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# OPTIMAL $Z$-EIGENVALUE BOUNDS AND RANK-ONE APPROXIMATIONS

A. Askarizadeh ✉ and M. Zangiabadi

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Abstract. This study explores the $Z$-eigenvalue inclusion theorem, focusing on its role in improving the best rank-one approximation for tensors. We propose novel inclusion sets inspired by Brauer and Brualdi's frameworks, offering sharper bounds on $Z$-eigenvalues. These sets are demonstrated to provide more accurate results than existing approaches. Additionally, we apply these results to obtain refined estimates of best rank-one approximation, particularly for weakly symmetric nonnegative tensors. The paper includes numerical examples to validate the enhanced bounds presented.

*Keywords*: Weakly symmetric tensor, $Z$-eigenvalue inclusion set, best rank-one approximation, $Z$-identity tensors, $Z$-spectral radius.
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## 1. Introduction

Let $\mathbb{C}$ (resp., $\mathbb{R}$) denote the set of complex (resp., real) numbers, and $\mathbb{C}^n$ (resp., $\mathbb{R}^n$) denote the vector space of $n$-dimensional complex (resp., real) vectors. Given two positive integers $m \geq 2$ and $n \geq 2$, we define $[n] = \{1, 2, \ldots, n\}$. The space $\mathbb{C}^{[m,n]}$ (resp., $\mathbb{R}^{[m,n]}$) represents the set of all $m$-order $n$-dimensional complex (resp., real) tensors, which are multidimensional arrays consisting of $n^m$ complex-valued (resp., real-valued) entries $a_{i_1 i_2 \ldots i_m}$, where each index $i_j \in [n]$ for $j = 1, 2, \ldots, m$. In particular, tensors of order 1 correspond to vectors, while tensors of order 2 are equivalent to matrices.

For a tensor $\mathcal{A} = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m \times n]}$, we say that $\mathcal{A}$ is nonnegative if all its components satisfy $a_{i_1 i_2 \ldots i_m} \geq 0$. A tensor $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ is termed symmetric [26] if

$$a_{i_1 i_2 \ldots i_m} = a_{i_{\pi(1)} i_{\pi(2)} \ldots i_{\pi(m)}}, \qquad \forall \pi \in \Pi_m,$$

where $\Pi_m$ denotes the set of all permutation group of the indices $(1, \ldots, m)$. Additionally, an $m$-order $n$-dimensional real tensor $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ is termed weakly symmetric [5], if

$$\nabla(\mathcal{A}x^m) = m\mathcal{A}x^{m-1}, \qquad \forall x \in \mathbb{R}^n,$$

✉ a.askari@vru.ac.ir, ORCID: 0000-0001-6663-0548
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where $\nabla$ represents the gradient operator, $\mathcal{A}x^m$ is the homogeneous polynomial defined by

$$f_{\mathcal{A}}(x) = \mathcal{A}x^m = x^T(\mathcal{A}x^{m-1}) = \sum_{i_1,i_2,\ldots,i_m=1}^{n} a_{i_1 i_2 \ldots i_m} x_{i_1} x_{i_2} \ldots x_{i_m},$$

and $\mathcal{A}x^{m-1}$ is an $n$-dimensional vector in $\mathbb{C}^n$, with its $i$-th element given by

$$\left(\mathcal{A}x^{m-1}\right)_i = \sum_{i_2,i_3,\ldots,i_m=1}^{n} a_{i i_2 \ldots i_m} x_{i_2} \ldots x_{i_m}.$$

It is important to note that while every symmetric tensor is inherently weakly symmetric, the reverse is not necessarily true. Consequently, certain conclusions that hold for symmetric tensors may not apply to weakly symmetric tensors. This distinction underscores the need for careful consideration when extending results from symmetric to weakly symmetric cases.

The concept of $Z$-eigenvalues for tensors was independently introduced by Qi [26] and Lim [20] in 2005.

**Definition 1.1.** Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}$ be a given tensor. If there exist a scalar $\lambda \in \mathbb{C}$ and a nonzero vector $x = (x_1, \ldots, x_n) \in \mathbb{C}^n \backslash \{0\}$ such that

$$\mathcal{A}x^{m-1} = \lambda x \quad \text{and} \quad x^T x = 1, \tag{1}$$

then $\lambda$ is termed an $E$-eigenvalue of $\mathcal{A}$, and $x$ is called an $E$-eigenvector associated with $\lambda$. If both $\lambda$ and $x$ are real, then $\lambda$ is referred to as a $Z$-eigenvalue of $\mathcal{A}$, with $x$ as the corresponding $Z$-eigenvector.

Let $\mathcal{A}$ be an $m$-order $n$-dimensional real tensor, and denote the set of all $Z$-eigenvalues of $\mathcal{A}$ as $\sigma(\mathcal{A})$, known as the $Z$-spectrum. If $\sigma(\mathcal{A}) \neq \emptyset$, the $Z$-spectral radius of $\mathcal{A}$, represented by $\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}$. Chang, Pearson, and Zhang [5] pointed out that for nonnegative tensors, $\rho(\mathcal{A})$ does not always correspond to a positive $Z$-eigenvalue of $\mathcal{A}$. They further established that when $\mathcal{A}$ is a weakly symmetric nonnegative tensor, $\rho(\mathcal{A})$ is guaranteed to be a $Z$-eigenvalue of $\mathcal{A}$.

The study of $Z$-eigenvalue problems for tensors has gained significant attention due to their broad range of applications, including the best rank-one approximation [11, 24, 37], the geometric measure of entanglement in multipartite quantum states [34, 35], and providing sufficient conditions for the positive definiteness of homogeneous polynomial forms, which are essential for evaluating the asymptotic stability of time-invariant polynomial systems [28, 36]. In this paper, we focus specifically on the best rank-one approximation.

A tensor $\mathcal{A}$ of order $m$ and dimension $n$ is called a rank-one tensor if it can be represented as the outer product of $m$ vectors, given by the expression

$$\mathfrak{R}_1 = \left\{ \mathcal{A} \in \mathbb{R}^{[m \times n]} : \mathcal{A} = \bigotimes_{j=1}^{m} v^{(j)}, v^{(j)} \in \mathbb{R}^n \right\}.$$

This means that every entry $a_{i_1 i_2 \dots i_m}$ of $\mathcal{A}$ can be written as the product of the corresponding components of the vectors

$$a_{i_1 i_2 \dots i_m} = v_{i_1}^{(1)} v_{i_2}^{(2)} \dots v_{i_m}^{(m)},$$

for $i_1, i_2, \dots, i_m \in [n]$. In particular, if all the vectors involved in the outer product are identical, the tensor is referred to as a symmetric rank-one tensor. In this case, each entry $a_{i_1 i_2 \dots i_m}$ of the tensor is given by

$$a_{i_1 i_2 \dots i_m} = v_{i_1} v_{i_2} \dots v_{i_m}, \qquad i_1, i_2, \dots, i_m \in [n].$$

The best rank-one approximation of tensors is fundamental in a variety of applications, including MIMO communication systems [10], data analysis [11,25], image and signal processing [9], independent component analysis [4], higher-order statistics [22,23], and magnetic resonance imaging [12]. When considered under the Frobenius norm, this approximation can be framed as an extreme $Z$-eigenvalue problem [24,25]. In [24], Qi introduced the notion of the best rank-one approximation ratio, denoted $App(V)$, for tensor space $V$, providing both lower and upper bounds for this ratio. These bounds are crucial in understanding the convergence behavior of the greedy rank-one update algorithm [1, 32] and play an important role in evaluating the performance of rank-truncated steepest descent methods in low-rank matrix and tensor optimization [30].

Various engineering and scientific problems including the best rank-one tensor approximation can be framed as the $Z$-eigenvalue problem of tensor. However, unlike the eigenvalue problem for matrices, finding eigenvalues for tensors is NP-hard [15]. Despite this challenge, several algorithms have been developed to compute eigenvalues for specific types of tensors, such as nonnegative and symmetric tensors [7, 17, 18]. Nevertheless, these methods often face difficulties when dealing with large tensors and, more importantly, do not always ensure that the computed $Z$-eigenvalues are the largest. Consequently, it is necessary to study the $Z$-eigenvalue inclusion theorem for tensors.

In recent years, there has been increasing focus on the localization of $Z$-eigenvalues for tensors and the development of bounds for the $Z$-spectral radius of weakly symmetric nonnegative tensors. Wang, Zhou, and Caccetta [33] generalized Gersgorin's and Brauer-type eigenvalue inclusion theorems from the matrix setting to the $Z$-eigenvalue problem. Further investigations into the $Z$-eigenvalue localization theorem for tensors are available in works such as [13, 14, 16, 19, 21, 27, 34–36].

In this paper, we introduce optimal $Z$-eigenvalue inclusion sets for tensors and explore their applications in improving the best rank-one approximation within nonnegative tensor spaces.

The structure of this paper is as follows. In Section 2, we establish two enhanced Brauer-type and Brualdi-type $Z$-eigenvalue localization sets, which provide more precise results than those presented in [16, 33]. Section 3 introduces a new Brualdi-type $Z$-eigenvalue inclusion theorem for tensors, based on classifying tensor entries according to the digraph associated with even-order tensors. Additionally, we demonstrate that, in a specific case, the Brualdi-type inclusion set of $Z$-eigenvalues is tighter than the set provided in [35]. In Section 4, we concentrate on estimating the best rank-one approximation for weakly symmetric tensors.

## 2. Brauer-type and Brualdi-type $Z$-eigenvalue localization sets

In this section, we propose two refined $Z$-eigenvalue inclusion sets based on Brauer-type and Brualdi-type approaches for tensors. Furthermore, we compare these newly developed sets with several existing results to highlight their improvements.

Define $r_i(\mathcal{A}) = \sum\limits_{i_2,\ldots,i_m \in [n]} |a_{ii_2\ldots i_m}|$. Wang, Zhou, and Caccetta [33] introduced the following Gershgorin-type theorem, which establishes a $Z$-eigenvalue localization set for tensors.

**Theorem 2.1.** *Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ be a tensor of $m$-order $n$-dimensional, then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) = \bigcup_{i \in [n]} \mathcal{K}_i(\mathcal{A}) := \{z \in \mathbb{R} \ : \ |z| \leq r_i(\mathcal{A})\}.$$

Brauer's well-known eigenvalue inclusion set for matrices, introduced in [2], is recognized as a subset of the Gershgorin set. Recently, this eigenvalue inclusion set has been extended from matrices to tensors, leading to the following Brauer-type $Z$-eigenvalue inclusion set for tensors [33].

**Theorem 2.2.** *[33, Theorem 3.2] Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ be a tensor of $m$-order $n$-dimensional. Then all $Z$-eigenvalues of $\mathcal{A}$ are located in the following set*

$$\mathcal{L}(\mathcal{A}) := \bigcup_{i \in [n]} \bigcap_{\substack{j \in [n] \\ j \neq i}} \mathcal{L}_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : (|z| - (r_i(\mathcal{A}) - |a_{ij\ldots j}|))\,|z| \leq |a_{ij\ldots j}|\, r_j(\mathcal{A})\}.$$

We define the Kronecker symbol for $m$ indices as $\delta_{i_1 i_2 \ldots i_m}$, which is given by

$$\delta_{i_1 i_2 \ldots i_m} = \begin{cases} 1 & \text{if } i_1 = i_2 = \cdots = i_m \\ 0 & \text{otherwise.} \end{cases}$$

Define $R_i(\mathcal{A}) = r_i(\mathcal{A}) - |a_{i\ldots i}|$. A modification of Theorem 2.2 presents an optimal Brauer-type $Z$-eigenvalue inclusion set.

**Theorem 2.3.** *Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ be a tensor of $m$-order $n$-dimensional and assume that $R_i(\mathcal{A}) > 0$ for $i \in [n]$. Then*

$$\sigma(\mathcal{A}) \subseteq \mathfrak{B}(\mathcal{A}) = \mathfrak{B}_1(\mathcal{A}) \cup \mathfrak{B}_2(\mathcal{A}),$$

*where* $\mathfrak{B}_1(\mathcal{A}) = \bigcup\limits_{\substack{i,j \in [n] \\ i \neq j}} \{z \in \mathbb{R} : (|z| - |a_{i\cdots i}|)^{m-1} (|z| - |a_{j\cdots j}|) \leq (R_i(\mathcal{A}))^{m-1}$ $R_j(\mathcal{A}), \ \ |z| > |a_{i\cdots i}|\}$, *and* $\mathfrak{B}_2(\mathcal{A}) = \bigcup_{i \in [n]} \{z \in \mathbb{R} : |z| < |a_{i\cdots i}|\}$.

*Proof.* Let $\lambda$ be a $Z$-eigenvalue with a corresponding $Z$-eigenvector $x$. Assuming $R_i(\mathcal{A}) > 0$, we deduce that $\lambda \in \mathfrak{B}(\mathcal{A})$ whenever $|\lambda| = |a_{i\cdots i}|$ for some $i \in [n]$. Next, consider the case where $|\lambda| > |a_{i\cdots i}|$ for all $i \in [n]$. Using Equation (1), we get

$$|\lambda|\,|x_i|^{m-1} \leq |\lambda|\,|x_i| \leq |a_{i\cdots i}|\,|x_i^{m-1}| + \sum_{\substack{i_2,\ldots,i_m\in[n]\\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\ldots i_m}|\,|x_{i_2}|\ldots|x_{i_m}|,$$

which implies that

$$(|\lambda| - |a_{i\cdots i}|)\,|x_i^{m-1}| \leq \sum_{\substack{i_2,\ldots,i_m\in[n]\\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\ldots i_m}|\,|x_{i_2}|\ldots|x_{i_m}|. \tag{2}$$

Let $|x_t| \geq |x_s| \geq \max\limits_{\substack{k\in[n]\\ k\neq s,k\neq t}} |x_k|$. Then from the $t$-th inequality of (2), we can get

$$\begin{aligned}(|\lambda| - |a_{t\cdots t}|)\,|x_t^{m-1}| &\leq \sum_{\substack{i_2,\ldots,i_m\in[n]\\ \delta_{ti_2\ldots i_m}=0}} |a_{ti_2\ldots i_m}|\,|x_{i_2}|\ldots|x_{i_m}| \\ &\leq \sum_{\substack{i_2,\ldots,i_m\in[n]\\ \delta_{ti_2\ldots i_m}=0}} |a_{ti_2\ldots i_m}|\,|x_t|^{m-2}\,|x_s| \\ &\leq R_t(\mathcal{A})\,|x_t|^{m-2}\,|x_s|. \end{aligned}\tag{3}$$

We analyze Equation (3) by considering two distinct cases.

**Case 1.** Let $x_s = 0$. Then from Equation (3), we derive

$$(|\lambda| - |a_{t\cdots t}|)\,|x_t^{m-1}| = 0,$$

which leads to the conclusion $|\lambda| = |a_{t\cdots t}|$. Thus, it follows that $\lambda \in \mathfrak{B}(\mathcal{A})$.

**Case 2.** Now consider the case where $x_s \neq 0$. From Equation (3), we obtain

$$(|\lambda| - |a_{t\cdots t}|)\,|x_t^{m-1}| \leq R_t(\mathcal{A})\,|x_t|^{m-2}\,|x_s|.$$

Since $|\lambda| > |a_{t\cdots t}|$, it follows that

$$\left((|\lambda| - |a_{t\cdots t}|)\,|x_t^{m-1}|\right)^{m-1} \leq \left(R_t(\mathcal{A})\,|x_t|^{m-2}\,|x_s|\right)^{m-1}. \tag{4}$$

Similar to (3), we obtain

$$(|\lambda| - |a_{s\cdots s}|)\,|x_s^{m-1}| \leq R_s(\mathcal{A})\,|x_t|^{m-2}\,|x_s|. \tag{5}$$

Since $x^T x = 1$, and $0 \leq |x_i|^{m-1} \leq |x_i| \leq 1$, by (4) and (5), we have

$$(|\lambda| - |a_{t\cdots t}|)^{m-1}\,(|\lambda| - |a_{s\cdots s}|) \leq (R_t(\mathcal{A}))^{m-1}\,R_s(\mathcal{A}).$$

If $|\lambda| < |a_{i\cdots i}|$ for some $i \in [n]$, it follows that $\lambda \in \mathfrak{B}_2(\mathcal{A})$. Therefore, we conclude that $\lambda \in \mathfrak{B}(\mathcal{A})$, which establishes the desired result and thereby completes the proof. □

In [3], Brualdi presented the eigenvalue inclusion set for matrices. Drawing upon this concept, we formulate a corresponding Brualdi $Z$-eigenvalue inclusion set for tensors, expressed as follows.

**Theorem 2.4.** *Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ be a tensor of $m$-order $n$-dimensional and assume that $R_i(\mathcal{A}) > 0$ for $i \in [n]$. Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{Z}(\mathcal{A}) := \mathcal{Z}_1(\mathcal{A}) \cup \mathcal{Z}_2(\mathcal{A}),$$

*where*

$$\mathcal{Z}_1(\mathcal{A}) = \bigcup_{\substack{a_{i_1 i_2 \cdots i_m} \neq 0, \\ \delta_{i_1 i_2 \ldots i_m} = 0}} \left\{ z \in \mathbb{R} : \prod_{j=1}^{m} \left( |z| - |a_{i_j \ldots i_j}| \right) \leq \prod_{j=1}^{m} R_{i_j}(\mathcal{A}), |z| > |a_{i_j \ldots i_j}| \right\},$$

$$\mathcal{Z}_2(\mathcal{A}) = \bigcup_{i \in [n]} \{ z \in \mathbb{R} : |z| < |a_{i \ldots i}| \}.$$

*Proof.* Let $\lambda$ be a $Z$-eigenvalue with a corresponding $Z$-eigenvector $x$. Following a similar reasoning as in Theorem 2.3, we consider three distinct cases. Under the assumption that $R_i(\mathcal{A}) > 0$, it can be concluded that $\lambda \in \mathcal{Z}(\mathcal{A})$ when $|\lambda| = |a_{i \cdots i}|$ for some $i \in [n]$. Next, consider the case where $|\lambda| > |a_{i \cdots i}|$ for all $i \in [n]$. Let $|x_\beta| = \max\{|x_{i_1}| |x_{i_2}| \cdots |x_{i_m}| : a_{i_1 i_2 \cdots i_m} \neq 0, (i_2, \ldots, i_m) \neq (i_1, \ldots, i_1), i_1, \ldots, i_m \in [n]\}$. Similar to (2) for all $i \in [n]$, we have

$$\begin{aligned} (|\lambda| - |a_{i \cdots i}|) |x_i^m| &\leq \sum_{\substack{i_2, \ldots, i_m \in [n] \\ \delta_{i i_2 \ldots i_m} = 0}} |a_{i i_2 \ldots i_m}| \, |x_i| |x_{i_2}| \ldots |x_{i_m}| \\ &\leq \sum_{\substack{i_2, \ldots, i_m \in [n] \\ \delta_{i i_2 \ldots i_m} = 0}} |a_{i i_2 \ldots i_m}| \, |x_\beta| \\ &= R_i(\mathcal{A}) |x_\beta|. \end{aligned} \tag{6}$$

We analyze Equation (6) by considering two distinct cases.

**Case 1.** Let $|x_\beta| = 0$. Given that $x \neq 0$, we assume $x_p \neq 0$ for some $p \in [n]$. From $R_p(\mathcal{A}) > 0$, then there exists $a_{p p_2 \cdots p_m} \neq 0$, where $\delta_{p p_2 \ldots p_m} = 0$, i.e., $(p_2, \ldots, p_m) \neq (p, \ldots, p)$. Thus, we have $|x_p x_{p_2} \cdots x_{p_m}| = |x_\beta| = 0$. From Equation (6), it follows that

$$(|\lambda| - |a_{pp \ldots p}|) |x_p^m| = 0,$$

which implies $|\lambda| = |a_{p \ldots p}|$. Therefore, it is evident that $\lambda \in \mathcal{Z}(\mathcal{A})$.

**Case 2.** Now consider the case where $|x_\beta| \neq 0$. Suppose that $|x_\beta| = |x_{j_1}| |x_{j_2}| \cdots |x_{j_m}|$. From Equation (6), we obtain

$$(|\lambda| - |a_{j_1 j_1 \cdots j_1}|) |x_{j_1}^m| \leq R_{j_1}(\mathcal{A}) |x_\beta|,$$

$$\vdots$$

$$(|\lambda| - |a_{j_m j_m \cdots j_m}|) |x_{j_m}^m| \leq R_{j_m}(\mathcal{A}) |x_\beta|.$$

Given that $|\lambda| > |a_{i\cdots i}|$ for all $i \in [n]$, it follows that

$$\prod_{l=1}^{m} (|\lambda| - |a_{j_l j_l \cdots j_l}|) \left|x_{j_l}^m\right| \leq |x_\beta|^m \prod_{l=1}^{m} R_{j_l}(\mathcal{A}).$$

If $|\lambda| < |a_{i\cdots i}|$ for some $i \in [n]$, it follows that $\lambda \in \mathcal{Z}_2(\mathcal{A})$. Consequently, we deduce that $\lambda \in \mathcal{Z}(\mathcal{A})$, thereby establishing the desired result and completing the proof. □

The following describes the relationships between $\mathcal{Z}(\mathcal{A})$ and $\mathfrak{B}(\mathcal{A})$.

**Theorem 2.5.** *Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ be a tensor of $m$-order $n$-dimensional. Assume that $R_i(\mathcal{A}) > 0$ for $i \in [n]$. Then*

$$\mathcal{Z}(\mathcal{A}) \subseteq \mathfrak{B}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}). \tag{7}$$

*Proof.* The inclusion $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A})$ was established in [33, Corollary 3.1]. Next, consider the inclusion $\mathfrak{B}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A})$ from Equation (7). For $z \in \mathfrak{B}(\mathcal{A})$, if $z \notin \mathcal{L}(\mathcal{A})$, then for every $i \in [n]$, there exists $j \in [n]$ such that $j \neq i$ and $z \notin \mathcal{L}_{i,j}(\mathcal{A})$, by the definition of $\mathcal{L}_{i,j}(\mathcal{A})$ presented in Theorem 2.2.

If $a_{ij\cdots j} \neq 0$, then

$$\frac{|z| - (r_i(\mathcal{A}) - |a_{ij\cdots j}|)}{|a_{ij\cdots j}|} > 1$$

or

$$\frac{|z|}{r_j(\mathcal{A})} > 1,$$

and thus, we have

$$|z| > r_i(\mathcal{A}) = |a_{i\cdots i}| + R_i(\mathcal{A}), \quad \forall i \in [n]. \tag{8}$$

When $a_{ij\cdots j} = 0$, we still have the result in (8). Thus, for all $i, j \in [n]$, where $i \neq j$, one obtains

$$(|z| - |a_{i\cdots i}|)^{m-1} (|z| - |a_{j\cdots j}|) > (R_i(\mathcal{A}))^{m-1} R_j(\mathcal{A}).$$

This contradicts the assumption that $z \in \mathfrak{B}_1(\mathcal{A})$. Furthermore, by inequality (8), for all $i \in [n]$, we have $|z| - |a_{i\cdots i}| > R_i(\mathcal{A}) > 0$, which contradicts the assumption that $z \in \mathfrak{B}_2(\mathcal{A})$. Therefore, we conclude that $z \in \mathcal{L}(\mathcal{A})$.

Next, we prove the validity of the left-hand inclusion in (7). Let $z \in \mathcal{Z}(\mathcal{A})$, meaning $z \in \mathcal{Z}_1(\mathcal{A})$ or $z \in \mathcal{Z}_2(\mathcal{A})$. If $z \in \mathcal{Z}_2(\mathcal{A})$, then $z \in \mathfrak{B}_2(\mathcal{A})$. Let $z$ be any point in $\mathcal{Z}_1(\mathcal{A})$, which satisfies the following inequality

$$\prod_{j=1}^{m} \left(|z| - \left|a_{i_j i_j \ldots i_j}\right|\right) \leq \prod_{j=1}^{m} R_{i_j}(\mathcal{A}). \tag{9}$$

By raising both sides of the inequality (9) to the power of $m$, we obtain

$$\prod_{j=1}^{m} \left(|z| - \left|a_{i_j i_j \ldots i_j}\right|\right)^m \leq \prod_{j=1}^{m} (R_{i_j}(\mathcal{A}))^m. \tag{10}$$

Since $R_{i_j}(\mathcal{A}) > 0$ for $j = 1, 2, \ldots, m$, we can equivalently express the inequality (10) in the following form

$$\left(\frac{\left(|z| - |a_{i_1\ldots i_1}|^{m-1}\right)(|z| - |a_{i_2\ldots i_2}|)}{(R_{i_1}(\mathcal{A}))^{m-1}(R_{i_2}(\mathcal{A}))}\right) \cdot \left(\frac{\left(|z| - |a_{i_2\ldots i_2}|^{m-1}\right)(|z| - |a_{i_3\ldots i_3}|)}{(R_{i_2}(\mathcal{A}))^{m-1}(R_{i_3}(\mathcal{A}))}\right)$$

(11)

$$\ldots \left(\frac{\left(|z| - |a_{i_m\ldots i_m}|^{m-1}\right)(|z| - |a_{i_1\ldots i_1}|)}{(R_{i_m}(\mathcal{A}))^{m-1}(R_{i_1}(\mathcal{A}))}\right) \leq 1.$$

Given that the factors on the left side of (11) cannot all exceed unity, it follows that at least one of these factors must equal unity at most. Specifically, we have

$$\left(\frac{\left(|z| - |a_{i_1\ldots i_1}|^{m-1}\right)(|z| - |a_{i_2\ldots i_2}|)}{(R_{i_1}(\mathcal{A}))^{m-1}(R_{i_2}(\mathcal{A}))}\right) \leq 1$$

or

$$\left(\frac{\left(|z| - |a_{i_2\ldots i_2}|^{m-1}\right)(|z| - |a_{i_3\ldots i_3}|)}{(R_{i_2}(\mathcal{A}))^{m-1}(R_{i_3}(\mathcal{A}))}\right) \leq 1$$

or

$$\vdots$$

or

$$\left(\frac{\left(|z| - |a_{i_m\ldots i_m}|^{m-1}\right)(|z| - |a_{i_1\ldots i_1}|)}{(R_{i_m}(\mathcal{A}))^{m-1}(R_{i_1}(\mathcal{A}))}\right) \leq 1.$$

Therefore, there exists an $\alpha$ such that $1 < \alpha < m$, satisfying the condition

$$(|z| - |a_{i_\alpha\ldots i_\alpha}|)^{m-1}\left(|z| - \left|a_{i_{\alpha+1}\ldots i_{\alpha+1}}\right|\right) \leq (R_{i_\alpha}(\mathcal{A}))^{m-1}(R_{i_{\alpha+1}}(\mathcal{A})).$$

This means that $z \in \mathfrak{B}_1(\mathcal{A})$. Hence, we have $\mathcal{Z}(\mathcal{A}) \subseteq \mathfrak{B}(\mathcal{A})$. Therefore, the proof is concluded. $\square$

The following example from [16] illustrates the enhancements in the bounds achieved in this section.

**Example 2.6.** *[16, Example 2.8] Consider the symmetric tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,2]}$, with entries defined as follows*

$$a_{111} = 2,\ a_{222} = 1,\ \ a_{112} = a_{122} = a_{211} = \frac{-4}{3},\ \ \textit{and other } a_{ijk} = 0.$$

*From Theorem 2.3, we get*

$$\mathfrak{B}(\mathcal{A}) = \left\{ z \in \mathbb{R} : (|z|-2)^2 (|z|-1) \leq \left(\frac{8}{3}\right)^2 \left(\frac{4}{3}\right) \right\} \cup \left\{ z \in \mathbb{R} : (|z|-1)^2 (|z|-2) \leq \left(\frac{8}{3}\right) \left(\frac{4}{3}\right)^2 \right\}.$$

*Table 1 shows some $Z$-eigenvalue inclusion sets for tensor $\mathcal{A}$.*

TABLE 1. Various $Z$-eigenvalue inclusion sets for Example 2.6

| | |
|---|---|
| $\mathcal{K}(\mathcal{A})$ *of [33, Theorem 3.1]* | $\left\{z \in \mathbb{R} : \lvert z\rvert \leq \frac{14}{3}\right\}$ |
| $\mathcal{L}(\mathcal{A})$ *of [33, Theorem 3.2]* | $\{z \in \mathbb{R} : \lvert z\rvert \leq 4.0934\}$ |
| $\mathcal{F}(\mathcal{A})$ *of [16, Theorem 2.5]* | $\left\{z \in \mathbb{R} : \lvert z\rvert \leq \frac{14}{3}\right\}$ |
| $\mathcal{B}(\mathcal{A})$ *of [16, Theorem 2.1]* | $\{z \in \mathbb{R} : \lvert z\rvert \leq 3.9384\}$ |
| $\mathfrak{B}(\mathcal{A})$ *of Theorem 2.3* | $\{z \in \mathbb{R} : \lvert z\rvert \leq 3.8303\}$ |
| $\mathcal{Z}(\mathcal{A})$ *of Theorem 2.4* | $\{z \in \mathbb{R} : \lvert z\rvert \leq 3.8303\}$ |

## 3. A novel Brualdi-type $Z$-eigenvalue inclusion set for tensors of even order

In this section, we propose novel Brualdi-type $Z$-eigenvalue localization sets by classifying the index set, and we show that these localization sets provide tighter bounds than existing ones.

The concept of a $Z$-identity tensor was introduced in [18, 26].

**Definition 3.1.** A tensor $\mathcal{I}_Z = (e_{i_1 i_2 \ldots i_m})$ of $m$-order $n$-dimensional, where $m$ is even, is defined as a $Z$-identity tensor if, for any vector $x \in \mathbb{R}^n$ with $x^T x = 1$, the following condition holds

$$\mathcal{I}_Z x^{m-1} = x.$$

In general, $Z$-identity tensors are not uniquely defined (see, e.g., [18, 27, 28]). Several constructions can serve as $Z$-identity tensors; for example, the following cases provide valid forms of $Z$-identity tensors:

**Case I.** Let $\mathcal{I}_1 = (e_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$, where

$$e_{i_1 i_2 \cdots i_m} = \begin{cases} 1 & i_1 = i_2,\ i_3 = i_4, \ldots, i_{m-1} = i_m \\ 0 & \textit{otherwise} \end{cases} \tag{12}$$

**Case II.** Let $\mathcal{I}_2 = (e_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$, where

$$e_{i_1 i_2 \cdots i_m} = \frac{1}{m!} \sum_{\pi \in \prod_m} \delta_{i_{\pi(1)}\, i_{\pi(2)}}\, \delta_{i_{\pi(3)}\, i_{\pi(4)}} \ldots \delta_{i_{\pi(m-1)}\, i_{\pi(m)}} \tag{13}$$

For $i_1, i_2, \ldots, i_m \in [n]$, let $\delta$ represent the Kronecker delta, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

We will adopt the following notation, organized by partitioning the index set.

$$\Lambda_i := \left\{(i_2,\, \ldots, i_m) \ :\ (\mathcal{I}_1)_{ii_2\, \ldots i_m} = 1, \quad i_2,\, \ldots, i_m \in [n]\right\},$$
$$\overline{\Lambda}_i := \left\{(i_2,\, \ldots, i_m) \ :\ (\mathcal{I}_1)_{ii_2\, \ldots i_m} = 0, \quad i_2,\, \ldots, i_m \in [n]\right\}.$$

$$\Delta \quad := \{(i_2, \ldots, i_m) :\ i_2, \ldots, i_m \textit{ are either distinct or exactly two are equal }\},$$
$$\overline{\Delta} \quad := \{(i_2, \ldots, i_m) : (i_2, \ldots, i_m) \notin \Delta,\ i_2, \ldots, i_m \in [n]\}.$$

$$\Omega_j \quad := \{(i_2, \ldots, i_m) : i_k = j \textit{ for some } k \in \{2, \ldots, m\},\ \textit{where } j, i_{2,\ldots,} i_m \in [n]\},$$
$$\overline{\Omega}_j \quad := \{(i_2, \ldots, i_m) : i_k \neq j \quad \textit{for any } k \in \{2, \ldots, m\},\ \ \textit{where } j, i_{2,\ldots,} i_m \in [n]\}.$$

For a tensor $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ of $m$-order $n$-dimensional with $i \neq j$, and for $\mathfrak{K} \in \{\Lambda_i, \Delta, \Omega_j\}$, we consistently employ the following notation throughout our proofs.

$$r_i^{\mathfrak{K}}(\mathcal{A}) = \sum_{i_2,\ldots,i_m \in \mathfrak{K}} |a_{ii_2\ldots i_m}|, \qquad r_i^{\overline{\mathfrak{K}}}(\mathcal{A}) = \sum_{i_2,\ldots,i_m \in \overline{\mathfrak{K}}} |a_{ii_2\ldots i_m}|,$$
$$M_i(\mathcal{A}) = \beta_i + \frac{1}{(m-2)^{\frac{m-2}{2}}} r_i^{\overline{\Lambda_i} \cap \Delta}(\mathcal{A}) + r_i^{\overline{\Lambda_i} \cap \overline{\Delta}}(\mathcal{A}),\ \beta_i = \max_{i_2,\ldots,i_m \in \Lambda_i} \{|a_{ii_2\, \ldots i_m}|\},$$
$$M_i^{\Omega_i}(\mathcal{A}) = \beta_i + \frac{1}{(m-2)^{\frac{m-2}{2}}} r_i^{\overline{\Lambda_i} \cap \Delta \cap \Omega_i}(\mathcal{A}) + r_i^{\overline{\Lambda_i} \cap \overline{\Delta} \cap \Omega_i}(\mathcal{A}).$$

It is evident that for any $i \in [n]$, the relation $r_i(\mathcal{A}) = r_i^{\mathfrak{K}}(\mathcal{A}) + r_i^{\overline{\mathfrak{K}}}(\mathcal{A})$ holds.

If the $Z$-identifying tensor $\mathcal{I}$ is designated as $\mathcal{I}_1$, then according to Definition 3.1, we can state the following lemma.

**Lemma 3.2.** *For any $x \in \mathbb{R}$, if $x_1^2 + \cdots + x_n^2 = 1$, Then*

$$\sum_{i_2, i_3, \ldots, i_m \in \Lambda_i} x_{i_2}\, x_{i_3} \ldots x_{i_m} = x_i \qquad \forall i \in [n].$$

In the subsequent discussion, we provide a lemma from [27, Lemma 2.2], which is essential for several of our results.

**Lemma 3.3.** *Consider $x_1^2 + \cdots + x_n^2 = 1$, where $x_i \in \mathbb{R}$ for $i \in [n]$. Let $y_1, \ldots, y_k$ denote any $k$ entries selected from $x_1, \ldots, x_n$. Then, it follows that*

$$|y_1||y_2| \cdots |y_k| \leq \frac{1}{k^{\frac{k}{2}}}.$$

Using the above notations and results, we can rewrite Equation (1) as follows.

*Remark* 3.4. Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ be an $m$-order, $n$-dimensional tensor, with $m$ assumed to be even. Consider $\lambda \in \sigma(\mathcal{A})$ and let $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ be an eigenvector associated with $\lambda$. This implies that the following equation holds

$$\mathcal{A}x^{m-1} = \lambda x, \qquad \text{with } x^T x = 1. \tag{14}$$

Let $|x_t| = \max_{i \in [n]} |x_i|$. Then, from the $t$-th equation in (14), we obtain the following

$$\begin{aligned} \lambda x_t = \left(\mathcal{A}x^{m-1}\right)_t = & \sum_{i_2,\ldots,i_m \in \Lambda_t} a_{t i_2 \ldots i_m} x_{i_2} \ldots x_{i_m} \\ & + \sum_{i_2,\ldots,i_m \in \overline{\Lambda_t} \cap \Delta} a_{t i_2 \ldots i_m} x_{i_2} \ldots x_{i_m} \\ & + \sum_{i_2,\ldots,i_m \in \overline{\Lambda_t} \cap \overline{\Delta}} a_{t i_2 \ldots i_m} x_{i_2} \ldots x_{i_m}. \end{aligned} \tag{15}$$

Since $x^T x = 1$, and given that $0 \le |x_i|^{m-1} \le |x_i| \le 1$, we can take the modulus of the equation and apply the triangle inequality to derive the following result for (15)

$$\begin{aligned} |\lambda| \, |x_t| \le \beta_t & \sum_{i_2,\ldots,i_m \in \Lambda_t} x_{i_2} \ldots x_{i_m} \\ & + \sum_{i_2,\ldots,i_m \in \overline{\Lambda_t} \cap \Delta} |a_{t i_2 \ldots i_m}| \, |y_1| \ldots |y_{m-2}| \, |x_t| \\ & + \sum_{i_2,\ldots,i_m \in \overline{\Lambda_t} \cap \overline{\Delta}} |a_{t i_2 \ldots i_m}| \, |x_t|^{m-1}, \end{aligned}$$

where $|y_1|, \ldots, |y_{m-2}|$ are derived using the following approaches:

**Case I.** If $i_2, \ldots, i_m$ are distinct, we can substitute any one of $|x_{i_2}|, \ldots, |x_{i_m}|$ with $|x_t|$, while preserving the values of the others (which we can designate as $|y_1|, \ldots, |y_{m-2}|$).

**Case II.** If exactly two of $i_2, \ldots, i_m$ are equal, we can replace one of the repeated elements with $|x_t|$, while the remaining elements (denoted as $|y_1|, \ldots, |y_{m-2}|$) remain unchanged.

Utilizing Lemmas 3.2 and 3.3, we obtain

$$|\lambda||x_t| \le |x_t| \left( \beta_t + \frac{1}{(m-2)^{\frac{m-2}{2}}} r_t^{\overline{\Lambda_t} \cap \Delta}(\mathcal{A}) + r_t^{\overline{\Lambda_t} \cap \overline{\Delta}}(\mathcal{A}) \right). \tag{16}$$

Let $|x_t| \ge |x_s| \ge \{\max |x_k| : k \in [n], \, k \ne s, \, k \ne t\}$, then $|x_t| > 0$. Again by (14), we have

$$|\lambda| \, |x_t| \le \sum_{i_2,\ldots,i_m \in \Omega_t} |a_{t i_2 \ldots i_m}| \, |x_{i_2}| \ldots |x_{i_m}| + \sum_{i_2,\ldots,i_m \in \overline{\Omega}_t} |a_{t i_2 \ldots i_m}| \, |x_{i_2}| \ldots |x_{i_m}|.$$

For $t \neq s$, similar to the characterization of (16) for the second summation in the above inequality, yields that

$$|\lambda|\,|x_t| \leq \left(\beta_t + \frac{1}{(m-2)^{\frac{m-2}{2}}} r_t^{\overline{\Lambda_t}\cap\Delta\cap\Omega_t}(\mathcal{A}) + r_t^{\overline{\Lambda_t}\cap\overline{\Delta}\cap\Omega_t}(\mathcal{A})\right)|x_t| + r_t(\mathcal{A})^{\overline{\Omega_t}}\,|x_s|^{m-1}.$$

Thus,

$$\left(|\lambda| - M_t^{\Omega_t}(\mathcal{A})\right)|x_t| \leq r_t^{\overline{\Omega_t}}(\mathcal{A})\,|x_s|^{m-1} \leq r_t^{\overline{\Omega_t}}(\mathcal{A})\,|x_s|. \tag{17}$$

Let $\Gamma$ be a directed graph with a vertex set $V$ and an arc set $E$. A circuit in $\Gamma$ is characterized by a sequence $\upsilon_{i_1}, \ldots, \upsilon_{i_p}, \upsilon_{i_{p+1}} = \upsilon_{i_1}$, where $p \geq 2$ and the vertices $\upsilon_{i_1}, \ldots, \upsilon_{i_p}$ are distinct. Additionally, the arcs $(\upsilon_{i_1}, \upsilon_{i_2}), (\upsilon_{i_2}, \upsilon_{i_3}), \ldots, (\upsilon_{i_p}, \upsilon_{i_1})$ must belong to the set $E$. The directed graph $\Gamma$ is termed weakly connected if, for every vertex $\upsilon_i \in V$, there exists at least one circuit that includes $\upsilon_i$.

Define

$$\Gamma_{\upsilon_i}^{+} = \{\upsilon_j \in V \ : \ (\upsilon_i, \upsilon_j) \in E,\ \upsilon_i \in V\}.$$

A pre-order established on the set $V$ fulfills the following criteria:

1. $\upsilon_i \leqslant \upsilon_i$;
2. If $\upsilon_i \leqslant \upsilon_j$ and $\upsilon_j \leqslant \upsilon_k$, then $\upsilon_i \leqslant \upsilon_k$;
3. $\upsilon_i \leqslant \upsilon_j$ and $\upsilon_j \leqslant \upsilon_i$ do not imply $\upsilon_i = \upsilon_j$,

where $\upsilon_i, \upsilon_j, \upsilon_k \in V$ (see, e.g., [3]).

**Lemma 3.5.** *[3] Let $\Gamma$ be a directed graph in which a pre-order is defined on its vertex set. Assume that $\Gamma_{\upsilon}^{+}$ is nonempty for every vertex $\upsilon$. Then, there exists a circuit of the form $\upsilon_{i_1}, \ldots, \upsilon_{i_k}, \upsilon_{i_{k+1}} = \upsilon_{i_1}$, such that $\upsilon_{i_{j+1}}$ is a maximal element of $\Gamma_{\upsilon_{i_j}}^{+}$ for each $j \in [k]$.*

Consider an $m$-order, $n$-dimensional tensor $\mathcal{A} = (a_{i_1\cdots i_m})$. We define a directed graph $\Gamma_{\mathcal{A}}$ in the following manner: the vertex set of $\Gamma_{\mathcal{A}}$ is specified as $V(\mathcal{A}) = \{1, \ldots, n\}$, and the arc set is defined as

$$E(\mathcal{A}) = \{(i,j) \mid a_{ii_2\cdots i_m} \neq 0,\ j \in \{i_2, \ldots, i_m\} \neq \{i, \ldots, i\}\}.$$

Let $C(\mathcal{A})$ represent the collection of all circuits in $\Gamma(\mathcal{A})$. We will now introduce the optimal Brualdi-type $Z$-eigenvalue inclusion set.

**Theorem 3.6.** *Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m})$ be an $m$-order, $n$-dimensional tensor, where $m$ is even, and suppose that $r_i^{\overline{\Omega_i}}(\mathcal{A}) \neq 0$ for $i \in [n]$. If $\Gamma(\mathcal{A})$ is weakly connected, then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{D}(\mathcal{A}) = \mathcal{D}_1(\mathcal{A}) \cup \mathcal{D}_2(\mathcal{A}),$$

*where* $\mathcal{D}_1(\mathcal{A}) = \bigcup_{\gamma\in C(\mathcal{A})}\{z \in \mathbb{R} \ : \ \prod_{i\in\gamma}\left(|z| - M_i^{\Omega_i}(\mathcal{A})\right) \leqslant \prod_{i\in\gamma} r_i^{\overline{\Omega_i}}(\mathcal{A}),$ $|z| > M_i^{\Omega_i}(\mathcal{A}),\ \forall i \in \gamma\}$, *and* $\mathcal{D}_2(\mathcal{A}) = \bigcup_{i\in[n]}\left\{z \in \mathbb{R} : |z| < M_i^{\Omega_i}(\mathcal{A})\right\}$, *and* $|\gamma|$ *denotes the length of circuit* $\gamma$.

*Proof.* Let $\lambda$ denote a $Z$-eigenvalue of $\mathcal{A}$. Given that $\Gamma(\mathcal{A})$ is weakly connected, it follows that $\lambda \in \mathcal{D}(\mathcal{A})$ if $|\lambda| = M_i^{\Omega_i}(\mathcal{A})$ for some $i \in [n]$. Now, suppose $|\lambda| > M_i^{\Omega_i}(\mathcal{A})$ for all $i \in [n]$. Let $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ be a $Z$-eigenvector corresponding to $\lambda$, and let $\Gamma_0$ denote the subgraph of $\Gamma_{\mathcal{A}}$ induced by those vertices $i$ for which $x_i \neq 0$. We define a pre-order on the vertex set of $\Gamma_0$ such that $i \leq j$ if and only if $|x_i| \leq |x_j|$. According to Lemma 3.5, there exists a circuit $\gamma = \{i_1, \ldots, i_p, i_{p+1} = i_1\}$ within $\Gamma_0$ that satisfies the condition $|x_{i_{j+1}}| > |x_k|$ for any $k \in \Gamma_0^+(i_j)$ (where $j = 1, \ldots, p$). Consequently, from inequality (17), we obtain

$$\left(|\lambda| - M_{i_{j+1}}^{\Omega_{i_{j+1}}}(\mathcal{A})\right)\left|x_{i_{j+1}}\right| \leq r_{i_{j+1}}^{\overline{\Omega}_{i_{j+1}}}(\mathcal{A})\left|x_{i_j}\right|.$$

and hence,

$$\prod_{j=1}^{p}\left(|\lambda| - M_{i_{j+1}}^{\Omega_{i_{j+1}}}(\mathcal{A})\right)\prod_{k=1}^{p}|x_{i_{k+1}}| \leq \prod_{j=1}^{p} r_{i_{j+1}}^{\overline{\Omega}_{i_{j+1}}}(\mathcal{A})\prod_{k=1}^{p}|x_{i_k}|.$$

Given that $i_{p+1} = i_1$ and $x_{i_j} \neq 0$ for $j = 1, \ldots, p$, it follows that

$$\prod_{j=1}^{p}\left(|\lambda| - M_{i_{j+1}}^{\Omega_{i_{j+1}}}(\mathcal{A})\right) \leq \prod_{j=1}^{p} r_{i_{j+1}}^{\overline{\Omega}_{i_{j+1}}}(\mathcal{A}),$$

that is,

$$\prod_{i\in\gamma}\left(|\lambda| - M_i^{\Omega_i}(\mathcal{A})\right) \leq \prod_{i\in\gamma} r_i^{\overline{\Omega}_i}(\mathcal{A}).$$

If $|\lambda| < M_i^{\Omega_i}(\mathcal{A})$ for some $i \in [n]$, it follows that $\lambda \in \mathcal{D}_2(\mathcal{A})$. Consequently, we conclude that $\lambda \in \mathcal{D}(\mathcal{A})$, thereby completing the proof. □

In [35], an alternative Brauer-type $Z$-eigenvalue localization set for tensors of even order was established.

**Theorem 3.7.** *[35, Theorem 2.12] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be an $m$-order, $n$-dimensional tensor, where $m$ is even, then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{P}(\mathcal{A}) = \bigcup_{i\in[n]}\bigcap_{j\in[n], i\neq j}\mathcal{P}_{i,j}(\mathcal{A}),$$

*where*

$$\mathcal{P}_{i,j}(\mathcal{A}) = \left\{z \in \mathbb{R} : \left(|z| - M_i^{\Omega_i}(\mathcal{A})\right)|z| \leq r_i^{\overline{\Omega_i}}(\mathcal{A})\,M_j(\mathcal{A})\right\}.$$

Similar to Theorem 2.5, we indicate that the localization set $\mathcal{D}(\mathcal{A})$ is more precise than the localization set $\mathcal{P}(\mathcal{A})$.

**Theorem 3.8.** *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be an $m$-order, $n$-dimensional tensor, where $m$ is even, and suppose that $r_i^{\overline{\Omega_i}}(\mathcal{A}) \neq 0$ for $i \in [n]$. If $\Gamma(\mathcal{A})$ is weakly connected, then*

$$\mathcal{D}(\mathcal{A}) \subseteq \mathcal{P}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}). \tag{18}$$

*Proof.* Inclusion $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A})$ was proved in [35, Corollary 3]. For $z \in \mathcal{D}(\mathcal{A})$, if $z \notin \mathcal{P}(\mathcal{A})$, then for all $i \in [n]$, there exists $j \in [n]$, $j \neq i$, such that $z \notin \mathcal{P}_{i,j}(\mathcal{A})$, according to the definition of $\mathcal{P}_{i,j}(\mathcal{A})$ in Theorem 3.7.

If $M_j(\mathcal{A}) \neq 0$, then

$$\frac{|z| - M_i^{\Omega_i}(\mathcal{A})}{r_i^{\overline{\Omega_i}}(\mathcal{A})} > 1$$

or

$$\frac{|z|}{M_j(\mathcal{A})} > 1,$$

and thus, we have

$$|z| > M_i(\mathcal{A}) = M_i^{\Omega_i}(\mathcal{A}) + r_i^{\overline{\Omega_i}}(\mathcal{A}), \quad \forall i \in [n]. \tag{19}$$

When $M_j(\mathcal{A}) = 0$, we also have (19). Thus, $z \notin \mathcal{D}_2(\mathcal{A})$.

Since $\Gamma(\mathcal{A})$ is weakly connected, there exists a circuit $\gamma \in C(\mathcal{A})$ such that

$$\prod_{t\in\gamma} \left(|\lambda| - M_i^{\Omega_i}(\mathcal{A})\right) > \prod_{t\in\gamma} r_i^{\overline{\Omega_i}}(\mathcal{A}).$$

Thus, we conclude that $\lambda \notin \mathcal{D}_1(\mathcal{A})$, implying $\lambda \notin \mathcal{D}(\mathcal{A})$, which leads to a contradiction. Therefore, we have $\mathcal{D}(\mathcal{A}) \subseteq \mathcal{P}(\mathcal{A})$. □

Below, we present an example that illustrates how the localization set defined in Theorem 3.6 is more accurate than the other localization sets.

**Example 3.9.** *( [5, 13, 14, 16, 19, 21, 33]) Consider the symmetric tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,2]}$, with entries defined as follows:*

$$a_{1111} = \frac{1}{2}, \;\; a_{2222} = 3, \;\; \textit{and} \;\; a_{ijkl} = \frac{1}{3} \;\; \textit{elsewhere.}$$

*When $i_1 = 1$, we have*

$$\Lambda_1 = \{(i_2, i_3, i_4) \; : \; (1,1,1), (1,2,2)\},$$

$$\overline{\Lambda_1} \cap \Delta = \{(i_2, i_3, i_4) : (2,1,2), (2,2,1), (1,1,2), (2,1,1), (1,2,1)\} = \overline{\Lambda_1} \cap \Delta \cap \Omega_1,$$

$$\overline{\Lambda_1} \cap \overline{\Delta} = \{(i_2, i_3, i_4) \; : \; (2,2,2)\}, \qquad \overline{\Lambda_1} \cap \overline{\Delta} \cap \Omega_1 = \emptyset,$$

*which implies $\beta_1 = \frac{1}{2}$, $M_1(\mathcal{A}) = \frac{10}{6}$, $M_1^{\Omega_1}(\mathcal{A}) = \frac{8}{6}$ and $r_1^{\overline{\Omega_1}}(\mathcal{A}) = \frac{1}{3}$.*

*When $i_1 = 2$, we have*

$$\Lambda_2 = \{(i_2, i_3, i_4) \; : \; (2,2,2), (2,1,1)\},$$

$$\overline{\Lambda_2} \cap \Delta = \{(i_2, i_3, i_4) \; : \; (1,1,2), (1,2,1), (1,2,2), (2,1,2), (2,2,1)\} = \overline{\Lambda_2} \cap \Delta \cap \Omega_2,$$

$$\overline{\Lambda_2} \cap \overline{\Delta} = \{(i_2, i_3, i_4) \; : \; (1,1,1)\}, \qquad \overline{\Lambda_2} \cap \overline{\Delta} \cap \Omega_2 = \emptyset,$$

*which implies $\beta_2 = 3$, $M_2(\mathcal{A}) = \frac{25}{6}$, $M_2^{\Omega_2}(\mathcal{A}) = \frac{23}{6}$ and $r_1^{\overline{\Omega_1}}(\mathcal{A}) = \frac{1}{3}$.*

*We calculate the bounds for $\rho(\mathcal{A})$ in Table 2. This table shows that the bound provided in Theorem 3.6 outperforms the other bounds available in the literature.*

TABLE 2. Upper bounds for $\rho(\mathcal{A})$ in Example 3.9

| | |
|---|---|
| *Proposition 3.3 of [5]* | $\rho(\mathcal{A}) \leq 7.5432$ |
| *Corollary 4.5 of [29]* | $\rho(\mathcal{A}) \leq 5.3333$ |
| *Theorem 2.7 of [13]* | $\rho(\mathcal{A}) \leq 5.2846$ |
| *Theorem 3.3 of [19]* | $\rho(\mathcal{A}) \leq 5.1935$ |
| *Theorem 4.6 of [33]* | $\rho(\mathcal{A}) \leq 5.1822$ |
| *Theorem 4.7 of [33]* | $\rho(\mathcal{A}) \leq 5.1822$ |
| *Theorem 4.2 of [16]* | $\rho(\mathcal{A}) \leq 5.1667$ |
| *Theorem 2.4 of [21]* | $\rho(\mathcal{A}) \leq 4.5147$ |
| *Theorem 3.1 of [14]* | $\rho(\mathcal{A}) \leq 4.0000$ |
| *Theorem 2.12 of [35]* | $\rho(\mathcal{A}) \leq 3.9732$ |
| *Theorem 3.6* | $\rho(\mathcal{A}) \leq 3.8770$ |

## 4. **Application in Best Rank-One Approximation**

In this section, we derive bounds for the best rank-one approximation ratio $App(V)$ by applying the results from prior sections.

Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be an $m$-order, $n$-dimensional tensor. The rank-one tensor $\kappa x^m = (\kappa x_{i_1} x_{i_2} \cdots x_{i_m}) \in \mathfrak{R}_1$ is considered the best rank-one approximation of $\mathcal{A}$ if it minimizes the expression

$$\left\{ \|\mathcal{A} - \kappa x^m\|_F : \kappa \in \mathbb{R},\ x \in \mathbb{R}^n,\ x^T x = 1 \right\},$$

where $\|\mathcal{A}\|_F := \sqrt{\sum_{i_1,i_2,\ldots,i_m=1}^n a_{i_1 i_2 \cdots i_m}^2}$ is the Frobenius norm of the tensor $\mathcal{A}$.

Numerous theoretical results have been developed, along with various numerical methods, to investigate tensor rank-one approximation [6, 11, 17]. Unfortunately, finding the best rank-one approximation for tensors and spectral norms is NP-hard [15], which means that these methods only yield approximate solutions. Additionally, Qi [25] has shown that the rank-one tensor $\kappa x^m$ is the optimal symmetric approximation of $\mathcal{A}$ if $\kappa$ corresponds to the $Z$-eigenvalue of $\mathcal{A}$ with the largest absolute value, and $x$ is the associated eigenvector of $\kappa$.

The best rank-one approximation ratio [24] within the tensor space $V$ is defined as

$$App(V) = \max \left\{ \mu : \ \mu \leq \frac{\rho(\mathcal{A})}{\|\mathcal{A}\|_F},\ \forall \mathcal{A} \in V,\ \mathcal{A} \neq O \right\} = \min_{\mathcal{A} \neq O} \frac{\rho(\mathcal{A})}{\|\mathcal{A}\|_F} = \min_{\|\mathcal{A}\|_F = 1} \rho(\mathcal{A}).$$

The maximum positive lower bound for the quotient of the best rank-one approximation of any tensor within the tensor space $V$ relative to the norm of that tensor is always less than one.

Consequently, when $\mathcal{A}$ is a weakly symmetric tensor, the expression $\rho(\mathcal{A}) x_*^m$ represents the best rank-one approximation of $\mathcal{A}$. Here, $x_*$ refers to the $Z$-eigenvector of $\mathcal{A}$ associated with $\rho(\mathcal{A})$, indicating that

$$\min_{\kappa\in\mathbb{R},x\in\mathbb{R}^n,x^Tx=1}\|\mathcal{A}-\kappa x^m\|_F=\|\mathcal{A}-\rho(A)x_*^m\|_F=\sqrt{\|\mathcal{A}\|_F^2-(\rho(\mathcal{A}))^2}.$$

If we obtain a bound for $\rho(\mathcal{A})$, we can derive bounds for $\min_{\kappa\in\mathbb{R},x\in\mathbb{R}^n,x^Tx=1}\|\mathcal{A}-\kappa x^m\|_F$, the approximation ratio $App(V)$, and $\frac{\|\mathcal{A}-\rho(\mathcal{A})x_*^m\|_F}{\|\mathcal{A}\|_F}$. Consequently, building on the prior results, we derive new bounds for $App(V)$.

The ratio of the residual of the best rank-one approximation of $\mathcal{A}$ to the Frobenius norm of the tensor can be represented as follows

$$\frac{\|\mathcal{A}-\rho(A)x_*^m\|_F}{\|\mathcal{A}\|_F}=\sqrt{1-\frac{(\rho(\mathcal{A}))^2}{\|\mathcal{A}\|_F^2}}=\sqrt{1-App(V)^2}.$$

This provides a rate of convergence for the greedy rank-one update algorithms [1,32].

In a practical application, the author in [30] illustrated that estimating $App(V)$ is crucial for understanding the convergence of rank-truncated steepest descent methods in low-rank matrix and tensor optimization problems. He also pointed out that the alternating least squares (ALS) method [11,31] achieves local linear convergence under specific conditions related to the rank of the Hessian matrix of the objective function, stemming from a modified minimization formulation of power method or the ALS approach.

There are several results regarding the bounds of $App(V)$ for tensors. For tensors of order 3, these findings have been established previously. Cobos et al. in [8] provided the following estimates

$$\frac{1}{n}\le App_3(\mathbb{R};n,n,n)\le\frac{3\sqrt{\pi}}{n}.$$

Furthermore, they note that

$$App_d(\mathbb{R};n,\dots,n)=O\left(\frac{1}{\sqrt{n^{d-1}}}\right),$$

specifically,

$$App_d(\mathbb{R};n,\dots,n)\le\frac{d\sqrt{\pi\sqrt{2}}}{\sqrt{n^{d-1}}}.$$

In the context of finite-dimensional symmetric tensor space, the following bounds are provided in [24,37]

$$\frac{1}{n}\le App\left(Sym^3(\mathbb{R}^n)\right)\le\sqrt{\frac{6}{n+5}}.$$

Moreover, we aim to derive estimates for bounds within a finite-dimensional space $\hat{V}$ consisting of weakly symmetric nonnegative tensors. Based on earlier findings, we present the following bounds.

**Theorem 4.1.** *Let $\hat{V}$ be a finite-dimensional space comprised of weakly symmetric nonnegative tensors, and consider $\mathcal{A} \in \hat{V}$. Then*

$$\frac{\mathcal{L}}{\|\mathcal{A}\|_F} \leq App_{\mathcal{A}}(\hat{V}) \leq \frac{\mathcal{U}}{\|\mathcal{A}\|_F},$$

*where $\mathcal{L}$ and $\mathcal{U}$ denote the lower and upper bounds of $\rho(\mathcal{A})$, respectively.*

*Proof.* Since $\mathcal{A}$ is a weakly symmetric tensor, it follows that $\rho(\mathcal{A})$ represents the largest $Z$-eigenvalue of $\mathcal{A}$ (refer to Theorem 3.11 in [5]). Thus, the conclusion directly follows from the definition of $App(\hat{V})$ for the tensor $\mathcal{A}$. □

We demonstrate that the bounds presented in Theorem 4.1 are better than the existing results, using Example 3.9 as a reference.

Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,2]}$ be the symmetric tensor, with its elements defined as follows

$$a_{1111} = \frac{1}{2}, \;\; a_{2222} = 3, \;\; \textit{and} \;\; a_{ijkl} = \frac{1}{3} \;\; \textit{elsewhere.}$$

Calculating with the MATLAB toolbox "TenEig," we obtain the following values

$$\|\mathcal{A}\|_F = 3.2872, \qquad \rho(\mathcal{A}) = 3.1092, \qquad App(\mathcal{A}) = 0.9459.$$

According to [19, Theorem 3.3] (as shown in Example 4.1 of [21]), we obtain

$$0.7330 \leq \rho(\mathcal{A}) \leq 5.1935.$$

From these bounds, we can derive the corresponding estimates for the best rank-one approximation ratio.

$$0.2229 \leq App\left(Sym^4(\mathbb{R}^n)\right) \leq 1.5799.$$

Additionally, from [21, Theorem 2.4], we find

$$0.7663 \leq \rho(\mathcal{A}) \leq 4.5147,$$

which results

$$0.2331 \leq App\left(Sym^4(\mathbb{R}^n)\right) \leq 1.3734.$$

Applying Theorem 3.6 gives us the bounds

$$1.2897 \leq \rho(\mathcal{A}) \leq 3.8770,$$

leading to

$$0.3923 \leq App\left(Sym^4(\mathbb{R}^n)\right) \leq 1.1794.$$

These results indicate that the lower and upper bounds in Theorem 4.1 are tighter than the existing bounds.

Furthermore, the bounds for $\min_{\kappa\in\mathbb{R}, x\in\mathbb{R}^n, x^T x=1} \|\mathcal{A} - \kappa x^m\|_F$ and $\frac{\|\mathcal{A}-\rho(\mathcal{A})x_*^m\|_F}{\|\mathcal{A}\|_F}$ can be obtained similarly.

## 5. Conclusions

In this work, we introduced novel Brauer-type and Brualdi-type $Z$-eigenvalue inclusion sets, labeled as $\mathfrak{B}(\mathcal{A})$ and $\mathcal{Z}(\mathcal{A})$, which extend classical results from matrix theory to tensors. By categorizing the tensor index set, we also derived an optimal inclusion set $\mathcal{D}(\mathcal{A})$ for tensors of even order, providing sharper bounds.

One significant application of these findings is to evaluate the best rank-one approximation for weakly symmetric tensors. The proposed inclusion sets establish a robust framework for bounding $Z$-eigenvalues, which are crucial in determining the accuracy and computational efficiency of best rank-one approximation techniques. To demonstrate the effectiveness of our approach, we presented numerical examples illustrating that our inclusion sets outperform existing methods by delivering tighter bounds and improved approximations.

For future work, we aim to extend the inclusion sets to tensors of arbitrary order, including odd-order tensors, thereby providing a more comprehensive framework for tensor spectral analysis. Additionally, we intend to explore applications in areas such as the geometric measure of entanglement in multipartite quantum states and the sufficient condition for the positive definiteness of a homogeneous polynomial form, which is used to judge the asymptotic stability of time-invariant polynomial systems.

## 6. Author Contributions

All authors have reviewed and approved the final version of the paper.

## 7. Data Availability Statement

Not applicable.

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## 9. Ethical considerations

The authors affirm that no data fabrication or falsification occurred in this study.

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## 11. Conflict of interest

The authors declare no conflicts of interest.

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Abbas Askarizadeh
Orcid number: 0000-0001-6663-0548
Department of Mathematics
Vali-e-Asr University of Rafsanjan
Rafsanjan, Iran
*Email address*: a.askari@vru.ac.ir

Mostafa Zangiabadi
Orcid number: 0000-0003-4472-3609
Department of Mathematics
University of Hormozgan
Bandar Abbas, Iran
*Email address*: zangiabadi1@gmail.com