Journal of Mahani Mathematical Research
Shahid Bahonar University of Kerman
Print ISSN: 2251-7952 Online ISSN: 2645-4505

A CERTAIN SUBCLASS OF STARLIKE FUNCTIONS DEFINED BY SUBORDINATION

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Article type: Research Article
(Received: 28 May 2024, Received in revised form 16 January 2025)
(Accepted: 25 March 2025, Published Online: 25 March 2025)

Abstract. In the present paper, we introduce and investigate a new result connected to subclasses of normalized and univalent functions in the open unit disk. Some majorization results and geometric properties such as radii of starlikeness, convexity, pre-Schwarzian norm and coefficient estimates are obtained.

Keywords: Subordination, Starlike function, Convex function, Radius problems, pre-Schwarzian norm, coefficient estimates.
2020 MSC: 30C45.

1. Introduction and Preliminaries

The geometric function theory is one of the important research subjects of complex analysis. Let $\mathcal{A}$ be the class of analytic and normalized functions $f(0) = 0 = f'(0) - 1)$ of the

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{S}$ be the class of all functions in $\mathcal{A}$ that are univalent (one-to-one) in Δ. Roberson [2] introduced the classes $\mathcal{S}^*(\gamma)$ and $\mathcal{K}(\gamma)$ of starlike and convex functions of order $\gamma \leq 1$, which are defined by

$$\mathcal{S}^*(\gamma) := \left\{ f \in \mathcal{A} : \ \mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \gamma, \ z \in \Delta \right\}, \tag{2}$$

$$\mathcal{K}(\gamma) := \left\{ f \in \mathcal{A} : \ \mathrm{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \gamma, \ z \in \Delta \right\}, \tag{3}$$

respectively. If $\gamma \in [0, 1)$, then a function in either of these classes is univalent. If $\gamma < 0$, it may fail to be univalent. Indeed $\mathcal{S}^*(0) = \mathcal{S}$ and $\mathcal{K}(0) = \mathcal{K}$ are the well-known starlike and convex functions respectively.

We also recall the well-known Alexander's Theorem $f \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$ [7].

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https://doi.org/10.22103/jmmr.2025.23520.1662
Publisher: Shahid Bahonar University of Kerman
How to cite: Kh. Sheikhi, Sh. Najafzadeh, *A certain subclass of starlike functions defined by subordination*, J. Mahani Math. Res. 2025; 14(2): 267-278.

The function $f(z)$ is said to be subordinate to $g(z)$ and write $f \prec g$ if there exists a Schwarz analytic function $w(z)$, with condition $w(0) = 0$ and $|w(z)| < 1$, such that

$$f(z) = g(w(z)). \tag{4}$$

Ma and Minda [17] introduced the family $\mathcal{S}^*(\varphi)$ as follows:

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{A} : \ \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}, \tag{5}$$

where φ is analytic and univalent in Δ and satisfies the following conditions:
a) $\varphi(\Delta) \subset \{\xi : \ \mathrm{Re}\, \xi > 0\}$.
b) φ is starlike with respect to $\varphi(0) = 1$.
c) $\varphi(\Delta)$ is symmetric with respect to real axis.
d) $\varphi'(0) > 0$.
Recently, several researchers in [9, 10, 13–15, 21] have defined a non-Ma-Minda class of functions as follows:

$$\mathcal{S}^*_c(\phi) = \left\{ f : \ \left(\frac{zf'(z)}{f(z)} - 1 \right) \prec \phi(z) \right\}, \tag{6}$$

where one or more of the condition (a)-(d) are not true by $\phi(z)$.

Furthermore many special cases of the class $\mathcal{S}^*(\varphi)$ were studied by many authors, see [3, 6, 8, 11, 22–26].

For $\alpha \in (0, 1]$, we consider the function

$$\psi_\alpha(z) := \frac{1}{\alpha} \log(1 + \alpha z), \qquad (z \in \Delta), \tag{7}$$

with series expansion:

$$\psi_\alpha(z) = z + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{\alpha^{n-1}}{n} z^n. \tag{8}$$

If $\alpha \to 0^+$, then ψ_α maps Δ onto itself while if $\alpha \to 1$, then $\psi_\alpha(\Delta)$ is an ellipse. See Figure 1.

Definition 1.1. For $\alpha \in (0, 1]$ and $\psi_\alpha(z)$ defined by (8), $f \in \mathcal{A}$ belongs to $\mathcal{S}^*_\psi(\alpha)$, if and only if

$$\left(\frac{zf'(z)}{f(z)} - 1 \right) \prec \psi_\alpha(z), \qquad (z \in \Delta). \tag{9}$$

In relation (5), if $\varphi(z) = (1 + (1 - 2\delta)z) / (1 - z)$, we get the class $\mathcal{M}(\delta)$, $(\delta > 1)$ which was introduced in [26] as follows:

$$\mathcal{M}(\delta) := \left\{ f \in \mathcal{A} : \ \mathrm{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < \delta, \ \delta > 1, z \in \Delta \right\}. \tag{10}$$

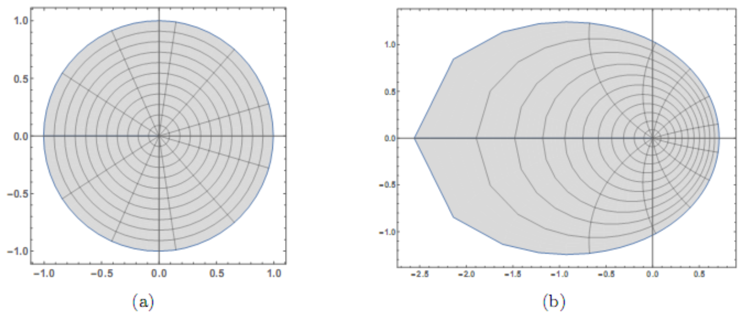

FIGURE 1. (a): The image of the unit disk under $\psi_{0.01}$, (b): The image of the unit disk under $\psi_{0.9}$.

It is an easy exercise to get

$$\frac{1}{\alpha}\ln(1-\alpha) \le \operatorname{Re}\left\{\psi_\alpha(e^{i\theta})\right\} \le \frac{1}{\alpha}\ln(1+\alpha), \tag{11}$$

$$-\frac{1}{\alpha}\arctan(\alpha) \le \operatorname{Im}\left\{\psi_\alpha(e^{i\theta})\right\} \le \frac{1}{\alpha}\arctan(\alpha). \tag{12}$$

For our main results, we need the following lemmas.

Lemma 1.2. *[16] Let $f \in \mathcal{A}$ and $0 < \alpha \le 1$. Then $f \in \mathcal{S}^*_\psi(\alpha)$ if and only if there exists an analytic function q with $q(z) \prec \psi_\alpha(z)$, $q(0) = \psi_\alpha(0)$, such that*

$$f(z) = z \exp \int_0^z \frac{q(t)}{t}dt, \qquad (z \in \Delta).$$

Lemma 1.3. *[5] For $0 < \alpha < 1$, the function $\psi_\alpha(z)$ is convex univalent in $|z| < r_c(\alpha)$, where $r_c(\alpha) \in (0, 1/2\alpha)$.*

Example 1.4. *If $q(t) = \psi_\alpha(t)$ in Lemma 1.2, then*

$$\begin{aligned} F_\alpha(z) &= z \exp \int_0^{\alpha z} \frac{\ln(1-t)}{t}dt \qquad 0 < \alpha \le 1 \\ &= z - \alpha z^2 + \frac{\alpha^2}{4}z^3 - \frac{\alpha^3}{36}z^4 - \frac{\alpha^4}{288}z^5 - \frac{23\alpha^5}{7200}z^6 + O(z^7), \end{aligned}$$

belongs to $\mathcal{S}^*_\psi$.

The aim of this paper is to study the class $\mathcal{S}^*_\psi(\alpha)$. We obtain the radii of starlikenees and convexity. Also, the radius connected to the majorization is investigated. Furthermore, we estimate the Pre-Schwarzian norm and coefficient bounds. For more details about such classes one may refer to [1, 4, 12, 18–20].

2. A Set of Main Results

The first result of this section is as follows, which shows that $\mathcal{S}^*_\psi(\alpha) \not\subset \mathcal{S}^*$.

Theorem 2.1. *Let $f \in \mathcal{S}^*_\psi(\alpha)$. Then the function f is not a starlike univalent function in the unit disk Δ.*

Proof. Let $f \in \mathcal{A}$ belong to the class $\mathcal{S}^*_\psi(\alpha)$, where $\alpha \in (0,1)$. Then by Definition 1.1 and subordination, there exists a Schwarz function $w(z)$ such that

$$\frac{zf'(z)}{f(z)} = 1 + \frac{1}{\alpha}\log(1+\alpha w(z)), \quad (z \in \Delta). \tag{13}$$

It is well-known that for $|z| \geq 1$, the following inequality holds true:

$$|\log z| \leq \sqrt{|z-1|^2 + \pi^2} \tag{14}$$

Now, it follows from $|w(z)| \leq |z|$, (13) and (14) that

$$\begin{aligned}\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} &= \operatorname{Re}\left\{1 + \frac{1}{\alpha}\log(1+\alpha w(z))\right\} \geq 1 - \frac{1}{\alpha}\left|\log(1+\alpha w(z))\right| \\ &\geq 1 - \frac{1}{\alpha}\sqrt{\alpha^2|z|^2 + \pi^2} = 1 - \frac{1}{\alpha}\sqrt{\alpha^2 r^2 + \pi^2} =: h(\alpha, r),\end{aligned}$$

where $|z| = r$. It can easily be seen that $h(\alpha, r) < 0$ for all $\alpha \in (0,1)$ and $r \in (0,1)$. Therefore,

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \not> 0$$

for all $z \in \Delta$. It means that f does not belong to the class of starlike univalent functions. The proof is now completed. $\square$

As a corollary of Definition 1.1, we have the following:

Corollary 2.2. *If a function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*_\psi(\alpha)$, then by* (11) *and Definition 1.1, we get*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \operatorname{Re}\left\{1 + \psi_\alpha(w(z))\right\} \geq 1 + \frac{1}{\alpha}\ln(1-\alpha) =: t(\alpha).$$

It is easy to see that $t(\alpha) < 0$ for all $\alpha \in (0,1)$, which shows that f is not a starlike function.

Remark 2.3. It follows from $\mathcal{K} \subset \mathcal{S}^*$ and Theorem 2.1 that if a function f belongs to the class $\mathcal{S}^*_\psi(\alpha)$, then it is not a convex univalent function, too.

Theorem 2.4. *If a function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*_\psi(\alpha)$, then it belongs to the class $\mathcal{M}(\delta(\alpha))$, where*

$$\delta(\alpha) := 1 + \frac{1}{\alpha}\sqrt{\alpha^2 + \pi^2}.$$

Proof. By (13) and (14) we have:

$$\begin{aligned}\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} &= \operatorname{Re}\left\{1 + \frac{1}{\alpha}\log(1+\alpha w(z))\right\} \leq 1 + \frac{1}{\alpha}\left|\log(1+\alpha w(z))\right| \\ &\leq 1 + \frac{1}{\alpha}\sqrt{\alpha^2|z|^2 + \pi^2} = 1 + \frac{1}{\alpha}\sqrt{\alpha^2 r^2 + \pi^2} =: g(\alpha, r), \quad (|z| = r).\end{aligned}$$

It is easy to check that $g(\alpha, r)$ gets its maximum at $r = 1$, thus $g(\alpha, r) < 1 + \sqrt{\alpha^2 + \pi^2}/\alpha$ which completes the proof. □

In order to find the radius of convexity, we need the following lemma which was verified by Nehari, see Ref. [19].

Lemma 2.5. *Let $\ell(z)$ be analytic in Δ and satisfying $|\ell(z)| \leq 1$ for all $z \in \Delta$. Then*

$$|\ell'(z)| \leq \frac{1 - |\ell(z)|^2}{1 - |z|^2}. \tag{15}$$

Theorem 2.6. *Let the function $f \in \mathcal{A}$ belong to the class $\mathcal{S}^*_\psi(\alpha)$, where $0 < \alpha \leq 1$. Then f is a convex univalent function in $|z| > r_c(\alpha)$, where $r_c(\alpha) < 1$ is the smallest positive root of the equation*

$$1 - \frac{1}{\alpha}\sqrt{\alpha^2 r^2 + \pi^2} - \left(\frac{1}{(1 - \alpha r)\left(1 - \frac{1}{\alpha}\sqrt{\alpha^2 r^2 + \pi^2}\right)} \right) \frac{r}{1 - r^2}.$$

On another hand, $\mathcal{S}^*_\psi(\alpha) \subset \mathcal{K}$ in $|z| > r_c(\alpha)$. The result is sharp.

Proof. Let the function $f \in \mathcal{A}$ belong to the class $\mathcal{S}^*_\psi(\alpha)$. Then, by (13) we obtain

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{1}{\alpha}\log(1 + \alpha w(z)) + \left(\frac{1}{(1 + \alpha w(z))\left(1 + \frac{1}{\alpha}\log(1 + \alpha w(z))\right)} \right) zw'(z).$$

If we calculate the real part of both sides of the last equality above and use the above Lemma **??**, then we get

$$\begin{aligned}
&\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \\
&= \mathrm{Re}\left\{1 + \frac{1}{\alpha}\log(1 + \alpha w(z)) + \left(\frac{1}{(1 + \alpha w(z))\left(1 + \frac{1}{\alpha}\log(1 + \alpha w(z))\right)} \right) zw'(z) \right\} \\
&\geq 1 - \left| \frac{1}{\alpha}\log(1 + \alpha w(z)) + \left(\frac{1}{(1 + \alpha w(z))\left(1 + \frac{1}{\alpha}\log(1 + \alpha w(z))\right)} \right) zw'(z) \right| \\
&\geq 1 - \frac{1}{\alpha}|\log(1 + \alpha w(z))| - \frac{1}{(1 - \alpha|w(z)|)\left(1 - \frac{1}{\alpha}|\log(1 + \alpha w(z))|\right)}|zw'(z)| \\
&\geq 1 - \frac{1}{\alpha}\sqrt{\alpha^2 r^2 + \pi^2} - \left(\frac{1}{(1 - \alpha r)\left(1 - \frac{1}{\alpha}\sqrt{\alpha^2 r^2 + \pi^2}\right)} \right) \frac{r}{1 - r^2} =: \lambda(\alpha, r).
\end{aligned}$$

It is an exercise that $\lambda(\alpha, r)$ has a root depending on α, denoted by $r_c(\alpha)$, which is less than one. It is easy to see that for $|z| > r_c(\alpha)$, the function $\lambda(\alpha, r)$ is positive. Figure 2 gives more details. □

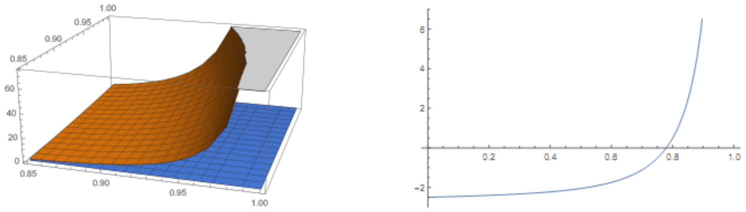

FIGURE 2. (a): The 3D plot of $\lambda(\alpha, r)$, where $0.085 < \alpha, r < 1$, (b): The plot of $\lambda(0.9, r)$, where $0 < r < 1$.

The next result gives the majorization radius. It should be remarked that if $f(z)$ is majorized by $g(z)$ in Δ and $g(0) = 0$, then

$$\max_{|z|=r} |f'(z)| \leq \max_{|z|=r} |g'(z)|$$

for each number r in the interval $[0, \sqrt{2} - 1]$ (see [16]).

It is easy to see that $g(z) = z$ satisfies the last inequality. Therefore we have the following (see [5]):

Lemma A. Let f be an analytic function in Δ with $|f(z)| \leq 1$ and $f(0) = 0$. Then $|f'(z)| \leq 1$ for $|z| \leq \sqrt{2} - 1$.

Motivated by the above, we obtain the radius of majorization for the class $\mathcal{S}^*_\psi(\alpha)$.

Theorem 2.7. *Let the function $g \in \mathcal{A}$ belong to the class $\mathcal{S}^*_\psi(\alpha)$, where $0 < \alpha \leq 1$. Also, let f be an analytic function in Δ. If f is majorized by g in Δ, then*

$$\max_{|z|=r_m} |f'(z)| \leq \max_{|z|=r_m} |g'(z)|$$

for each r_m, where r_m is the smallest positive root of the equation

$$(1 - r^2)\sqrt{\alpha^2 r^2 + \pi^2} - 2\alpha r - \alpha r^2 = 0, \quad (0 < r < 1).$$

The result is sharp.

Proof. Let the function $g \in \mathcal{A}$ be given by (1). Also, let g belong to the class $\mathcal{S}^*_\psi(\alpha)$ $(0 < \alpha \leq 1)$. Then there exists a Schwarz function $w(z)$ such that

$$\frac{zg'(z)}{g(z)} = \frac{\alpha + \log(1 + \alpha w(z))}{\alpha}, \quad (z \in \Delta),$$

or equivalently

$$\frac{g(z)}{g'(z)} = \frac{\alpha z}{\alpha + \log(1 + \alpha w(z))}, \quad (z \in \Delta). \tag{16}$$

By (14), and since $|w(z)| \leq |z|$ for all $z \in \Delta$, the relation (16) implies

$$\left|\frac{g(z)}{g'(z)}\right| \leq \frac{\alpha r}{\alpha + \sqrt{\alpha^2 r^2 + \pi^2}}, \quad (|z| = r < 1). \tag{17}$$

By using the definition of majorization, there exists an analytic function η in Δ with $|\eta(z)| \leq 1$ such that

$$f(z) = \eta(z) g(z), \quad (z \in \Delta). \tag{18}$$

Taking the differentiation of (18) gives

$$f'(z) = \eta(z) g'(z) + \eta'(z) g(z) = \left(\eta(z) + \eta'(z)\frac{g(z)}{g'(z)}\right) g'(z). \tag{19}$$

It follows from Lemma **??**, inequality (17), and (19) that

$$\begin{aligned} |f'(z)| &\leq \left(|\eta(z)| + \frac{1 - |\eta(z)|^2}{1 - r^2} \cdot \frac{\alpha r}{\alpha + \sqrt{\alpha^2 r^2 + \pi^2}}\right) |g'(z)| \\ &= \left(\xi + \frac{1 - \xi^2}{1 - r^2} \cdot \frac{\alpha r}{\alpha + \sqrt{\alpha^2 r^2 + \pi^2}}\right) |g'(z)|, \end{aligned}$$

where $|\eta(z)| =: \xi \in [0, 1]$. We define the function ℓ as follows:

$$\ell(\xi, r, \alpha) := \xi + \frac{1 - \xi^2}{1 - r^2} \cdot \frac{\alpha r}{\alpha + \sqrt{\alpha^2 r^2 + \pi^2}}, \quad (0 \leq \xi \leq 1, \, 0 < r < 1).$$

We are aiming to find the radius r such that

$$r_m := \max\{r \in [0, 1) : \ell(\xi, r, \alpha) \leq 1 \text{ for all } \xi \in [0, 1]\}.$$

Because $(1 - r^2)(\alpha + \sqrt{\alpha^2 r^2 + \pi^2})$ is positive for all $0 < \alpha \leq 1$ and for all $r \in (0, 1)$ we obtain

$$\begin{aligned} &\ell(\xi, r, \alpha) \leq 1 \\ &\Leftrightarrow \xi + \frac{1 - \xi^2}{1 - r^2} \cdot \frac{\alpha r}{\alpha + \sqrt{\alpha^2 r^2 + \pi^2}} \leq 1 \\ &\Leftrightarrow \xi(1 - r^2)\left(\alpha + \sqrt{\alpha^2 r^2 + \pi^2}\right) + (1 - \xi^2)(\alpha r) \leq (1 - r^2)\left(\alpha + \sqrt{\alpha^2 r^2 + \pi^2}\right) \\ &\Leftrightarrow (1 - \xi)\left((1 - r^2)\sqrt{\alpha^2 r^2 + \pi^2} - \alpha r(1 + \xi) - \alpha r^2\right) \geq 0 \\ &\Leftrightarrow (1 - \xi) L(\alpha, \xi, r) \geq 0, \end{aligned}$$

where

$$L(\alpha, \xi, r) := (1 - r^2)\sqrt{\alpha^2 r^2 + \pi^2} - \alpha r(1 + \xi) - \alpha r^2.$$

Since $1 - \xi \geq 0$ so, we are looking for those $r \in (0, 1)$ such that $L(\alpha, \xi, r) \geq 0$. It is easy to see that

$$\frac{\partial}{\partial \xi} L(\alpha, \xi, r) = -\alpha r < 0.$$

Thus, $L(\alpha,\xi,r)$ is a decreasing function with respect to ξ. Moreover, L gets its minimum at $\xi = 1$. Namely,

$$\min\{L(\alpha,\xi,r) : \xi \in [0,1]\} = L(\alpha,1,r) =: l(\alpha,r),$$

where

$$l(\alpha,r) := (1-r^2)\sqrt{\alpha^2r^2+\pi^2} - 2\alpha r - \alpha r^2.$$

A simple check gives $\lim_{r\to 0+} l(\alpha,r) = \pi > 0$ and $\lim_{r\to 1-} l(\alpha,r) = -3\alpha < 0$. We conclude that there exists at least one root, denoted by r_0, in the interval $(0,1)$ such that $l(\alpha,r) \geq 0$ for all $r \leq r_0$. Figure 3 gives more details about r_0. Therefore, the proof is complete. □

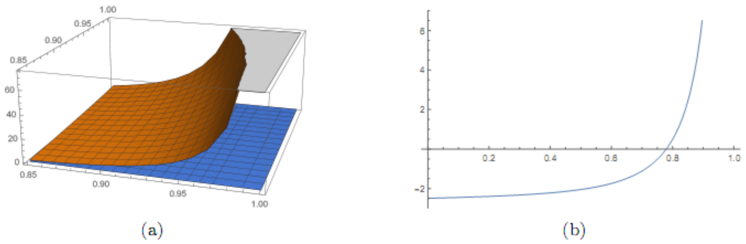

FIGURE 3. (a): The 3D plot of $l(\alpha,r)$, where $0 < \alpha, r < 1$, (b): The plot of $l(0.8,r)$, where $0 < r < 1$.

In both the realms of Teichmuller spaces and locally univalent functions, the pre-Schwarzian derivative, denoted as, $T_f(z) = f''(z)/f'(z)$, plays a pivotal role with numerous applications. For a locally univalent holomorphic function, we define the norm of T_f as follows:

$$\|f\| = \sup_{z\in\Delta}(1-|z|^2)|T_f|.$$

Furthermore, if f is univalent within the domain Δ, then it follows that $\|f\| \leq 6$. In the context of functions belonging to the class $\mathcal{A}$, we can assert that $\|f\| \leq 1$ implies that f belongs to the class $\mathcal{S}$. It's worth noting that these bounds are proven to be sharp [3]. Moving forward, we will delve into this problem for functions that are part of the class $\mathcal{S}^*_\psi(\alpha)$.

Theorem 2.8. *Let $\alpha \in (0,1]$. If a function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*_\psi(\alpha)$, then an upper bound for $\|f\|$ does not exist.*

Proof. Let $\alpha \in (0,1]$, and the function $f \in \mathcal{A}$ be given by (1) belong to the class $\mathcal{S}^*_\psi(\alpha)$. Then it satisfies (13). The logarithmic derivative of equation (13) can be derived by taking the derivative of both sides and simplifying the result, therefore

(20)
$$\frac{f''(z)}{f'(z)} = \frac{1}{\alpha z}\log(1+\alpha w(z)) + \frac{\alpha w'(z)}{(1+\alpha w(z))(\alpha + \log(1+\alpha w(z)))}, \quad (z \in \Delta).$$

By utilizing equation (20), applying the triangle inequality, and considering Lemma ??, it can be inferred that

$$\left|\frac{f''(z)}{f'(z)}\right| = \frac{1}{\alpha|z|}|\log(1+\alpha w(z))| + \frac{\alpha|w'(z)|}{(1-\alpha|w(z)|)(\alpha-|\log(1+\alpha w(z))|)}, \quad (z\in\Delta)$$

$$\leq \frac{\sqrt{\alpha^2|w(z)|^2+\pi^2}}{\alpha|z|} + \left(\frac{\alpha}{(1-\alpha|z|)(\alpha-\sqrt{\alpha^2|w(z)|^2+\pi^2})}\right)\frac{1-|w(z)|^2}{1-|z|^2}$$

$$\leq \frac{\sqrt{\alpha^2|z|^2+\pi^2}}{\alpha|z|} + \left(\frac{\alpha}{(1-\alpha|z|)(\alpha-\sqrt{\alpha^2|z|^2+\pi^2})}\right)\frac{1}{1-|z|^2}. \tag{21}$$

By multiplying equation (21) by $(1-|z|^2)$, we obtain

$$\|f\| \leq \frac{\sqrt{\alpha^2|z|^2+\pi^2}\,(1-|z|^2)}{\alpha|z|} + \left(\frac{\alpha}{(1-\alpha|z|)(\alpha-\sqrt{\alpha^2|z|^2+\pi^2})}\right). \tag{22}$$

If we allow $|z|$ to approach 1 from the left, then equation (22) results in

$$\|f\| \leq \frac{\alpha}{(1-\alpha)(\alpha-\sqrt{\alpha^2+\pi^2})}.$$

It is easy to see that $\alpha-\sqrt{\alpha^2+\pi^2}<0$, and therefore there is no upper bound for $\|f\|$. The proof now is complete. □

In the following, we will be dedicated to exploring certain coefficient-related inquiries concerning the function f within the class $\mathcal{S}^*_\psi(\alpha)$. The following lemma will be useful in our investigation.

Lemma 2.9. *[?, p. 172] Assume that w is a Schwarz function so that $w(z)=\sum_{n=1}^{\infty} w_n z^n$. Then*

$$|w_1|\leq 1 \quad \textit{and} \quad |w_n|\leq 1-|w_1|^2, \quad (n=2,3,\ldots).$$

Lemma 2.10. *[2, Lemma 1] If $w(z)=w_1z+w_2z^2+\cdots$ is a Schwarz function, then*

$$|w_2-tw_1^2|\leq \begin{cases} -t, & t\leq -1; \\ 1 & -1\leq t\leq 1; \\ t, & t\geq 1. \end{cases}$$

All inequalities are sharp.

The first result of this section is as follows:

Theorem 2.11. *Consider a function f in the class $\mathcal{A}$, which is expressed in the form (1), and assume that it belongs to the class $\mathcal{S}^*_\psi(\alpha)$ for all α in the interval $(0,1]$. Then*

$$|a_2|\leq 1 \quad \textit{and} \quad |a_3|\leq \frac{1}{2}\left(1+\frac{\alpha}{2}\right). \tag{23}$$

All inequalities are sharp.

Proof. Let $\alpha \in (0,1]$. If a function f belonging to the class $\mathcal{A}$ also falls within the class $\mathcal{S}^*_\psi(\alpha)$, then, as per the definition of subordination, there exists a Schwarz function $w(z) = w_1 z + w_2 z^2 + \cdots$ such that equation (13) is satisfied. Upon substituting the Taylor series expansions of both f and w into equation (13), we derive the following:

$$a_2 z + (2a_3 - a_2^2)z^2 + (3a_4 - 3a_2 a_3 + a_2^3)z^3 + \cdots = w_1 z + \left(w_2 - \frac{\alpha}{2} w_1^2\right) z^2 + \cdots .$$

Setting the coefficients of the respective terms in the last relation equal to each other results in

$$a_2 = w_1, \quad \text{and} \quad 2a_3 - a_2^2 = w_2 - \frac{\alpha}{2} w_1^2. \tag{24}$$

From the initial equality in (24) and, by referencing Lemma 2.9, we can conclude that $|a_2| = |w_1| \leq 1$. By utilizing Lemma (2.10), we obtain

$$a_3 = \frac{1}{2}\left(w_2 - \left(1 + \frac{\alpha}{2}\right) w_1^2\right) \Longrightarrow |a_3| \leq \frac{1}{2}\left(1 + \frac{\alpha}{2}\right),$$

which gives the second inequality of (23). □

Based on the following series expansion, the logarithmic coefficients γ_n of $f \in \mathcal{S}$ can be calculated:

$$\log \frac{f(z)}{z} = \sum_{n=1}^{\infty} \gamma_n(f) z^n, \quad (z \in \Delta). \tag{25}$$

In the theory of univalent functions, these coefficients are crucial for various estimates, and note that we utilize γ_n instead of $\gamma_n(f)$. Calculations give us

$$\begin{cases} 2\gamma_1 = a_2, \\ 2\gamma_2 = a_3 - \frac{1}{2} a_2^2, \\ 2\gamma_3 = a_4 - a_2 a_3 + \frac{1}{3} a_2^3. \end{cases} \tag{26}$$

For the first two logarithmic coefficients, we have sharp estimates:

$$|\gamma_1| \leq 1 \quad \text{and} \quad |\gamma_2| \leq \frac{1}{2}(1 + 2e^{-2}) \approx 0.635.$$

Despite this, the sharp estimate of $|\gamma_n|$ when $n \geq 3$ for $f \in \mathcal{S}$ is still open. Following, we estimate the initial logarithmic coefficients of $f \in \mathcal{S}^*_\psi(\alpha)$, where $\alpha > 1$.

Theorem 2.12. *Let the function f be of the form* (1) *and belong to the class $\mathcal{S}^*_\psi(\alpha)$, where $\alpha > 1$. Then the logarithmic coefficients of f satisfy*

$$|\gamma_1| \leq \frac{1}{2} \quad \textit{and} \quad |\gamma_2| \leq \frac{1}{4}\left(2 + \frac{\alpha}{2}\right).$$

Both inequalities are sharp.

Proof. The first inequality follows from Theorem 2.11. With a simple calculation, we obtain,

$$\gamma_2 = \frac{1}{4}\left(w_2 - \left(2 + \frac{\alpha}{2}\right) w_1^2\right).$$

The desired inequality follows now from Lemma 2.10. □

3. Conclusion

Here, in our present investigation, we have introduced and investigated various interesting properties of some new subclasses of analytic and univalent functions associated with Ma-Minda class functions defined in the open unit disk. We have settled the radii properties and majorization structure. Studies of continue to motivate researchers in Geometric Function Theory on various subclasses as well as the ongoing usages of the operators in the study of other meromorphic or multivalent functions classes.

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