RO(\beth, \gimel))(\Upsilon \cap \Gamma \neq \emptyset)\} = \{\gimel | (\forall \Upsilon \in O(\beth, \gimel))(\prod(\coprod(\Upsilon)) \cap \Gamma \neq \emptyset)\}$,

(c) δ-$\coprod(\Gamma^{c}) = (\delta$-$\prod(\Gamma))^{c}$,

(d) δ-$\prod(\Gamma^{c}) = (\delta$-$\coprod(\Gamma))^{c}$.

Definition 2.2. [21] A mapping $\coprod : 2^{\beth} \rightarrow 2^{\beth}$ is called a generalized closure operator (or simply, closure operator) if for any $\Gamma, \Lambda \subseteq \beth$, $\coprod$ satisfies the following three axioms:
(1) $\Gamma \subseteq \coprod(\Gamma)$;
(2) $\Gamma \subseteq \Lambda \Rightarrow \coprod(\Gamma) \subseteq \coprod(\Lambda)$;
(3) $\coprod(\coprod(\Gamma) = \coprod(\Gamma))$.

Definition 2.3. [10] Let $\daleth$ be a non-empty set. A collection $\mathcal{G} \subseteq 2^{\daleth}$ is called a generalized topological space on $\daleth$ if it satisfies the following conditions:

(1) $\emptyset \in \mathcal{G}$,
(2) The arbitrary union of $\Gamma_i \in \mathcal{G}$, for $i \in I \neq \emptyset$ is belongs to $\mathcal{G}$.

The pair, $(\daleth, \mathcal{G})$ (briefly, $\daleth$) is a GTS unless otherwise indicated. This space's elements are identified as $\mathcal{G}$-open and their own complements are identified as $\mathcal{G}$-closed. $\coprod_{\mathcal{G}}(\Gamma)$ represents the closure of $\Gamma \subseteq \daleth$, which is described as the intersection of all $\mathcal{G}$-closed sets that contain $\mathcal{G}$. $\prod_{\mathcal{G}}(\Gamma)$ represents the interior of $\Gamma \subseteq \daleth$, which is described as the as the union of all $\mathcal{G}$-open sets that contained in $\mathcal{G}$. Also, $\coprod_{\mathcal{G}}(\coprod_{\mathcal{G}}(\Gamma)) = \coprod_{\mathcal{G}}(\Gamma)$, $\prod_{\mathcal{G}}(\prod_{\mathcal{G}}(\Gamma)) = \prod_{\mathcal{G}}(\Gamma)$ and $\prod_{\mathcal{G}}(\Gamma) \subseteq \Gamma \subseteq \coprod_{\mathcal{G}}(\Gamma)$. Γ is $\mathcal{G}$-open if $\Gamma = \prod_{\mathcal{G}}(\Gamma)$, Γ is $\mathcal{G}$-closed if $\Gamma = \coprod_{\mathcal{G}}(\Gamma)$ and $\coprod_{\mathcal{G}}(\Gamma) = \daleth \setminus \left(\prod_{\mathcal{G}}(\daleth \setminus \Gamma)\right)$.

Definition 2.4. [10] Let $\daleth$ be a non-empty set. A collection $\mathcal{L} \subseteq 2^{\daleth}$ is called an ideal on $\daleth$ if it satisfies the following conditions:

(a) $\phi \in \mathcal{L}$,
(b) if $\Gamma \in \mathcal{L}$ and $\Lambda \subseteq \Gamma$, then $\Lambda \in \mathcal{L}$,
(c) if $\Gamma, \Lambda \in \mathcal{L}$, then $\Gamma \cup \Lambda \in \mathcal{L}$.

3. The $(\cdot)^{\bullet}$ operator and its topology $\Delta^{\bullet}$

A novel operator in ideal-TSs is presented and studied in this section. Along with providing several counterexamples, we also derive some basic features of this new operator. As of right now, the operator $(\cdot)^{\bullet}$ has the definition shown below. Furthermore, We characterize and study a novel operator that resembles a generalized closure operator. This leads to a new generalized topology that is finer than the topology Δ_{δ}, which we name $\Delta^{\bullet}$.

Definition 3.1. Let $(\daleth, \Delta, \mathcal{L})$ be an ideal-TS. An operator $(\cdot)^{\bullet} : 2^{\daleth} \to 2^{\daleth}$ defined by $\Gamma^{\bullet}(\daleth, \Delta, \mathcal{L}) = \{\gimel \in \daleth : (\forall \Upsilon \in \Delta(\gimel))(\Gamma^c \cup \Upsilon^c \in \mathcal{L})\}$ is called the $\bullet$-local function of any subset Γ of $\daleth$ with respect to an ideal $\mathcal{L}$ and a topology Δ on $\daleth$. We can also write $\Gamma^{\bullet}_{\mathcal{L}}$ as $\Gamma^{\bullet}(\daleth, \Delta, \mathcal{L})$ to specify the ideal as per our requirements.

Remark 3.2. Let $(\daleth, \Delta, \mathcal{L})$ be an ideal-TS. For any subset Γ of $\daleth$, $\Gamma^{\bullet} \subseteq \Gamma$ or $\Gamma \subseteq \Gamma^{\bullet}$ need not be true as shown by the following example.

Example 3.3. (1) *Let* $\daleth = \{\gimel_1, \gimel_2, \gimel_3\}$ *and* $\Delta = \{\emptyset, \{\gimel_1\}, \{\gimel_2\}, \{\gimel_1, \gimel_2\}, \daleth\}$. *We consider an ideal* $\mathcal{L} = \{\emptyset, \{\gimel_1\}, \{\gimel_2\}, \{\gimel_1, \gimel_2\}\}$ *on* $\daleth$. *Now, if* $\Gamma = \{\gimel_1, \gimel_2\}$, *then* $\emptyset = \Gamma^{\bullet} \subseteq \Gamma = \{\gimel_1, \gimel_2\}$.

(2) *Let* $\daleth = \{\gimel_1, \gimel_2, \gimel_3\}$ *with the indiscrete topology. We consider an ideal* $\mathcal{L} = \{\emptyset, \{\gimel_1\}, \{\gimel_2\}, \{\gimel_1, \gimel_2\}\}$ *on* $\daleth$. *Now, if* $\Gamma = \{\gimel_3\}$, *then* $\{\gimel_3\} = \Gamma \subseteq \Gamma^{\bullet} = \daleth$.

Theorem 3.4. *Let* $(\daleth, \Delta, \mathcal{L})$ *be an ideal-TS and* $\Gamma, \Lambda \subseteq \daleth$. *Then, the following statements hold:*

(*i*) *If* $\Gamma^c \in \Delta, \gimel \notin \Gamma$ *and* $\gimel \in \Gamma^\bullet$, *then* $\mathcal{L} = 2^{\beth}$,

(*ii*) $\coprod(\Gamma^\bullet) = \Gamma^\bullet$,

(*iii*) $(\Gamma^\bullet)^\bullet \subseteq \Gamma^\bullet$,

(*iv*) *if* $\Gamma \subseteq \Lambda$, *then* $\Gamma^\bullet \subseteq \Lambda^\bullet$,

(*v*) $\Gamma^\bullet \cup \Lambda^\bullet \subseteq (\Gamma \cup \Lambda)^\bullet$,

(*vi*) $(\Gamma \cap \Lambda)^\bullet = \Gamma^\bullet \cap \Lambda^\bullet$,

(*vii*) *if* $\Gamma^c \in \Delta$ *and* $\mathcal{L} \neq 2^{\beth}$, *then* $\Gamma^\bullet \subseteq \Gamma$.

Proof. (i) Let $\Gamma^c \in \Delta$ and $\gimel \in \Gamma^\bullet$. Since $\gimel \notin \Gamma$, $\Gamma^c \in \Delta(\gimel)$. Since $\gimel \in \Gamma^\bullet$, $\Gamma^c \cup \Upsilon^c \in \mathcal{L}$ for all $\Upsilon \in \Delta(\gimel)$. Therefore, $\beth = \Gamma \cup \Gamma^c = (\Gamma^c)^c \cup \Gamma^c \in \mathcal{L}$. This means that $\mathcal{L} = 2^{\beth}$.

(ii) We have always $\Gamma^\bullet \subseteq \coprod(\Gamma^\bullet)$. Conversely, let $\gimel \in \coprod(\Gamma^\bullet)$ and $\Upsilon \in \Delta(\gimel)$. Then, $\Upsilon \cap \Gamma^\bullet \neq \emptyset$. Therefore, there exists $y \in \beth$ such that $y \in \Upsilon$ and $y \in \Gamma^\bullet$. Then, we have $\daleth^c \cup \Gamma^c \in \mathcal{L}$ for all $\daleth \in \Delta(y)$. Thus, we get $\Upsilon^c \cup \Gamma^c \in \mathcal{L}$. This means that $\gimel \in \Gamma^\bullet$. Hence, $\coprod(\Gamma^\bullet) \subseteq \Gamma^\bullet$. Therefore, $\Gamma^\bullet$ is closed in $\beth$.

(iii) It is obvious from (i) and (ii).

(iv) Let $\Gamma \subseteq \Lambda$ and $\gimel \in \Gamma^\bullet$. Then, we have $\Gamma^c \cup \Upsilon^c \in \mathcal{L}$ for all $\Upsilon \in \Delta(\gimel)$. Thus, $\Lambda^c \cup \Upsilon^c \in \mathcal{L}$ since $\Gamma \subseteq \Lambda$. It follows that , $\gimel \in \Lambda^\bullet$. Hence, $\Gamma^\bullet \subseteq \Lambda^\bullet$.

(v) We can get from (iv) that $\Gamma^\bullet \subseteq (\Gamma \cup \Lambda)^\bullet$ and $\Lambda^\bullet \subseteq (\Gamma \cup \Lambda)^\bullet$. Hence, $\Gamma^\bullet \cup \Lambda^\bullet \subseteq (\Gamma \cup \Lambda)^\bullet$.

(vi) We can get from (iv) that $\Gamma^\bullet \supseteq (\Gamma \cap \Lambda)^\bullet$ and $\Lambda^\bullet \supseteq (\Gamma \cap \Lambda)^\bullet$. Hence, $\Gamma^\bullet \cap \Lambda^\bullet \supseteq (\Gamma \cap \Lambda)^\bullet$. Conversely, let $\gimel \in \Gamma^\bullet \cap \Lambda^\bullet$. Then $\gimel \in \Gamma^\bullet$ and $\gimel \in \Lambda^\bullet$. Then, $\forall \Upsilon \in \Delta(\gimel)$, we have $\Gamma^c \cup \Upsilon^c \in \mathcal{L}$ and $\Lambda^c \cup \Upsilon^c \in \mathcal{L}$. Hence, we have $(\Gamma \cap \Lambda)^c \cup \Upsilon^c \in \mathcal{L}$ since $\mathcal{L}$ is an ideal. This means that $\gimel \in (\Gamma \cap \Lambda)^\bullet$. Thus, $(\Gamma \cap \Lambda)^\bullet \supseteq \Gamma^\bullet \cap \Lambda^\bullet$. Hence, $(\Gamma \cap \Lambda)^\bullet = \Gamma^\bullet \cap \Lambda^\bullet$.

(vii) Let $\Gamma^c \in \Delta$ and $\gimel \in \Gamma^\bullet$. Suppose that $\gimel \notin \Gamma$. Then, $\Gamma^c \in \Delta(\gimel)$. Since $\gimel \in \Gamma^\bullet$, $\Gamma^c \cup \Upsilon^c \in \mathcal{L}$ for all $\Upsilon \in \Delta(\gimel)$. Therefore, $\beth = \Gamma \cup \Gamma^c = (\Gamma^c)^c \cup \Gamma^c \in \mathcal{L}$. This contradicts with $\beth \notin \mathcal{L}$. Hence, $\Gamma^\bullet \subseteq \Gamma$.

□

Remark 3.5. The equality in Theorem 3.4 (v) need not be true, as seen in the following example.

Example 3.6. *Let* $\beth = \{\gimel_1, \gimel_2, \gimel_3\}$ *and* $\Delta = \{\emptyset, \{\gimel_1\}, \{\gimel_2\}, \{\gimel_1, \gimel_2\}, \beth\}$. *We consider an ideal* $\mathcal{L} = \{\emptyset, \{\gimel_3\}\}$ *on* $\beth$. *Now, if* $\Gamma = \{\gimel_1\}$ *and* $\Lambda = \{\gimel_2\}$, *then* $\emptyset = \Gamma^\bullet = \Lambda^\bullet$. *Hence,* $\Gamma^\bullet \cup \Lambda^\bullet = \emptyset \neq (\Gamma \cup \Lambda)^\bullet = (\{\gimel_1, \gimel_2\})^\bullet = \{\gimel_3\}$,

Definition 3.7. Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS. We consider a map $\coprod^{\bullet} : 2^{\beth} \to 2^{\beth}$ as $\coprod^{\bullet}(\Gamma) = \Gamma \cup \Gamma^{\bullet}$, where Γ is any proper subset of $\beth$ and $\coprod^{\bullet}(\emptyset) = \emptyset$.

Theorem 3.8. *Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS and $\Gamma, \Lambda \subseteq \beth$. Then, the following statements hold:*

(*i*) $\coprod^{\bullet}(\beth) = \beth$,

(*ii*) $\Gamma \subseteq \coprod^{\bullet}(\Gamma)$,

(*iiii*) *if* $\Gamma \subseteq \Lambda$, *then* $\coprod^{\bullet}(\Gamma) \subseteq \coprod^{\bullet}(\Lambda)$,

(*iv*) $\coprod^{\bullet}(\Gamma) \cup \coprod^{\bullet}(\Lambda) \subseteq \coprod^{\bullet}(\Gamma \cup \Lambda)$,

(*v*) $\coprod^{\bullet}(\Gamma) \cap \coprod^{\bullet}(\Lambda) = \coprod^{\bullet}(\Gamma \cap \Lambda)$,

(*vi*) $\coprod^{\bullet}(\coprod^{\bullet}(\Gamma)) = \coprod^{\bullet}(\Gamma)$.

Proof. Let $\Gamma, \Lambda \subseteq \beth$.

(*i*) Since $\beth \cup \beth^{\bullet} = \beth$, we have $\coprod^{\bullet}(\beth) = \beth$.

(*ii*) Since $\coprod^{\bullet}(\Gamma) = \Gamma \cup \Gamma^{\bullet}$, we have $\Gamma \subseteq \coprod^{\bullet}(\Gamma)$.

(*iii*) Let $\Gamma \subseteq \Lambda$. We get from (iv) of Theorem 3.4 that $\Gamma^{\bullet} \subseteq \Lambda^{\bullet}$. Therefore, we have $\Gamma \cup \Gamma^{\bullet} \subseteq \Lambda \cup \Lambda^{\bullet}$ which means that $\coprod^{\bullet}(\Gamma) \subseteq \coprod^{\bullet}(\Lambda)$.

(*iv*) It is obvious from the definition of the operator $\coprod^{\bullet}$ and (v) of Theorem 3.4.

(*v*) It is obvious from the definition of the operator $\coprod^{\bullet}$ and (vi) of Theorem 3.4.

(*vi*) It is obvious from (ii) that $\coprod^{\bullet}(\Gamma) \subseteq \coprod^{\bullet}(\coprod^{\bullet}(\Gamma))$.

Conversely, let $\daleth \in \coprod^{\bullet}(\coprod^{\bullet}(\Gamma)) = \coprod^{\bullet}(\Gamma) \cup \left(\coprod^{\bullet}(\Gamma)\right)^{\bullet}$ and $\Upsilon \in \Delta(\daleth)$. Then, we have the following two cases:

(a) $\daleth \in \coprod^{\bullet}(\Gamma)$. This means that $\coprod^{\bullet}(\coprod^{\bullet}(\Gamma)) \subseteq \coprod^{\bullet}(\Gamma)$.

(b) Let $\daleth \notin \coprod^\bullet(\Gamma)$. Then, we have

$$\begin{aligned}\daleth \in (\Gamma \cup \Gamma^\bullet)^\bullet \quad &\Rightarrow \quad [\Upsilon^c \cup (\Gamma \cup \Gamma^\bullet)^c \in \mathcal{L}\ \forall \Upsilon \in \Delta(\daleth)][\exists\ \Upsilon \in \Delta(\daleth):\ \Upsilon^c \cup (\Gamma)^c \notin \mathcal{L}] \\ &\Rightarrow \quad \Upsilon^c \in \mathcal{L} \text{ and } \Gamma^c \notin \mathcal{L} \\ &\Rightarrow \quad \Upsilon^c \cup \Gamma^c \notin \mathcal{L} \text{ for all } \Upsilon \in \Delta(\daleth^*), \daleth^* \in \beth \\ &\Rightarrow \quad \Gamma^\bullet = \emptyset \Rightarrow \daleth \in \coprod^\bullet(\Gamma), \text{ which is a contradiction} \\ &\Rightarrow \quad \coprod^\bullet(\coprod^\bullet(\Gamma)) \subseteq \coprod^\bullet(\Gamma).\end{aligned}$$

Thus, from (a), (b) we have $\coprod^\bullet(\coprod^\bullet(\Gamma)) \subseteq \coprod^\bullet(\Gamma)$.
Hence, we have $\coprod^\bullet(\coprod^\bullet(\Gamma)) = \coprod^\bullet(\Gamma)$.

□

Remark 3.9. The equality in Theorem 3.8 (iv) need not be true, as seen in the following example.

Example 3.10. *Let* $\beth = \{\daleth_1, \daleth_2, \daleth_3\}$ *and* $\Delta = \{\emptyset, \{\daleth_2\}, \{\daleth_3\}, \{\daleth_2, \daleth_3\}, \beth\}$. *We consider an ideal* $\mathcal{L} = \{\emptyset, \{\daleth_1\}\}$ *on* $\beth$. *Now, if* $\Gamma = \{\daleth_2\}$ *and* $\Lambda = \{\daleth_3\}$, *then* $\Gamma = \coprod^\bullet(\Gamma)$ *and* $\Lambda = \coprod^\bullet(\Lambda)$. *Hence,* $\coprod^\bullet(\Gamma) \cup \coprod^\bullet(\Lambda) = \{\daleth_2, \daleth_3\} \neq \coprod^\bullet(\Gamma \cup \Lambda) = (\{\daleth_2, \daleth_3\})^\bullet = \beth$.

Corollary 3.11. *Let* $(\beth, \Delta, \mathcal{L})$ *be an ideal-TS. Then the function* $\coprod^\bullet : 2^\beth \to 2^\beth$ *defined by* $\coprod^\bullet(\Gamma) = \Gamma \cup \Gamma^\bullet$, *where* Γ *is any subset of* $\beth$, *is a generalized closure operator. But it does not satisfy the four Kuratowski's closure operator conditions.*

Definition 3.12. Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS. Then, the collection $\Delta^\bullet = \{\Gamma \subseteq \beth | \coprod^\bullet(\Gamma^c) = \Gamma^c\}$ is a generalized topology on $\beth$ induced by topology Δ and ideal $\mathcal{L}$. It is called generalized ideal topology on $\beth$. We can also write $\Delta^\bullet_{\mathcal{L}}$ instead of $\Delta^\bullet$ to specify the ideal as per our requirements.

Remark 3.13. Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS. Then the generalized ideal topology $\Delta^\bullet$ and the topology Δ are incomparable.

Example 3.14. (1) *Let* $\beth = \{\daleth_1, \daleth_2, \daleth_3\}$ *and* $\Delta = \{\emptyset, \{\daleth_3\}, \beth\}$. *We consider an ideal* $\mathcal{L} = \{\emptyset, \{\daleth_1\}, \{\daleth_2\}, \{\daleth_3\}, \{\daleth_1, \daleth_2\}, \{\daleth_1, \daleth_3\}, \{\daleth_2, \daleth_3\}, \beth\}$ *on* $\beth$. *By computing, for all* $\Gamma \subseteq \beth$ *we have* $\Gamma^\bullet = \beth$, *then* $\Delta^\bullet = \{\phi, \beth\}$.

(2) *Let* $\beth = \{\daleth_1, \daleth_2, \daleth_3\}$ *and* $\Delta = \{\emptyset, \{\daleth_2\}, \beth\}$. *We consider an ideal* $\mathcal{L} = \{\emptyset, \{\daleth_1\}\}$ *on* $\beth$. *By computing,* $\Delta^\bullet = \{\phi, \{\daleth_2\}, \{\daleth_3\}, \{\daleth_2, \daleth_3\}, \{\daleth_1, \daleth_2\}, \{\daleth_1, \daleth_3\}, \beth\}$.

Theorem 3.15. *Let* $(\beth, \Delta, \mathcal{L})$ *be an ideal-TS and* $\mathcal{L} \neq 2^\beth$. *Then the generalized ideal topology* $\Delta^\bullet$ *is finer than* Δ.

Proof. Let $\Gamma \in \Delta$. Then, Γ^c is Δ-closed in $\beth$. From (vii) of Theorem 3.4, we get $(\Gamma^c)^\bullet \subseteq \Gamma^c$. Thus, $\coprod^\bullet(\Gamma^c) = \Gamma^c \cup (\Gamma^c)^\bullet \subseteq \Gamma^c$. Since $\Gamma^c \subseteq \coprod^\bullet(\Gamma^c)$ is always true for any subset Γ of $\beth$, we obtain $\coprod^{\bullet}(\Gamma^c) = \Gamma^c$. This means that $\Gamma \in \Delta^{\bullet}$. Thus, we have $\Delta \subseteq \Delta^{\bullet}$. □

Theorem 3.16. *Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS. Then, the following statements hold:*

(i) If $\mathcal{L} = \{\emptyset\}$, then $\Delta^{\bullet} = 2^{\beth}$.

(ii) If $\mathcal{L} = 2^{\beth}$, then $\Delta \supseteq \Delta^{\bullet}$.

Proof. (i) Let $\Gamma \in 2^{\beth}$. Since $\mathcal{L} = \{\emptyset\}$, we have $\Gamma^{\bullet} = \emptyset$ for any subset Γ of $\beth$. Therefore, $\coprod^{\bullet}(\Gamma^c) = \Gamma^c$. This means that $\Gamma \in \Delta^{\bullet}$. Hence, $2^{\beth} \subseteq \Delta^{\bullet}$. Thus, we have $\Delta^{\bullet} = 2^{\beth}$.
(ii) Let $\Gamma \in \Delta^{\bullet}$. Then $\Gamma^c \cup (\Gamma^c)^{\bullet} = \Gamma^c$ which means that $(\Gamma^c)^{\bullet} \subseteq \Gamma^c$. Now, let $\gimel \notin (\Gamma^c)^{\bullet}$. Then, there exists $\Upsilon \in \Delta(\gimel)$ such that $\Upsilon^c \cup (\Gamma^c)^c = \Upsilon^c \cup \Gamma \notin \mathcal{L}$. This contradicts with $\mathcal{L} = 2^{\beth}$. Thus, $\Gamma = \beth$. Therefore, $\Delta^{\bullet} = \{\phi, \beth\}$. Hence, $\Delta^{\bullet} \subseteq \Delta$. □

Theorem 3.17. *Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS and $\Gamma \subseteq \beth$. Then the following hold:*

(i) $\Gamma \in \Delta^{\bullet}$ if and only if for all $\gimel$ in Γ, there exists an open set Υ containing $\gimel$ such that $\Upsilon^c \cup \Gamma \notin \mathcal{L}$,

(ii) if $\Gamma \notin \mathcal{L}$, then $\Gamma \in \Delta^{\bullet}$.

Proof. (i) Let $\Gamma \in \Delta^{\bullet}$.

$$\begin{aligned}
\Gamma \in \Delta^{\bullet} &\Leftrightarrow \coprod^{\bullet}(\Gamma^c) = \Gamma^c \\
&\Leftrightarrow \Gamma^c \cup (\Gamma^c)^{\bullet} = \Gamma^c \\
&\Leftrightarrow (\Gamma^c)^{\bullet} \subseteq \Gamma^c \\
&\Leftrightarrow \Gamma \subseteq ((\Gamma^c)^{\bullet})^c \\
&\Leftrightarrow (\forall \gimel \in \Gamma)(\gimel \notin (\Gamma^c)^{\bullet}) \\
&\Leftrightarrow (\forall \gimel \in \Gamma)(\exists \Upsilon \in \Delta(\gimel))(\Upsilon^c \cup (\Gamma^c)^c = \Upsilon^c \cup \Gamma \notin \mathcal{L}).
\end{aligned}$$

(ii) Let $\Gamma \notin \mathcal{L}$ and $\gimel \in \Gamma$. Put $\Upsilon = \beth$. Then, Υ is a Δ-open set containing $\gimel$. Since $\Gamma \notin \mathcal{L}$ and $\Upsilon^c \cup \Gamma = \Gamma$, we have $\Upsilon^c \cup \Gamma \notin \mathcal{L}$. From (i), we get $\Gamma \in \Delta^{\bullet}$. □

Theorem 3.18. *Let $(\beth, \Delta, \mathcal{L})$ and $(\beth, \Delta, \mathcal{Q})$ be two ideal-TSs. If $\mathcal{L} \subseteq \mathcal{Q}$, then $\Delta^{\bullet}_{\mathcal{Q}} \subseteq \Delta^{\bullet}_{\mathcal{L}}$.*

Proof. Let $\Gamma \in \Delta^{\bullet}_{\mathcal{Q}}$. Then $\Gamma^c \cup (\Gamma^c)^{\bullet}_{\mathcal{Q}} = \Gamma^c$ which means that $(\Gamma^c)^{\bullet}_{\mathcal{Q}} \subseteq \Gamma^c$. Now, let $\gimel \notin \Gamma^c$. Then, we get $\gimel \notin (\Gamma^c)^{\bullet}_{\mathcal{Q}}$ and so there exists $\Upsilon \in \Delta(\gimel)$ such that $\Upsilon^c \cup (\Gamma^c)^c = \Upsilon^c \cup \Gamma \notin \mathcal{Q}$. Since $\mathcal{L} \subseteq \mathcal{Q}$, we have $\Upsilon^c \cup \Gamma \notin \mathcal{L}$. Therefore, $\gimel \notin (\Gamma^c)^{\bullet}_{\mathcal{L}}$. Thus, $(\Gamma^c)^{\bullet}_{\mathcal{L}} \subseteq \Gamma^c$ and so $\coprod^{\bullet}(\Gamma^c) = \Gamma^c \cup (\Gamma^c)^{\bullet}_{\mathcal{L}} = \Gamma^c$. Hence, $\Gamma \in \Delta^{\bullet}_{\mathcal{L}}$. Consequently, we have $\Delta^{\bullet}_{\mathcal{Q}} \subseteq \Delta^{\bullet}_{\mathcal{L}}$. □

Lemma 3.19. *Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS. If $\Gamma^c \notin \mathcal{L}$, then $\Gamma^{\bullet} = \emptyset$.*

Proof. Suppose that $\gimel \in \Gamma^\bullet$. Then, for any open set Υ containing $\gimel$ we have $\Upsilon^c \cup \Gamma^c \in \mathcal{L}$. Since $\Gamma^c \notin \mathcal{L}$, $\Upsilon^c \cup \Gamma^c \notin \mathcal{L}$ for some open set Υ containing $\gimel$. This is a contradiction. Hence, $\Gamma^\bullet = \emptyset$. □

Theorem 3.20. *Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS and $\Gamma, \Lambda \subseteq \beth$. If Γ is open in $\beth$, then $\Gamma \cap \Lambda^\bullet \subseteq (\Gamma \cap \Lambda)^\bullet$.*

Proof. Let $\Gamma \in \Delta$ and $\gimel \in \Gamma \cap \Lambda^\bullet$. Therefore, $\gimel \in \Gamma$ and $\gimel \in \Lambda^\bullet$. Then, we have $\Lambda^c \cup \Upsilon^c \in \mathcal{L}$ for all $\Upsilon \in \Delta(\gimel)$. Since $\Gamma \in \Delta$, we get $(\Gamma \cap \Lambda)^c \cup \Upsilon^c = \Lambda^c \cup (\Gamma \cap \Upsilon)^c \in \mathcal{L}$ for all $\Upsilon \in \Delta(\gimel)$. This means that $\gimel \in (\Gamma \cap \Lambda)^\bullet$. Thus, $\Gamma \cap \Lambda^\bullet \subseteq (\Gamma \cap \Lambda)^\bullet$. □

Corollary 3.21. *Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS and $\Gamma, \Lambda \subseteq \beth$. If Γ is open in $\beth$, then $\Gamma \cap \Lambda^\bullet = \Gamma \cap (\Gamma \cap \Lambda)^\bullet \subseteq (\Gamma \cap \Lambda)^\bullet$.*

Proof. We have $\Gamma \cap (\Gamma \cap \Lambda)^\bullet \subseteq \Gamma \cap \Lambda^\bullet$. Let $\gimel \in \Gamma \cap \Lambda^\bullet$ and $\daleth \in \Delta(\gimel)$. Then $\Gamma \cap \daleth \in \Delta(\gimel)$ and $\gimel \in \Lambda^\bullet$, then $(\Gamma \cap \daleth)^c \cup \Lambda^c \in \mathcal{L}$, i.e. $(\Gamma \cap \Lambda)^c \cup \daleth^c \in \mathcal{L}$, then $\gimel \in (\Gamma \cap \Lambda)^\bullet$ and $\gimel \in \Gamma \cap (\Gamma \cap \Lambda)^\bullet$. Hence by Theorem 3.20, we have $\Gamma \cap \Lambda^\bullet = \Gamma \cap (\Gamma \cap \Lambda)^\bullet \subseteq (\Gamma \cap \Lambda)^\bullet$. □

Lemma 3.22. *Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS and Γ, Λ be subsets of $\beth$. Then, $\Gamma^\bullet - \Lambda^\bullet \supseteq (\Gamma - \Lambda)^\bullet - \Lambda^\bullet$.*

Proof. We have by Theorem 3.4, $(\Gamma - \Lambda)^\bullet \subseteq \Gamma^\bullet$ and hence, $(\Gamma - \Lambda)^\bullet - \Lambda^\bullet \subseteq \Gamma^\bullet - \Lambda^\bullet$. □

Corollary 3.23. *Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS and Γ, Λ be subsets of $\beth$ with $\Lambda^c \notin \mathcal{L}$. Then $(\Gamma \cup \Lambda)^\bullet \supseteq \Gamma^\bullet \supseteq (\Gamma - \Lambda)^\bullet$.*

Proof. Since $\Lambda^c \notin \mathcal{L}$, $\Lambda^\bullet = \emptyset$. Again by Lemma 3.22, $\Gamma^\bullet \supseteq (\Gamma - \Lambda)^\bullet$ and by Theorem 3.4, $(\Gamma \cup \Lambda)^\bullet \supseteq \Gamma^\bullet \cup \Lambda^\bullet \supseteq \Gamma^\bullet$. □

Remark 3.24. Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS and $\Gamma, \Lambda \subseteq \beth$. The equality in Corollary 3.23 need not be true, as seen in the following example.

Example 3.25. *Let $\beth = \{\gimel_1, \gimel_2, \gimel_3\}$ and $\Delta = \{\emptyset, \{\gimel_2\}, \{\gimel_3\}, \{\gimel_2, \gimel_3\}, \beth\}$. We consider an ideal $\mathcal{L} = \{\emptyset, \{\gimel_1\}\}$ on $\beth$. Now, if $\Gamma = \{\gimel_2\}$ and $\Lambda = \{\gimel_3\}$, then $\Gamma^\bullet = \emptyset$ and $\Lambda^\bullet = \emptyset$. Hence, $\Gamma^\bullet = \emptyset \neq (\Gamma \cup \Lambda)^\bullet = (\{\gimel_2, \gimel_3\})^\bullet = \{\gimel_1\}$. Now, let $\Gamma = \{\gimel_2, \gimel_3\}$ and $\Lambda = \{\gimel_3\}$. Then, $(\Gamma - \Lambda)^\bullet = (\{\gimel_2\})^\bullet = \emptyset \neq \Gamma^\bullet = (\{\gimel_2, \gimel_3\})^\bullet = \{\gimel_1\}$.*

Theorem 3.26. *Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS and $\mathcal{L} \neq 2^\beth$, Then, the collection $\beta_\mathcal{L} = \{T \cap P | T \in \Delta$ and $P \notin \mathcal{L}\}$ is a base for the ideal topology $\Delta^\bullet$ on $\beth$.*

Proof. Let $\Lambda \in \beta_\mathcal{L}$. Then, there exist $T \in \Delta$ and $P \notin \mathcal{L}$ such that $\Lambda = T \cap P$. Since $\Delta \subseteq \Delta^\bullet$, we get $T \in \Delta^\bullet$. On the other hand, since $\coprod^\bullet(P^c) = P^c$, we have $P \in \Delta^\bullet$. Therefore, $\Lambda \in \Delta^\bullet$. Consequently, $\beta_\mathcal{L} \subseteq \Delta^\bullet$. Now, let $\Gamma \in \Delta^\bullet$ and $\gimel \in \Gamma$. Then, from Theorem 3.17 (i), there exists $\Upsilon \in \Delta(\gimel)$ such that $\Upsilon^c \cup \Gamma \notin \mathcal{L}$. Now, let $\Lambda = \Upsilon \cap (\Upsilon^c \cup \Gamma)$. Hence, we have $\Lambda \in \beta_\mathcal{L}$ such that $\gimel \in \Lambda \subseteq \Gamma$. Therefore, $\beta_\mathcal{L}$ is a base for $\Delta^\bullet$. □

4. The $(\cdot)^{\bullet}_{R}$ operator and its topology $\Delta^{\bullet}_{R}$

In this section, we present and study a novel operator in ideal-TSs called $(\cdot)^{\bullet}_{R}$. Along with providing several counterexamples, we also derive some basic properties of this new operator. Furthermore, We characterize and study a novel operator that is a generalized closure operator. This leads to a new generalized topology that is finer than Δ_{δ}, which we name $\Delta^{\bullet}_{R}$.

Definition 4.1. Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS. An operator $(\cdot)^{\bullet}_{R} : 2^{\beth} \to 2^{\beth}$ given by $\Gamma^{\bullet}_{R} = \{\gimel \in \beth : (\forall \Upsilon \in RO(\beth, \gimel))(\Gamma^{c} \cup \Upsilon^{c} \in \mathcal{L})\}$ is called the $\bullet_{R}$-local function of any subset Γ of $\beth$ with respect to an ideal $\mathcal{L}$ and a topology Δ on $\beth$. We can also use the notation $\Gamma^{\bullet}_{R}$ as $\Gamma^{\bullet}_{R}(\beth, \Delta, \mathcal{L})$ to indicate the ideal and the topology as per our requirements.

Corollary 4.2. *Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS and $\Gamma \subseteq \beth$. The sets $\Gamma^{\bullet}_{R}$ and $\Gamma^{\bullet}$ coincide in regular topological spaces because the families of all open sets and all regular open sets are the same in those spaces.*

Corollary 4.3. *Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS and $\Gamma \subseteq \beth$. If $\beth$ is a compact Hausdorff space, then the sets $\Gamma^{\bullet}_{R}$ and $\Gamma^{\bullet}$ coincide.*

Proof. Follows from the fact that every compact Hausdorf space is regular. □

Remark 4.4. Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS and $\Gamma \subseteq \beth$. The inclusions of $\Gamma^{\bullet}_{R} \subseteq \Gamma$ or $\Gamma \subseteq \Gamma^{\bullet}_{R}$ need not always be true as shown by the following examples.

Example 4.5. *Let $\beth = \{\gimel_1, \gimel_2, \gimel_3\}$, $\Delta = \{\emptyset, \{\gimel_2\}, \{\gimel_3\}, \{\gimel_2, \gimel_3\}, \{\gimel_1, \gimel_3\}, \beth\}$ and $\mathcal{L} = \{\emptyset, \{\gimel_2\}, \{\gimel_3\}, \{\gimel_2, \gimel_3\}\}$. For the subset $\Gamma = \{\gimel_2, \gimel_3\}$, we get $\Gamma^{\bullet}_{R} = \emptyset$ but $\Gamma = \{\gimel_2, \gimel_3\} \nsubseteq \emptyset = \Gamma^{\bullet}_{R}$.*

Example 4.6. *Let $\beth = \{\gimel_1, \gimel_2, \gimel_3\}$, $\Delta = \{\emptyset, \beth\}$ and $\mathcal{L} = \{\emptyset, \{\gimel_2\}, \{\gimel_3\}, \{\gimel_2, \gimel_3\}\}$. For the subset $\Gamma = \{\gimel_1\}$, we get $\Gamma^{\bullet}_{R} = \beth$ but $\Gamma^{\bullet}_{R} = \beth \nsubseteq \{\gimel_1\} = \Gamma$.*

Theorem 4.7. *Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS and $\Gamma, \Lambda \subseteq \beth$. Then, the following statements hold:*

a) $\Gamma^{\bullet} \subseteq \Gamma^{\bullet}_{R}$,

b) *if* $\Gamma \in \delta C(\beth)$, *then* $\Gamma^{\bullet}_{R} \subseteq \Gamma$,

c) *if* $\mathcal{L} \neq 2^{\beth}$, *then* $\emptyset^{\bullet}_{R} = \emptyset$,

d) $\Gamma^{\bullet}_{R} \in \delta C(\beth)$,

e) $(\Gamma^{\bullet}_{R})^{\bullet}_{R} \subseteq \Gamma^{\bullet}_{R}$,

f) *if* $\Gamma \subseteq \Lambda$, *then* $\Gamma^{\bullet}_{R} \subseteq \Lambda^{\bullet}_{R}$, *g)* $\Gamma^{\bullet}_{R} \cup \Lambda^{\bullet}_{R} \subseteq (\Gamma \cup \Lambda)^{\bullet}_{R}$,

h) $(\Gamma \cap \Lambda)^{\bullet}_{R} = \Gamma^{\bullet}_{R} \cap \Lambda^{\bullet}_{R}$.

Proof. *a)* It is obvious since every regular open set in topological spaces is open.

b) Let $\Gamma \in \delta C(\beth)$ and $\gimel \notin \Gamma$. Our aim is to show that $\gimel \notin \Gamma^{\bullet}_{R}$.

$$\begin{aligned} \Gamma \in \delta C(\beth) \quad &\Rightarrow \quad \Gamma^c \in \delta O(\beth) \Rightarrow (\exists\, \Gamma \subseteq RO(\beth))\,(\Gamma^c = \bigcup \Gamma)\,(\gimel \notin \Gamma \Rightarrow \gimel \in \Gamma^c) \\ &\Rightarrow \quad (\exists \Lambda \in RO(\beth))(\gimel \in \Lambda \subseteq \Gamma^c) \\ &\Rightarrow \quad (\Lambda \in RO(\beth, \gimel))(\Lambda^c \cup \Gamma^c \supseteq (\Gamma^c)^c \cup \Gamma^c = \Gamma \cup \Gamma^c = \beth \notin \mathcal{L}) \\ &\Rightarrow \quad (\Lambda \in RO(\beth, \gimel))(\Lambda^c \cup \Gamma^c \notin \mathcal{L}) \\ &\Rightarrow \quad \gimel \notin \Gamma^{\bullet}_{R}. \end{aligned}$$

c) $\emptyset^{\bullet}_{R} = \{\gimel \in \beth : (\forall \Upsilon \in RO(\beth, \gimel))(\Upsilon^c \cup \emptyset^c = \Upsilon^c \cup \beth = \beth \in \mathcal{L})\} = \emptyset$.

d) Our aim is to show that $\Gamma^{\bullet}_{R} = \delta\text{-}\coprod(\Gamma^{\bullet}_{R})$. We have always $\Gamma^{\bullet}_{R} \subseteq \delta\text{-}\coprod(\Gamma^{\bullet}_{R})$.

Conversely, now let $\gimel \in \delta\text{-}\coprod(\Gamma^{\bullet}_{R})$ and $\Upsilon \in RO(\beth, \gimel)$.

$$\begin{aligned} (\Upsilon \in RO(\beth, \gimel))(\gimel \in \delta\text{-}\textstyle\coprod(\Gamma^{\bullet}_{R})) \quad &\Rightarrow \quad \Upsilon \cap \Gamma^{\bullet}_{R} \neq \emptyset \Rightarrow (\exists y \in \beth)(y \in \Upsilon \cap \Gamma^{\bullet}_{R}) \\ &\Rightarrow \quad (\exists y \in \beth)(y \in \Upsilon)(y \in \Gamma^{\bullet}_{R}) \\ &\Rightarrow \quad (\Upsilon \in RO(\beth, y))(y \in \Gamma^{\bullet}_{R}) \\ &\Rightarrow \quad \Upsilon^c \cup \Gamma^c \in \mathcal{L}. \end{aligned}$$

Then, we have $\gimel \in \Gamma^{\bullet}_{R}$. Thus, $\delta\text{-}\coprod(\Gamma^{\bullet}_{R}) \subseteq \Gamma^{\bullet}_{R}$. Therefore, $\delta\text{-}\coprod(\Gamma^{\bullet}_{R}) = \Gamma^{\bullet}_{R}$. Hence, $\Gamma^{\bullet}_{R} \in \delta C(\beth)$.

e) It is obvious from (*b*) and (*d*).

f) Let $\Gamma \subseteq \Lambda$ and $\gimel \in \Gamma^{\bullet}_{R}$. Our aim is to show that $\gimel \in \Lambda^{\bullet}_{R}$.

$$\begin{aligned} \gimel \in \Gamma^{\bullet}_{R} \quad &\Rightarrow \quad \left.\begin{array}{r} (\forall \Upsilon \in RO(\beth, \gimel))(\Upsilon^c \cup \Gamma^c \in \mathcal{L}) \\ \Gamma \subseteq \Lambda \Rightarrow \Lambda^c \subseteq \Gamma^c \end{array}\right\} \\ &\Rightarrow \quad \left.\begin{array}{r} (\forall \Upsilon \in RO(\beth, \gimel))(\Upsilon^c \cup \Lambda^c \subseteq \Upsilon^c \cup \Gamma^c \in \mathcal{L}) \\ \mathcal{L} \text{ is an ideal on } \beth \end{array}\right\} \\ &\Rightarrow \quad (\forall \Upsilon \in RO(\beth, \gimel))(\Upsilon^c \cup \Lambda^c \in \mathcal{L}) \Rightarrow \gimel \in \Lambda^{\bullet}_{R}. \end{aligned}$$

g) Let $\Gamma, \Lambda \subseteq \beth$.

$$\left.\begin{array}{l} \Gamma, \Lambda \subseteq \beth \Rightarrow \Gamma \subseteq \Gamma \cup \Lambda \Rightarrow \Gamma^{\bullet}_{R} \subseteq (\Gamma \cup \Lambda)^{\bullet}_{R} \\ \\ \Gamma, \Lambda \subseteq \beth \Rightarrow \Lambda \subseteq \Gamma \cup \Lambda \Rightarrow \Lambda^{\bullet}_{R} \subseteq (\Gamma \cup \Lambda)^{\bullet}_{R} \end{array}\right\} \Rightarrow \Gamma^{\bullet}_{R} \cup \Lambda^{\bullet}_{R} \subseteq (\Gamma \cup \Lambda)^{\bullet}_{R}.$$

h) It is clear from (*f*) that $\Gamma^{\bullet}_{R} \cap \Lambda^{\bullet}_{R} \supseteq (\Gamma \cap \Lambda)^{\bullet}_{R}$.
Conversely, let $\gimel \in \Gamma^{\bullet}_{R} \cap \Lambda^{\bullet}_{R}$.

$$\begin{array}{rcl} \gimel \in \Gamma^{\bullet}_{R} \cap \Lambda^{\bullet}_{R} & \Rightarrow & (\gimel \in \Gamma^{\bullet}_{R})(\gimel \in \Lambda^{\bullet}_{R}) \Rightarrow (\forall \Upsilon \in RO(\beth, \gimel))(\Upsilon^c \cup \Gamma^c \in \mathcal{L})(\Upsilon^c \cup \Lambda^c \in \mathcal{L}) \\ & \Rightarrow & (\Upsilon \in RO(\beth, \gimel))(\Upsilon^c \cup \Gamma^c \in \mathcal{L})(\Upsilon^c \cup \Lambda^c \in \mathcal{L}) \\ & \Rightarrow & (\Upsilon \in RO(\beth, \gimel))(\Upsilon^c \cup (\Gamma \cap \Lambda)^c = (\Upsilon^c \cup \Gamma^c) \cup (\Upsilon^c \cup \Lambda^c) \in \mathcal{L}) \\ & \Rightarrow & \gimel \in (\Gamma \cap \Lambda)^{\bullet}_{R}. \end{array}$$

Therefore $(\Gamma \cap \Lambda)^{\bullet}_{R} \supseteq \Gamma^{\bullet}_{R} \cap \Lambda^{\bullet}_{R}$. Hence, we have $(\Gamma \cap \Lambda)^{\bullet}_{R} = \Gamma^{\bullet}_{R} \cap \Lambda^{\bullet}_{R}$. □

Theorem 4.8. *Let* $(\beth, \Delta, \mathcal{L})$ *be an ideal-TS and* $\Gamma, \Lambda \subseteq \beth$. *If* $\Gamma \in \delta O(\beth)$, *then* $\Gamma \cap \Lambda^{\bullet}_{R} \subseteq (\Gamma \cap \Lambda)^{\bullet}_{R}$.

Proof. Let $\Gamma \in \delta O(\beth)$ and $\gimel \in \Gamma \cap \Lambda^{\bullet}_{R}$.

$$\left.\begin{array}{r} \gimel \in \Gamma \cap \Lambda^{\bullet}_{R} \Rightarrow (\gimel \in \Gamma)(\gimel \in \Lambda^{\bullet}_{R}) \Rightarrow (\gimel \in \Gamma)(\forall \Upsilon \in RO(\beth, \gimel))(\Upsilon^c \cup \Lambda^c \in \mathcal{L}) \\ \\ \Gamma \in \delta O(\beth) \Rightarrow (\exists\ \mathcal{A} \subseteq RO(\beth))\,(\Gamma = \bigcup \mathcal{A}) \Rightarrow (\exists \daleth \in RO(\beth))(\daleth \subseteq \Gamma) \end{array}\right\}$$

So $\forall \Upsilon \in RO(\beth, \gimel)$, we have $\Upsilon \cap \daleth \in RO(\beth, \gimel)$. Therefore $\Upsilon^c \cup (\Gamma \cap \Lambda)^c = (\Upsilon \cap \Gamma)^c \cup \Lambda^c \subseteq (\Upsilon \cap \daleth)^c \cup \Lambda^c \in \mathcal{L}$. It follows that

$$\Upsilon^c \cup (\Gamma \cap \Lambda)^c \in \mathcal{L} \text{ for all } \Upsilon \in RO(\beth, \gimel).$$

Hence $\gimel \in (\Gamma \cap \Lambda)^{\bullet}_{R}$.

□

Theorem 4.9. *Let* $(\beth, \Delta, \mathcal{L})$ *be an ideal-TS. Then, the following statements are equivalent:*

a) $\beth^{\bullet}_{R} = \beth$;

b) $RC(\beth) \setminus \{\beth\} \subseteq \mathcal{L}$;

c) $\Gamma \subseteq \Gamma^{\bullet}_{R}$ *for all regular open subsets* Γ *of* $\beth$.

Proof. $(a) \Rightarrow (b)$: Let $\beth^{\bullet}_{R} = \beth$.

$$\begin{array}{rcl} \beth^{\bullet}_{R} = \beth & \Rightarrow & (\forall \gimel \in \beth)(\gimel \in \beth^{\bullet}_{R}) \\ & \Rightarrow & (\forall \gimel \in \beth)(\forall \Upsilon \in RO(\beth, \gimel))(\Upsilon^c \cup \beth^c = \Upsilon^c \in \mathcal{L}) \\ & \Rightarrow & (\forall \daleth \in RC(\beth) \setminus \{\beth\})(\daleth \in \mathcal{L}) \\ & \Rightarrow & RC(\beth) \setminus \{\beth\} \subseteq \mathcal{L}. \end{array}$$

$(b) \Rightarrow (a)$: Let $\gimel \in \beth$ and $\Upsilon \in RO(\beth, \gimel)$.

$$\left.\begin{array}{r} \Upsilon \in RO(\beth, \gimel) \Rightarrow \Upsilon^c \in RC(\beth) \setminus \{\beth\} \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \Upsilon^c \cup \beth^c = \Upsilon^c \cup \emptyset = \Upsilon^c \in \mathcal{L}.$$

Then, we have $\gimel \in \beth^{\bullet}_R$. Thus, $\beth \subseteq \beth^{\bullet}_R \subseteq \beth$ and so $\beth^{\bullet}_R = \beth$.

$(b) \Rightarrow (c)$: Let $\Gamma \in RO(\beth)$.

$$\left.\begin{array}{r} \Gamma \in RO(\beth) \overset{\text{Theorem 4.8}}{\Rightarrow} \Gamma \cap \beth^{\bullet}_R \subseteq (\Gamma \cap \beth)^{\bullet}_R = \Gamma^{\bullet}_R \\ RC(\beth) \setminus \{\beth\} \subseteq \mathcal{L} \Rightarrow \beth^{\bullet}_R = \beth \end{array}\right\} \Rightarrow \Gamma \subseteq \Gamma^{\bullet}_R.$$

$(c) \Rightarrow (b)$: Let $\Gamma \in RC(\beth) \setminus \{\beth\}$.

$$\Gamma \in RC(\beth) \setminus \{\beth\} \quad \Rightarrow \quad \left.\begin{array}{r} \Gamma^c \in RO(\beth) \setminus \{\emptyset\} \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \Gamma^c \subseteq (\Gamma^c)^{\bullet}_R \Rightarrow (\forall \gimel \in \Gamma^c)(\gimel \in (\Gamma^c)^{\bullet}_R)$$

$$\Rightarrow \quad \left.\begin{array}{r} (\forall \gimel \in \Gamma^c)(\forall \Upsilon \in RO(\beth, \gimel))(\Upsilon^c \cup (\Gamma^c)^c = \Upsilon^c \cup \Gamma \in \mathcal{L}) \\ \Gamma \subseteq \Upsilon^c \cup \Gamma \end{array}\right\}$$

Then, $\Gamma \in \mathcal{L}$. Hence, we have $RC(\beth) \setminus \{\beth\} \subseteq \mathcal{L}$. □

Theorem 4.10. *Let $\mathcal{L}$ be an ideal on TS $(\beth, \Delta)$ and $\Gamma \subseteq \beth$. If $\Gamma^{\bullet}_R \neq \emptyset$, then $\Gamma^c \in \mathcal{L}$.*

Proof. Let $\Gamma^{\bullet}_R \neq \emptyset$.

$$\left.\begin{array}{r} \Gamma^{\bullet}_R \neq \emptyset \Rightarrow (\exists \gimel \in \beth)(\gimel \in \Gamma^{\bullet}_R) \Rightarrow (\forall \Upsilon \in RO(\beth, \gimel))(\Gamma^c \subseteq \Upsilon^c \cup \Gamma^c \in \mathcal{L}) \\ \mathcal{L} \text{ is an ideal on } \beth \end{array}\right\} \Rightarrow \Gamma^c \in \mathcal{L}.$$

□

Corollary 4.11. *Let $\mathcal{L}$ be an ideal on TS $(\beth, \Delta)$ and $\Gamma \subseteq \beth$. If $\Gamma^c \notin \mathcal{L}$, then $\Gamma^{\bullet}_R = \emptyset$.*

Proof. It is obvious from Theorem 4.10. □

Theorem 4.12. *Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS and $\Gamma, \Lambda \subseteq \beth$. Then, $\Gamma^{\bullet}_R \setminus \Lambda^{\bullet}_R \supseteq (\Gamma \setminus \Lambda)^{\bullet}_R \setminus \Lambda^{\bullet}_R$.*

Proof. Let $\Gamma, \Lambda \subseteq \beth$.

$$\begin{array}{rl} \Gamma, \Lambda \subseteq \beth & \Rightarrow \quad \Gamma \setminus \Lambda \subseteq \Gamma \\ & \Rightarrow \quad (\Gamma \setminus \Lambda)^{\bullet}_R \subseteq \Gamma^{\bullet}_R \\ & \Rightarrow \quad (\Gamma \setminus \Lambda)^{\bullet}_R \setminus \Lambda^{\bullet}_R \subseteq \Gamma^{\bullet}_R \setminus \Lambda^{\bullet}_R. \end{array}$$

□

Theorem 4.13. *Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS and $\Gamma, \Lambda \subseteq \beth$. If $\Lambda^c \notin \mathcal{L}$, then $(\Gamma \cup \Lambda)^{\bullet}_R \supseteq \Gamma^{\bullet}_R \supseteq \Gamma \setminus \Lambda)^{\bullet}_R$.*

Proof. Let $\Gamma, \Lambda \subseteq \beth$.

$$\left.\begin{array}{r}\Gamma, \Lambda \subseteq \beth \overset{\text{Theorem 4.12}}{\Rightarrow} \Gamma^{\bullet}_{R} \setminus \Lambda^{\bullet}_{R} \supseteq (\Gamma \setminus \Lambda)^{\bullet}_{R} \setminus \Lambda^{\bullet}_{R} \\ \Lambda^{c} \notin \mathcal{L} \overset{\text{Corollary 4.11}}{\Rightarrow} \Lambda^{\bullet}_{R} = \emptyset\end{array}\right\} \Rightarrow \Gamma^{\bullet}_{R} \supseteq (\Gamma \setminus \Lambda)^{\bullet}_{R}.$$

$$\left.\begin{array}{r}\Gamma, \Lambda \subseteq \beth \overset{\text{Theorem 3.4}}{\Rightarrow} (\Gamma \cup \Lambda)^{\bullet}_{R} \supseteq \Gamma^{\bullet}_{R} \cup \Lambda^{\bullet}_{R} \\ \Lambda^{c} \notin \mathcal{L} \overset{\text{Corollary 4.11}}{\Rightarrow} \Lambda^{\bullet}_{R} = \emptyset\end{array}\right\} \Rightarrow (\Gamma \cup \Lambda)^{\bullet}_{R} \supseteq \Gamma^{\bullet}_{R}.$$

Hence $(\Gamma \cup \Lambda)^{\bullet}_{R} \supseteq \Gamma^{\bullet}_{R} \supseteq (\Gamma \setminus \Lambda)^{\bullet}_{R}$. □

Definition 4.14. Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS. We define an operator $\coprod^{\bullet}_{R} : 2^{\beth} \to 2^{\beth}$ as $\coprod^{\bullet}_{R}(\Gamma) = \Gamma \cup \Gamma^{\bullet}_{R}$ and $\coprod^{\bullet}_{R}(\phi) = \phi$, where $\Gamma \neq \phi$ is any subset of $\beth$.

Corollary 4.15. *Let $\beth$ be a TS and $\Gamma \subseteq \beth$. If $\beth$ is a regular space, then the sets $\coprod^{\bullet}_{R}(\Gamma)$ and $\coprod^{\bullet}(\Gamma)$ coincide.*

Proof. It is obvious from Corollary 4.2 and Definition 4.14. □

Corollary 4.16. *Let $\beth$ be a TS and $\Gamma \subseteq \beth$. If $\beth$ is a compact Hausdorff space, then the sets $\coprod^{\bullet}_{R}(\Gamma)$ and $\coprod^{\bullet}(\Gamma)$ coincide.*

Proof. It is obvious from Corollary 4.3. □

Theorem 4.17. *Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS and $\Gamma, \Lambda \subseteq \beth$. Then, the following statements hold:*

a) $\coprod^{\bullet}_{R}(\emptyset) = \emptyset$,

b) $\coprod^{\bullet}_{R}(\beth) = \beth$,

c) $\Gamma \subseteq \coprod^{\bullet}(\Gamma) \subseteq \coprod^{\bullet}_{R}(\Gamma)$,

d) if $\Gamma \subseteq \Lambda$, *then* $\coprod^{\bullet}_{R}(\Gamma) \subseteq \coprod^{\bullet}_{R}(\Lambda)$,

e) $\coprod^{\bullet}_{R}(\Gamma \cup \Lambda) \supseteq \coprod^{\bullet}_{R}(\Gamma) \cup \coprod^{\bullet}_{R}(\Lambda)$,

f) $\coprod^{\bullet}_{R}(\coprod^{\bullet}_{R}(\Gamma)) = \coprod^{\bullet}_{R}(\Gamma)$.

g)$\coprod^{\bullet}_{R}(\Gamma \cap \Lambda) = \coprod^{\bullet}_{R}(\Gamma) \cap \coprod^{\bullet}_{R}(\Lambda)$.

Proof. *a)* It is obvious from Definition 4.14.

b) Since $\beth^{\bullet}_{R} \subseteq \beth$, we have $\coprod^{\bullet}_{R}(\beth) = \beth \cup \beth^{\bullet}_{R} = \beth$.

c) Since $\coprod^{\bullet}_{R}(\Gamma) = \Gamma \cup \Gamma^{\bullet}_{R}$, we have $\Gamma \subseteq \coprod^{\bullet}_{R}(\Gamma)$. Also, since $\Gamma^{\bullet} \subseteq \Gamma^{\bullet}_{R}$, we have $\coprod^{\bullet}(\Gamma) \subseteq \coprod^{\bullet}_{R}(\Gamma)$.

d) Let $\Gamma \subseteq \Lambda$.

$$\Gamma \subseteq \Lambda \Rightarrow \Gamma_R^{\bullet} \subseteq \Lambda_R^{\bullet} \Rightarrow \Gamma \cup \Gamma_R^{\bullet} \subseteq \Lambda \cup \Lambda_R^{\bullet} \Rightarrow \coprod_R^{\bullet}(\Gamma) \subseteq \coprod_R^{\bullet}(\Lambda).$$

e) It is obvious from the definition of the operator $\coprod_R^{\bullet}$ in Definition 4.14 and (g) of Theorem 4.7.

f) Let $\Gamma \subseteq \beth$. It is obvious from (*c*) that $\coprod_R^{\bullet}(\Gamma) \subseteq \coprod_R^{\bullet}(\coprod_R^{\bullet}(\Gamma))$.

Conversely, let $\gimel \in \coprod_R^{\bullet}(\coprod_R^{\bullet}(\Gamma)) = \coprod_R^{\bullet}(\Gamma) \cup \left(\coprod_R^{\bullet}(\Gamma)\right)_R^{\bullet}$ and $\Upsilon \in \Delta(\gimel)$. Then, we have the following two cases:

(i) $\gimel \in \coprod_R^{\bullet}(\Gamma)$. This means that $\coprod_R^{\bullet}(\coprod_R^{\bullet}(\Gamma)) \subseteq \coprod_R^{\bullet}(\Gamma)$.

(ii) Let $\gimel \notin \coprod_R^{\bullet}(\Gamma)$. Then, we have

$$\begin{aligned}
\gimel \in (\Gamma \cup \Gamma_R^{\bullet})_R^{\bullet} \quad &\Rightarrow \quad [\Upsilon^c \cup (\Gamma \cup \Gamma_R^{\bullet})^c \in \mathcal{L} \ \forall \Upsilon \in \Delta(\gimel)][\exists\ \Upsilon \in \Delta(\gimel):\ \Upsilon^c \cup (\Gamma)^c \notin \mathcal{L}] \\
&\Rightarrow \quad \Upsilon^c \in \mathcal{L} \text{ and } \Gamma^c \notin \mathcal{L} \\
&\Rightarrow \quad \Upsilon^c \cup \Gamma^c \notin \mathcal{L} \text{ for all } \Upsilon \in \Delta(\gimel^*), \gimel^* \in \beth \\
&\Rightarrow \quad \Gamma_R^{\bullet} = \emptyset \Rightarrow \gimel \in \coprod_R^{\bullet}(\Gamma), \text{ which is a contradiction} \\
&\Rightarrow \quad \coprod_R^{\bullet}(\coprod_R^{\bullet}(\Gamma)) \subseteq \coprod_R^{\bullet}(\Gamma).
\end{aligned}$$

Thus, from (i), (ii) we have $\coprod_R^{\bullet}(\coprod_R^{\bullet}(\Gamma)) \subseteq \coprod_R^{\bullet}(\Gamma)$.

Consequentially $\coprod_R^{\bullet}(\coprod_R^{\bullet}(\Gamma)) = \coprod_R^{\bullet}(\Gamma)$.

g) Let $\Gamma, \Lambda \subseteq \beth$.

$$\begin{aligned}
\coprod_R^{\bullet}(\Gamma \cap \Lambda) &= (\Gamma \cap \Lambda) \cup (\Gamma \cap \Lambda)_R^{\bullet} \\
&= (\Gamma \cap \Lambda) \cup (\Gamma_R^{\bullet} \cap \Lambda_R^{\bullet}) \\
&\subseteq (\Gamma \cup \Gamma_R^{\bullet}) \cap (\Lambda \cup \Lambda_R^{\bullet}) \\
&= \coprod_R^{\bullet}(\Gamma) \cap \coprod_R^{\bullet}(\Lambda).
\end{aligned}$$

Conversely, let $\gimel \in \coprod_R^{\bullet}(\Gamma) \cap \coprod_R^{\bullet}(\Lambda)$.

$$\begin{aligned}
\gimel \in \coprod_R^{\bullet}(\Gamma) \cap \coprod_R^{\bullet}(\Lambda) \quad &\Rightarrow \quad (\gimel \in \Gamma)(\gimel \in \Gamma_R^{\bullet}) \text{ and } (\gimel \in \Lambda)(\gimel \in \Lambda_R^{\bullet}) \\
&\Rightarrow \quad (\forall \Upsilon \in RO(\beth, \gimel))(\gimel \in \Gamma \text{ and } \gimel \in \Lambda)(\Upsilon^c \cup \Gamma^c \in \mathcal{L})(\Upsilon^c \cup \Lambda^c \in \mathcal{L}) \\
&\Rightarrow \quad (\Upsilon \in RO(\beth, \gimel))(\gimel \in \Gamma \text{ and } \gimel \in \Lambda)(\Upsilon^c \cup \Gamma^c \in \mathcal{L})(\Upsilon^c \cup \Lambda^c \in \mathcal{L}) \\
&\Rightarrow \quad (\Upsilon \in RO(\beth, \gimel))(\gimel \in \Gamma \text{ and } \gimel \in \Lambda) \\
&\Rightarrow \quad (\Upsilon^c \cup (\Gamma \cap \Lambda)^c = (\Upsilon^c \cup \Gamma^c) \cup (\Upsilon^c \cup \Lambda^c) \in \mathcal{L}) \\
&\Rightarrow \quad \gimel \in \Gamma \cap \Lambda \text{ and } \gimel \in (\Gamma \cap \Lambda)_R^{\bullet}.
\end{aligned}$$

Thus $\gimel \in \coprod_R^{\bullet}(\Gamma \cap \Lambda)$. Therefore $\coprod_R^{\bullet}(\Gamma \cap \Lambda) \supseteq \coprod_R^{\bullet}(\Gamma) \cap \coprod_R^{\bullet}(\Lambda)$. Hence, we have $(\Gamma \cap \Lambda)_R^{\bullet} = \Gamma_R^{\bullet} \cap \Lambda_R^{\bullet}$. □

Corollary 4.18. *Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS. Then the operator $\coprod_R^\bullet : 2^\beth \to 2^\beth$ defined by $\coprod_R^\bullet(\Gamma) = \Gamma \cup \Gamma_R^\bullet$, where Γ is any subset of $\beth$, is a generalized closure operator. But it is not a Kuratowski's closure operator.*

Definition 4.19. Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS. Then, the collection $\Delta_R^\bullet = \{\Gamma \subseteq \beth : \coprod_R^\bullet(\Gamma^c) = \Gamma^c\}$ is a generalized topology on $\beth$ introduced by the generalized topology Δ and ideal $\mathcal{L}$. Also, we can use the notation $\Delta_{R(\mathcal{L})}^\bullet$ instead of $\Delta_R^\bullet$ to illustrate the ideal as per our requirements.

Corollary 4.20. *Let $\beth$ be a TS and $\Gamma \subseteq \beth$. If $\beth$ is a regular space, then $\Delta_R^\bullet$ and $\Delta^\bullet$ coincide.*

Proof. It is obvious from Corollary 4.16. □

Corollary 4.21. *Let $\beth$ be a TS and $\Gamma \subseteq \beth$. If $\beth$ is a compact Hausdorff space, then the generalized topologies $\Delta_R^\bullet$ and $\Delta^\bullet$ coincide.*

Proof. It is obvious since every compact Hausdorff space is regular. □

Theorem 4.22. *Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS. Then, the following statements hold:*

a) $\Delta_\delta \subseteq \Delta_R^\bullet$, *where Δ_δ is the collection of all δ-open sets in a TS $(\beth, \Delta)$.*

b) $\Delta_R^\bullet \subseteq \Delta^\bullet$.

Proof. *a)* Let $\Gamma \in \Delta_\delta$. Our aim is to show that $\Gamma \in \Delta_R^\bullet$.

$$\begin{aligned} \Gamma \in \Delta_\delta \quad &\Rightarrow \quad \Gamma^c \in \delta C(\beth) \Rightarrow (\Gamma^c)_R^\bullet \subseteq \Gamma^c \\ &\Rightarrow \quad (\Gamma^c \cup (\Gamma^c)_R^\bullet = \Gamma^c)(\textstyle\coprod_R^\bullet(\Gamma^c) = \Gamma^c \cup (\Gamma^c)_R^\bullet) \\ &\Rightarrow \quad \textstyle\coprod_R^\bullet(\Gamma^c) = \Gamma^c \Rightarrow \Gamma \in \Delta_R^\bullet. \end{aligned}$$

b) Let $\Gamma \in \Delta_R^\bullet$. Our aim is to show that $\Gamma \in \Delta^\bullet$.

$$\begin{aligned} \Gamma \in \Delta_R^\bullet \quad &\Rightarrow \quad \textstyle\coprod_R^\bullet(\Gamma^c) = \Gamma^c \\ &\Rightarrow \quad \Gamma^c \cup (\Gamma^c)_R^\bullet = \Gamma^c \\ &\overset{\text{Theorem 4.17}}{\Rightarrow} \quad (\Gamma^c)^\bullet \subseteq (\Gamma^c)_R^\bullet \subseteq \Gamma^c \\ &\Rightarrow \quad (\Gamma^c)^\bullet \cup \Gamma^c = \Gamma^c \\ &\Rightarrow \quad \textstyle\coprod^\bullet(\Gamma^c) = \Gamma^c \\ &\Rightarrow \quad \Gamma \in \Delta^\bullet. \quad \square \end{aligned}$$

Remark 4.23. Let $(\daleth, \Delta, \mathcal{L})$ be an ideal-TS. If $\mathcal{L} \neq 2^{\daleth}$, then we have the following diagram from Theorem 4.22.

$$\begin{array}{ccccc} & & \Delta^{\bullet}\text{-}open & & \\ & \nearrow & & \nwarrow & \\ \Delta^{\bullet}_{R}\text{-}open & & & & \Delta\text{-}open \\ & \nwarrow & & \nearrow & \\ & & \Delta_{\delta}\text{-open} & & \end{array}$$

Remark 4.24. Let $(\daleth, \Delta, \mathcal{L})$ be an ideal-TS. The following examples demonstrate that the reverses of the implications stated in the above diagram need not be true. Furthermore, there is no interdependence between the concepts of $\Delta^{\bullet}_{R}$-open and Δ-open.

Example 4.25. *Let* $\daleth = \{\gimel_1, \gimel_2, \gimel_3\}$ *with the topology* $\Delta = \{\emptyset, \daleth, \{\gimel_1, \gimel_2\}, \{\gimel_2, \gimel_3\}, \{\gimel_2\}\}$ *and let* $\mathcal{L} = \{\emptyset, \{\gimel_1\}, \{\gimel_2\}, \{\gimel_1, \gimel_2\}\}$. *Simple calculations show that* $\Delta_{\delta} = \{\emptyset, \daleth\}$, $\Delta^{\bullet}_{R} = \{\emptyset, \daleth, \{\gimel_3\}, \{\gimel_1, \gimel_3\}, \{\gimel_2, \gimel_3\}\}$, *and* $\Delta^{\bullet} = 2^{\daleth}$. *By computing, we find the following results:*

1) *The set* $\{\gimel_3\}$ *is* $\Delta^{\bullet}_{R}$*-open but it is not* Δ_{δ}*-open.*

2) *The set* $\{\gimel_1\}$ *is* $\Delta^{\bullet}$*-open but it is not* $\Delta^{\bullet}_{R}$*-open.*

3) *The set* $\{\gimel_3\}$ *is* $\Delta^{\bullet}_{R}$*-open but it is not* Δ*-open.*

4) *The set* $\{\gimel_2\}$ *is* Δ*-open but it is not* $\Delta^{\bullet}_{R}$*-open.*

Theorem 4.26. *Let* $(\daleth, \Delta, \mathcal{L})$ *be an ideal-TS and* $\Gamma \subseteq \daleth$. *Then, the following statements hold:*

a) $\Gamma \in \Delta^{\bullet}_{R}$ *if and only if for all* $\gimel$ *in* Γ, *there exists a regular open set* Υ *containing* $\gimel$ *such that* $\Upsilon^{c} \cup \Gamma \notin \mathcal{L}$,

b) if $\Gamma \notin \mathcal{L}$, *then* $\Gamma \in \Delta^{\bullet}_{R}$.

Proof. a) Let $\Gamma \in \Delta^{\bullet}_{R}$.

$$\begin{aligned} \Gamma \in \Delta^{\bullet}_{R} &\Leftrightarrow \textstyle\coprod^{\bullet}_{R}(\Gamma^{c}) = \Gamma^{c} \\ &\Leftrightarrow \Gamma^{c} \cup (\Gamma^{c})^{\bullet}_{R} = \Gamma^{c} \\ &\Leftrightarrow (\Gamma^{c})^{\bullet}_{R} \subseteq \Gamma^{c} \\ &\Leftrightarrow \Gamma \subseteq ((\Gamma^{c})^{\bullet}_{R})^{c} \\ &\Leftrightarrow (\forall \gimel \in \Gamma)(\gimel \notin (\Gamma^{c})^{\bullet}_{R}) \\ &\Leftrightarrow (\forall \gimel \in \Gamma)(\exists \Upsilon \in RO(\daleth, \gimel))(\Upsilon^{c} \cup (\Gamma^{c})^{c} = \Upsilon^{c} \cup \Gamma \notin \mathcal{L}). \end{aligned}$$

b) Let $\Gamma \notin \mathcal{L}$ and $\gimel \in \Gamma$. We will make use of $(\gimel_1)$.

$$\left.\begin{array}{r}(\Upsilon := \sqsupset)(\gimel \in \Gamma) \Rightarrow (\Upsilon \in RO(\sqsupset, \gimel))(\Gamma = \Upsilon^c \cup \Gamma) \\ \Gamma \notin \mathcal{L}\end{array}\right\} \Rightarrow \Upsilon^c \cup \Gamma \notin \mathcal{L}. \qquad \square$$

Remark 4.27. Let $(\sqsupset, \Delta, \mathcal{L})$ be an ideal-TS. The reverse of Theorem 4.26 (b) need not be always true as shown by the following example.

Example 4.28. *Let* $\sqsupset = \{\gimel_1, \gimel_2, \gimel_3\}$ *with the topology* $\Delta = \{\emptyset, \sqsupset, \{\gimel_1\}, \{\gimel_2\}, \{\gimel_1, \gimel_2\}\}$ *and* $\mathcal{L} = \{\emptyset, \{\gimel_1\}, \{\gimel_2\}, \{\gimel_1, \gimel_2\}\}$. *Simple calculations show that* $\Delta^\bullet_R = 2^{\sqsupset}$. *It is obvious that the set* $\{\gimel_2\}$ *belongs to both* $\Delta^\bullet_R$ *and* $\mathcal{L}$.

Theorem 4.29. *Let* $(\sqsupset, \Delta, \mathcal{L})$ *be an ideal-TS. Then, if* $\mathcal{L} = \{\emptyset\}$, *then* $\Delta^\bullet_R = 2^{\sqsupset}$.

Proof. We have always $\Delta^\bullet_R \subseteq 2^{\sqsupset}$. Now, let $\Gamma \in 2^{\sqsupset}$. Our aim is to show that $\Gamma \in \Delta^\bullet_R$.

$$\left.\begin{array}{l}\coprod^\bullet_R(\Gamma^c) = \Gamma^c \cup (\Gamma^c)^\bullet_R \\ \mathcal{L} = \{\emptyset\} \Rightarrow (\Gamma^c)^\bullet_R = \emptyset\end{array}\right\} \Rightarrow \coprod^\bullet_R(\Gamma^c) = \Gamma^c \Rightarrow \Gamma \in \Delta^\bullet_R.$$

Then, we have $2^{\sqsupset} \subseteq \Delta^\bullet_R$. Hence $\Delta^\bullet_R = 2^{\sqsupset}$. $\square$

Remark 4.30. Let $(\sqsupset, \Delta, \mathcal{L})$ be an ideal-TS. The reverses of Theorem 3.16(a) and Theorem 4.29 need not be true as shown by the following example.

Example 4.31. *Let* $\sqsupset = \{\gimel_1, \gimel_2, \gimel_3\}$ *with the topology* $\Delta = \{\emptyset, \sqsupset, \{\gimel_1\}, \{\gimel_2\}, \{\gimel_1, \gimel_2\}\}$ *and* $\mathcal{L} = \{\emptyset, \{\gimel_1\}, \{\gimel_2\}, \{\gimel_1, \gimel_2\}\}$. *Simple calculations show that* $\Delta^\bullet_R = 2^{\sqsupset}$, *but* $\mathcal{L} \neq \{\emptyset\}$.

Theorem 4.32. *Let* $(\sqsupset, \Delta, \mathcal{L})$ *be an ideal-TS. Then, the collection* $\beta = \{T \cap P | (T \in \Delta_\delta)(P \notin \mathcal{L})\}$ *is a base for the topology* $\Delta^\bullet_R$ *on* $\sqsupset$.

Proof. Let $\Lambda \in \beta$.

$$\left.\begin{array}{r}\Lambda \in \beta \Rightarrow (\exists T \in \Delta_\delta)(\exists P \notin \mathcal{L})(\Lambda = T \cap P) \\ \Delta_\delta \subseteq \Delta^\bullet_R\end{array}\right\} \overset{\text{Theorem 4.26}}{\Rightarrow} (T, P \in \Delta^\bullet_R)(\Lambda = T \cap P) \Rightarrow \Lambda \in \Delta^\bullet_R.$$

Then, we have $\beta \subseteq \Delta^\bullet_R$. Now, let $\Gamma \in \Delta^\bullet_R$ and $\gimel \in \Gamma$. Our aim is to find $\Lambda \in \beta$ such that $\gimel \in \Lambda \subseteq \Gamma$.

$$\left.\begin{array}{r}\gimel \in \Gamma \in \Delta^\bullet_R \overset{\text{Theorem 4.26}}{\Rightarrow} (\exists \Upsilon \in RO(\sqsupset, \gimel))(\Upsilon^c \cup \Gamma \notin \mathcal{L}) \\ \Lambda := \Upsilon \cap (\Upsilon^c \cup \Gamma)\end{array}\right\} \Rightarrow (\Lambda \in \beta)(\gimel \in \Lambda \subseteq \Gamma).$$

Hence, β is a base for the topology $\Delta^\bullet_R$ on $\sqsupset$. $\square$

Theorem 4.33. *Let* $(\sqsupset, \Delta, \mathcal{L})$ *and* $(\sqsupset, \Delta, \mathcal{Q})$ *be two ideal-TSs. If* $\mathcal{L} \subseteq \mathcal{Q}$, *then* $\Delta^\bullet_{R(\mathcal{Q})} \subseteq \Delta^\bullet_{R(\mathcal{L})}$.

Proof. Let $\Gamma \in \Delta^{\bullet}_{R(\mathcal{Q})}$.

$$\left.\begin{array}{r}\Gamma \in \Delta^{\bullet}_{R(\mathcal{Q})} \Rightarrow (\forall \gimel \in \Gamma)(\exists \Upsilon \in RO(\beth, \gimel))(\Upsilon^c \cup \Gamma \notin \mathcal{Q}) \\ \mathcal{L} \subseteq \mathcal{Q}\end{array}\right\} \Rightarrow (\forall \gimel \in \Gamma)(\exists \Upsilon \in RO(\beth, \gimel))(\Upsilon^c \cup \Gamma \notin \mathcal{L}) \Rightarrow \Gamma \in \Delta^{\bullet}_{R(\mathcal{L})}.$$ □

5. Topology suitable for an ideal

In this section, we study various properties and introduce topology compatible for an ideal on an ideal-TS.

Definition 5.1. Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS. Then, Δ is said to be suitable for the ideal $\mathcal{L}$ if $\Gamma^c \cup \Gamma^{\bullet} \notin \mathcal{L}$, for all $\Gamma \subseteq \beth$.

Example 5.2. *Let $\beth = \{\gimel_1, \gimel_2, \gimel_3\}$ and $\Delta = \{\emptyset, \{\gimel_1\}, \{\gimel_2\}, \{\gimel_1, \gimel_2\}, \beth\}$. We consider an ideal $\mathcal{L} = \{\emptyset, \{\gimel_1\}, \{\gimel_2\}, \{\gimel_1, \gimel_2\}\}$ on $\beth$. By computing, we obtain the following table.*

Table 1. Illustration of Definition 5.1

$\Gamma \subseteq \beth$	Γ^c	$\Gamma^{\bullet}$	$\Gamma^c \cup \Gamma^{\bullet}$	$\Gamma^c \cup \Gamma^{\bullet} \notin \mathcal{L}$?
$\emptyset$	$\beth$	$\emptyset$	$\beth$	*Yes*
$\{\gimel_1\}$	$\{\gimel_2, \gimel_3\}$	$\emptyset$	$\{\gimel_2, \gimel_3\}$	*Yes*
$\{\gimel_2\}$	$\{\gimel_1, \gimel_3\}$	$\emptyset$	$\{\gimel_1, \gimel_3\}$	*Yes*
$\{\gimel_3\}$	$\{\gimel_1, \gimel_2\}$	$\gimel_3$	$\beth$	*Yes*
$\{\gimel_1, \gimel_2\}$	$\{\gimel_3\}$	$\emptyset$	$\{\gimel_3\}$	*Yes*
$\{\gimel_1, \gimel_3\}$	$\{\gimel_2\}$	$\gimel_3$	$\{\gimel_2, \gimel_3\}$	*Yes*
$\{\gimel_2, \gimel_3\}$	$\{\gimel_1\}$	$\gimel_3$	$\{\gimel_1, \gimel_3\}$	*Yes*
$\beth$	$\emptyset$	$\gimel_3$	$\{\gimel_3\}$	*Yes*

From Table 1, the topology $\Delta = \{\emptyset, \{\gimel_1\}, \{\gimel_2\}, \{\gimel_1, \gimel_2\}, \beth\}$ is suitable for the ideal $\mathcal{L} = \{\emptyset, \{\gimel_1\}, \{\gimel_2\}, \{\gimel_1, \gimel_2\}\}$. Clearly, if $\mathcal{L} = 2^{\beth}$, then the topology Δ is not suitable for the ideal $\mathcal{L}$.

We now give some equivalent descriptions of this definition.

Theorem 5.3. *For an ideal-TS $(\beth, \Delta, \mathcal{L})$, the following are equivalent:*

(1) *Δ is suitable for the ideal $\mathcal{L}$,*
(2) *for any $\Delta^{\bullet}$-closed subset Γ of $\beth$, $\Gamma^c \cup \Gamma^{\bullet} \notin \mathcal{L}$,*
(3) *whenever for any $\Gamma \subseteq \beth$ and each $\gimel \in \Gamma$ there corresponds some $U_x \in \Delta(\gimel)$ with $U^c_{\gimel} \cup \Gamma^c \notin \mathcal{L}$, it follows that $\Gamma^c \notin \mathcal{L}$,*
(4) *for $\Gamma \subseteq \beth$ and $\Gamma \cap \Gamma^{\bullet} = \emptyset$, it follows that $\Gamma^c \notin \mathcal{L}$.*

Proof. (1) $\Rightarrow$ (2): It is trivial. (2) $\Rightarrow$ (3): Let $\Gamma \subseteq \beth$ and assume that for every $\gimel \in \Gamma$ there exists $\Upsilon \in \Delta(\gimel)$ such that $\Upsilon^c \cup \Gamma^c \notin \mathcal{L}$. Then, $\gimel \notin \Gamma^{\bullet}$ so that $\Gamma \cap \Gamma^{\bullet} = \emptyset$. Since $\Gamma \cup \Gamma^{\bullet}$ is $\Delta^{\bullet}$-closed, by (2) we have $(\Gamma \cup \Gamma^{\bullet})^c \cup (\Gamma \cup \Gamma^{\bullet})^{\bullet} \notin \mathcal{L}$. That is, $(\Gamma \cup \Gamma^{\bullet})^c \cup (\Gamma^{\bullet} \cup (\Gamma^{\bullet})^{\bullet}) \notin \mathcal{L}$ by Theorem 3.4, i.e., $(\Gamma \cup \Gamma^{\bullet})^c \cup \Gamma^{\bullet} \notin \mathcal{L}$ by Theorem 3.4, i.e., $\Gamma^c \notin \mathcal{L}$ (as $\Gamma \cap \Gamma^{\bullet} = \emptyset$).

(3) $\Rightarrow$ (4): If $\Gamma \subseteq \beth$ and $\Gamma \cap \Gamma^\bullet = \emptyset$, then $\Gamma \subseteq \beth \setminus \Gamma^\bullet$. Let $\daleth \in \Gamma$. Then $\daleth \notin \Gamma^\bullet$. So there exists $\Upsilon \in \Delta(\daleth)$ such that $\Upsilon^c \cup \Gamma^c \notin \mathcal{L}$. Then by (3), $\Gamma^c \notin \mathcal{L}$.

(4) $\Rightarrow$ (1): Let $\Gamma \subseteq \beth$. We first claim that $(\Gamma \setminus \Gamma^\bullet) \cap (\Gamma \setminus \Gamma^\bullet)^\bullet = \emptyset$. In fact, if $\daleth \in (\Gamma \setminus \Gamma^\bullet) \cap (\Gamma \setminus \Gamma^\bullet)^\bullet$, then $\daleth \in \Gamma \setminus \Gamma^\bullet$. Thus $\daleth \in \Gamma$ and $\daleth \notin \Gamma^\bullet$. Then, there exists $\Upsilon \in \Delta(\daleth)$ such that $\Upsilon^c \cup \Gamma^c \notin \mathcal{L}$. Now, $\Upsilon^c \cup \Gamma^c \subseteq \Upsilon^c \cup (\Gamma \setminus \Gamma^\bullet)^c$ since $\mathcal{L}$ is an ideal, $\Upsilon^c \cup (\Gamma \setminus \Gamma^\bullet)^c \notin \mathcal{L}$. Hence, $\daleth \notin (\Gamma \setminus \Gamma^\bullet)^\bullet$, which is a contradiction. Hence, by (4), $(\Gamma \setminus \Gamma^\bullet)^c = \Gamma^c \cup \Gamma^\bullet \notin \mathcal{L}$ and Δ is suitable for the ideal $\mathcal{L}$. □

Theorem 5.4. *For an ideal-TS $(\beth, \Delta, \mathcal{L})$, the following conditions are equivalent and any of these three conditions is necessary for Δ to be suitable for the ideal $\mathcal{L}$.*

(1) *For any $\Gamma \subseteq \beth$, $\Gamma \cap \Gamma^\bullet = \emptyset$, then $\Gamma^\bullet = \emptyset$,*
(2) *for any $\Gamma \subseteq \beth$, $(\Gamma \setminus \Gamma^\bullet)^\bullet = \emptyset$,*
(3) *for any $\Gamma \subseteq \beth$, $(\Gamma \cap \Gamma^\bullet)^\bullet = \Gamma^\bullet$.*

Proof. (1) $\Rightarrow$ (2): It follows by noting that $(\Gamma \setminus \Gamma^\bullet) \cap (\Gamma \setminus \Gamma^\bullet)^\bullet = \emptyset$, for all $\Gamma \subseteq \beth$.

(2) $\Rightarrow$ (3): Since $\Gamma = (\Gamma \setminus (\Gamma \cap \Gamma^\bullet)) \cup (\Gamma \cap \Gamma^\bullet)$, we have $\Gamma^\bullet = (\Gamma \setminus (\Gamma \cap \Gamma^\bullet))^\bullet \cup (\Gamma \cap \Gamma^\bullet)^\bullet = (\Gamma \setminus \Gamma^\bullet)^\bullet \cup (\Gamma \cap \Gamma^\bullet)^\bullet = (\Gamma \cap \Gamma^\bullet)^\bullet$ by (2).

(3) $\Rightarrow$ (1): Let $\Gamma \subseteq \beth$ and $\Gamma \cap \Gamma^\bullet = \emptyset$. Then by (3), $\Gamma^\bullet = (\Gamma \cap \Gamma^\bullet)^\bullet = \emptyset^\bullet = \emptyset$. □

Corollary 5.5. *If $(\beth, \Delta, \mathcal{L})$ is an ideal-TS such that Δ is suitable for $\mathcal{L}$., then the operator $\bullet$ is an idempotent operator, i.e., $\Gamma^\bullet = (\Gamma^\bullet)^\bullet$ for any $\Gamma \subseteq \beth$.*

Proof. By Theorem 3.4 (iv), we have $(\Gamma^\bullet)^\bullet \subseteq \Gamma^\bullet$. By Theorem 5.3 and Theorem 3.4 (v), we get $\Gamma^\bullet = (\Gamma \cap \Gamma^\bullet)^\bullet \subseteq (\Gamma^\bullet)^\bullet$. □

Theorem 5.6. *Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS such that Δ is suitable for $\mathcal{L}$. Then, a subset Γ of $\beth$ is $\Delta^\bullet$-closed if and only if it can be expressed as a union of a set which is closed in $(\beth, \Delta)$ and a complement not in $\mathcal{L}$.*

Proof. Let Γ be a $\Delta^\bullet$-closed subset of $\beth$. Then, $\Gamma^\bullet \subseteq \Gamma$. Now, $\Gamma = \Gamma^\bullet \cup (\Gamma \setminus \Gamma^\bullet)$. Since Δ is suitable for $\mathcal{L}$, by Theorem 5.3 $(\Gamma \setminus \Gamma^\bullet)^c \notin \mathcal{L}$ and by Theorem 3.4 (ii), $\Gamma^\bullet$ is closed.

Conversely, let $\Gamma = F \cup \Lambda$, where F is closed and $\Lambda^c \notin \mathcal{L}$. Then, $\Gamma^\bullet = (F \cup \Lambda)^\bullet = F^\bullet$ by Corollary 3.23, and hence by Theorem 3.4(ii) $\Gamma^\bullet = (F \cup \Lambda)^\bullet = F^\bullet = \coprod(F) = F \subseteq \Gamma$. Hence, Γ is $\Delta^\bullet$-closed. □

Corollary 5.7. *Let the topology Δ on a space $\beth$ be suitable for an ideal $\mathcal{L}$ on $\beth$. Then, $\beta_{\mathcal{L}} = \{\Upsilon \cap P : (\Upsilon \in \Delta)(P \notin \mathcal{L})\}$ is a topology on $\beth$ and hence, $\beta_{\mathcal{L}} = \Delta^\bullet$*

Proof. Let $\Upsilon \in \Delta^\bullet$. Then by Theorem 5.6, $\beth \setminus \Upsilon = F \cup \Lambda$, where F is closed and $\Lambda^c \notin \mathcal{L}$. Then, $\Upsilon = \beth \setminus (F \cup \Lambda) = (\beth \setminus F) \cap (\beth \setminus \Lambda) = \urcorner \cap P$, where $\urcorner = F^c \in \Delta$ and $P = \Lambda^c \notin \mathcal{L}$. Thus, every $\Delta^\bullet$-open set is of the form $\urcorner \cap P$, where $\urcorner \in \Delta$ and $P \notin \mathcal{L}$. The rest follows from Theorem 3.26. □

Theorem 5.8. *Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS and Γ be any subset of $\beth$ such that $\Gamma \subseteq \Gamma^\bullet$. Then, $\coprod(\Gamma) = \coprod^\bullet(\Gamma) = \coprod(\Gamma^\bullet) = \Gamma^\bullet$.*

Proof. Since $\Delta^\bullet$ is finer than Δ, $\coprod^\bullet(\Gamma) \subseteq \coprod(\Gamma)$ for any subset Γ of $\beth$. Now $\gimel \notin \coprod^\bullet(\Gamma)$, there exist $\daleth \in \Delta$ and $\Lambda \in \mathcal{L}$ such that $\gimel \in \daleth \cap \Lambda$ and $(\daleth \cap \Lambda) \cap \Gamma = \emptyset$, then $[(\daleth \cap \Lambda) \cap \Gamma]^\bullet = \emptyset$. Thus $[(\daleth \cap \Gamma) \setminus \Lambda^c]^\bullet = \emptyset$, hence by Corollary 4.10 we have $(\daleth \cap \Gamma)^\bullet = \emptyset$. By Theorem 3.20, we get $\daleth \cap (\Gamma)^\bullet = \emptyset$ and $\daleth \cap \Gamma = \emptyset$ (as $\Gamma \subseteq \Gamma^\bullet$), then $\gimel \notin \coprod(\Gamma)$. Thus, $\coprod(\Gamma) = \coprod^\bullet(\Gamma)$. Now, by Theorem 3.4 (ii), $\Gamma^\bullet = \coprod(\Gamma^\bullet)$. Now, let $\gimel \notin \coprod(\Gamma)$. Then, there exists $\Upsilon \in \Delta(\gimel)$ such that $\Upsilon \cap \Gamma = \emptyset$. Thus, $(\Upsilon \cap \Gamma)^c = \Upsilon^c \cup \Gamma^c = \beth \notin \mathcal{L}$. So, $\gimel \notin \Gamma^\bullet$ and hence $\Gamma^\bullet \subseteq \coprod(\Gamma)$. Again as $\Gamma^\bullet \subseteq \coprod(\Gamma)$, so we have $\coprod(\Gamma^\bullet) \subseteq \coprod(\coprod(\Gamma)) = \coprod(\Gamma)$. Also, $\Gamma \subseteq \Gamma^\bullet$, then $\coprod(\Gamma) \subseteq \coprod(\Gamma^\bullet)$. Thus, $\coprod(\Gamma) = \coprod(\Gamma^\bullet) = \Gamma^\bullet$. □

Theorem 5.9. *Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS such that Δ is suitable for $\mathcal{L}$. Then, for every $\Gamma \in \Delta$ and any $\Lambda \subseteq \beth$, $(\Gamma \cap \Lambda)^\bullet = (\Gamma \cap \Lambda^\bullet)^\bullet = \coprod(\Gamma \cap \Lambda^\bullet)$.*

Proof. Let $\Gamma \in \Delta$. Then by Corollary 3.21, $\Gamma \cap \Lambda^\bullet = \Gamma \cap (\Gamma \cap \Lambda)^\bullet \subseteq (\Gamma \cap \Lambda)^\bullet$ and hence, $(\Gamma \cap \Lambda^\bullet)^\bullet \subseteq [(\Gamma \cap \Lambda)^\bullet]^\bullet = (\Gamma \cap \Lambda)^\bullet$ by Corollary 5.5.

Now by using Corollary 3.21 and Theorem 5.4, we obtain $[\Gamma \cap (\Lambda \setminus \Lambda^\bullet)]^\bullet = \Gamma \cap (\Lambda \setminus \Lambda^\bullet)^\bullet = \Gamma \cap \emptyset = \emptyset$.

Also, $(\Gamma \cap \Lambda)^\bullet \setminus (\Gamma \cap \Lambda^\bullet)^\bullet \subseteq [(\Gamma \cap \Lambda) \setminus (\Gamma \cap \Lambda^\bullet)]^\bullet = [\Gamma \cap (\Lambda \setminus \Lambda^\bullet)]^\bullet = \emptyset$ by Lemma 3.22. Hence, $(\Gamma \cap \Lambda)^\bullet \subseteq (\Gamma \cap \Lambda^\bullet)^\bullet$ and, we get $(\Gamma \cap \Lambda)^\bullet = (\Gamma \cap \Lambda^\bullet)^\bullet$.

Again $(\Gamma \cap \Lambda)^\bullet = (\Gamma \cap \Lambda^\bullet)^\bullet \subseteq \coprod(\Gamma \cap \Lambda^\bullet)$, since $\Delta^\bullet$ is finer than Δ. Due to $\Gamma \cap \Lambda^\bullet \subseteq (\Gamma \cap \Lambda)^\bullet$, we have $\coprod(\Gamma \cap \Lambda^\bullet) \subseteq \coprod((\Gamma \cap \Lambda)^\bullet) = (\Gamma \cap \Lambda)^\bullet$. Hence, $(\Gamma \cap \Lambda)^\bullet = \coprod(\Gamma \cap \Lambda^\bullet)$. □

Corollary 5.10. *Let $(\beth, \Delta, \mathcal{L})$ be an ideal-TS such that Δ is suitable for $\mathcal{L}$. If $\Gamma \in \Delta$ and $\Gamma^c \notin \mathcal{L}$, then $\Gamma \subseteq \beth \setminus \beth^\bullet$.*

Proof. Taking $\Lambda = \beth$ in Theorem 5.9, we get $(\Gamma \cap \beth)^\bullet = \coprod(\Gamma \cap \beth^\bullet)$. Thus, $\Gamma^\bullet = \coprod(\Gamma \cap \beth^\bullet)$, for all $\Gamma \in \Delta$. Now if $\Gamma^c \notin \mathcal{L}$, then $\Gamma^\bullet = \emptyset$. Thus, $(\Gamma \cap \beth)^\bullet = \coprod(\Gamma \cap \beth^\bullet) = \emptyset$. So, $\Gamma \cap \beth^\bullet = \emptyset$ by Theorem 3.20 and hence, $\Gamma \subseteq \beth \setminus \beth^\bullet$. □

6. Conclusion

Coinciding with the great spread of many literatures in various fields important mathematical structures appeared in the theory of topology coinciding with this scientific revolution. For example, the concept of generalized topology appeared, which is based on the concept of "generalized open set" and this space was more generalized than the topological space as the intersection condition was neglected. This space has been extensively studied; much literature has been written about it, and many properties and theories have been studied about it.

Using the concept of "ideal" and the notion of "generalized open set", we create and examine two new local functions in this research, $(\cdot)^\bullet$ and $(\cdot)^\bullet_R$, in order to define two new closure operators, $\coprod^\bullet$ and $\coprod^\bullet_R(\cdot)$. Thus, we were able to derive two new, finer generalized topologies than Δ_δ denoted by $\Delta^\bullet$ and $\Delta^\bullet_R$. On the other hand, we demonstrated the independence of the concepts of $\Delta^\bullet_R$-open and Δ-open. Moreover, we proved several fundamental results related to the operators $(\cdot)^\bullet$, $(\cdot)^\bullet_R$, $\coprod^\bullet(\cdot)$, and $\coprod^\bullet_R(\cdot)$. Furthermore, we give not only some relationships but also several examples. Finally, we present topology appropriate for an ideal and establish some matching conditions.

In future work, we will define more concepts that are related to the generated operators and their ideal-Ts $(\daleth, \Delta, \mathcal{L})$. Based on the proposed generalized closure operators, we will introduce new operators and explore their main characteristics. Using these new operators, we will create new versions of topology. Additionally, we will present some results related to compatibility. Further, we will propose and study pre-local functions, semi-local functions, $\theta-$local functions and $\alpha-$local functions with respect to an ideal-Ts $(\daleth, \Delta, \mathcal{L})$ and study there related closure operators. Furthermore, we will propose some topological notions such as separation axioms and connectedness of these spaces. If possible, we are looking forward to connecting the proposed operators with some ideas like supra-topology and infra-topology.

7. Data Availability Statement

No data were used to support this study.

8. Aknowledgement

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9. Conflict of interest

The authors declare that they have no conflicts of interes

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