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VANISHING AND LOCALIZATION OF $(d,\mathfrak{b})$-IDEAL TRANSFORMS

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Abstract. Let R be a commutative Noetherian ring, M an R-module and d a non-negative integer. Let Σ denote the set of ideals $\mathfrak{J}$ of R such that $\dim(R/\mathfrak{J}) \leq d$. For an ideal $\mathfrak{b}$ of R, we define the $(d,\imath\mathfrak{b})$-transform $D_{d,\imath\imath\mathfrak{b}}(M)$ and study its properties. Then a criterion for $D_{d,\mathfrak{b}}(R) = \bigcap_{\mathfrak{p}\notin W(d,\mathfrak{b})} R_{\mathfrak{p}}$ will be given, where $W(d,\mathfrak{b})$ contains all ideals $\mathfrak{a}$ of R such that $\mathfrak{J} \subseteq \mathfrak{a}+\mathfrak{b}$ for some $\mathfrak{J} \in \Sigma$. For each $i \geq 0$, let $D^i_{d,\mathfrak{b}}(-)$ denote the i-th right derived functor of $D_{d,\mathfrak{b}}(M)$. We study the localization of the module $D^i_{d,\mathfrak{b}}(M)$ and prove that $D^i_{d,\mathfrak{b}}(M)_{\mathfrak{p}} \cong D^i_{d-\dim(R/(\mathfrak{p}+\mathfrak{b})),\mathfrak{b}_{\mathfrak{p}}}(M_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ and all $i \geq 0$. Finally, we establish vanishing theorems for $D^i_{d,\mathfrak{b}}(M)$.

Keywords: local cohomology, ideal transforms, finitely generated, localization, associated prime.
2020 MSC: 13C05, 13C12, 13D45, 16D70.

1. Introduction

In this paper, we assume that R is a commutative Noetherian ring with identity and d is a non-negative integer. Let $\mathcal{I}(R)$ be the set of all ideals of R, and let $\mathfrak{b} \in \mathcal{I}(R)$. The study of vanishing and localization is a significant topic in local cohomology, as discussed in [2]. In [1], the authors introduced a specific type of local cohomology and examined its vanishing and localization under certain conditions. Further extensive investigations into this concept can be found in [6] and [7]. Here, I introduce a generalization of these specific modules and explore their key properties, including vanishing and localization aspects. Put $\Sigma := \{\mathfrak{J} \in \mathcal{I}(R) | \dim(R/\mathfrak{J}) \leq d\}$, $\tilde{W}(d,\mathfrak{b}) := \{\mathfrak{a} \in \mathcal{I}(R) : \exists \mathfrak{J} \in \Sigma, \mathfrak{J} \subseteq \mathfrak{a}+\mathfrak{b}\}$ and $W(d,\mathfrak{b}) := \tilde{W}(d,\mathfrak{b}) \cap \operatorname{Spec}(R)$. Then, with the reverse inclusion, both Σ and $\tilde{W}(d,\mathfrak{b})$ are systems of ideals in R in the sense of [2, p. 21]. For an R-module M, $L_d(M) = \{x \in M | \exists \mathfrak{J} \in \Sigma; \mathfrak{J}x = 0\}$ and $H^i_d(-)$ is the i-th right derived functor of $L_d(-)$ which was introduced in [1]. Also, we denote by $\Gamma_{d,\mathfrak{b}}(M)$ the set of elements $x \in M$ such that $\mathfrak{a}x \subseteq \mathfrak{b}x$ for some $\mathfrak{a} \in \Sigma$. Then, $\Gamma_{d,\mathfrak{b}}(M)$ is a submodule of M and $\Gamma_{d,\mathfrak{b}}(-)$ constitutes an additive left exact functor on the category of R-modules. Moreover, $x \in \Gamma_{d,\mathfrak{b}}(M)$ if and

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only if $\mathfrak{J} \subseteq (0 :_R x) + \mathfrak{b}$ for some $\mathfrak{J} \in \Sigma$. Also, $L_d(M) = \Gamma_{d,0}(M)$. Clearly, $\Gamma_{d,\mathfrak{b}}(M)$ is a submodule of M, called the $(d, \mathfrak{b})$-torsion submodule of M and we say that M is $(d, \mathfrak{b})$-torsion (resp. $(d, \mathfrak{b})$-torsion-free) if $\Gamma_{d,\mathfrak{b}}(M) = M$ (resp. $\Gamma_{d,\mathfrak{b}}(M) = 0$). For a homomorphism $f : M \to N$ of R-modules, we have $f(\Gamma_{d,\mathfrak{b}}(M)) \subseteq \Gamma_{d,\mathfrak{b}}(N)$, and $\Gamma_{d,\mathfrak{b}}(-)$ is an additive left exact functor on the category of R-modules. Now, for an integer i, the i-th right derived functor of $\Gamma_{d,\mathfrak{b}}(-)$ is denoted by $H^i_{d,\mathfrak{b}}(-)$, and for an R-module M, $H^i_{d,\mathfrak{b}}(M)$ is the i-th local cohomology module of M with respect to $(d, \mathfrak{b})$. Analogous to these objects are the i-th $(d, \mathfrak{b})$-transforms $D^i_{d,\mathfrak{b}}(-)$, so that for each $i \geq 0$, $D^i_{d,\mathfrak{b}}(-)$ is the i-th right derived functor of $D_{d,\mathfrak{b}}(-) = \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} \operatorname{Hom}_R(\mathfrak{a}, -)$. After collecting some elementary results in the next section and in Section 3, we present some properties of the right derived functors of $\Gamma_{d,\mathfrak{b}}(-)$ and $D_{d,\mathfrak{b}}(-)$, denoted by $H^i_{d,\mathfrak{b}}(-)$ and $D^i_{d,\mathfrak{b}}(-)$, respectively, where the special case ($\mathfrak{b} = 0$ and M is a finitely generated module) was examined in [7]. Also, we prove that $D_{d,\mathfrak{b}}(R) = \cap_{\mathfrak{p}\notin W(d,\mathfrak{b})} R_\mathfrak{p}$, where R is a domain.
In Section 4, we continue our study and see that when R is a catenary and biequidimensional ring and M is a finitely generated R-module, then for each prime ideal $\mathfrak{p}$ and each $i \geq 0$, we have $H^i_{d,\mathfrak{b}}(M)_\mathfrak{p} \cong H^i_{d-\dim(R/(\mathfrak{p}+\mathfrak{b})),\mathfrak{b}_\mathfrak{p}}(M_\mathfrak{p})$ and $D^i_{d,\mathfrak{b}}(M)_\mathfrak{p} \cong D^i_{d-\dim(R/(\mathfrak{p}+\mathfrak{b})),\mathfrak{b}_\mathfrak{p}}(M_\mathfrak{p})$. In Section 5, we present some conditions for the vanishing of $D^i_{d,\mathfrak{b}}(M)$ for all $i \geq 0$.

2. Preliminaries

Definition 2.1. For any R-module M, we define

$$D_{d,\mathfrak{b}}(M) = \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} \operatorname{Hom}(\mathfrak{a}, M).$$

The functor $D_{d,\mathfrak{b}}(-)$ is an additive, covariant, R-linear functor, and it is also left exact.

Using a similar argument as in [2, Theorem 2.2.6], one can prove the following lemma.

Lemma 2.2. *Let M be an R-module. Then the sequence*

$$0 \to \Gamma_{d,\mathfrak{b}}(M) \to M \to D_{d,\mathfrak{b}}(M) \to H^1_{d,\mathfrak{b}}(M) \to 0$$

is exact. Moreover, $D^i_{d,\mathfrak{b}}(M) \cong H^{i+1}_{d,\mathfrak{b}}(M)$ for all $i \geq 1$.

Theorem 2.3. *If E is an injective R-module, then $\Gamma_{d,\mathfrak{b}}(E)$ is also an injective R-module.*

Proof. Similar to [6, Lemma 2.1(6)], it is easy to see that $\Gamma_{d,\mathfrak{b}}(E) \cong \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} \Gamma_\mathfrak{a}(E)$. On the other hand, by [2, Proposition 2.1.4], the module $\Gamma_\mathfrak{a}(E)$ is injective for each $\mathfrak{a} \in \mathcal{I}(R)$. Now, since R is Noetherian, the result follows from [4, Exercise 5.23 (p.255)]. □

Corollary 2.4. *If E is an injective R-module, then $D_{d,\mathfrak{b}}(E)$ is also an injective R-module.*

Proof. Since E is an injective module, $H^1_{d,\mathfrak{b}}(E) = \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Ext}^1_R(R/\mathfrak{a}, E) = 0$. Now, the exact sequence

$$0 \longrightarrow \Gamma_{d,\mathfrak{b}}(E) \longrightarrow E \longrightarrow D_{d,\mathfrak{b}}(E) \longrightarrow 0$$

implies that $D_{d,\mathfrak{b}}(E)$ is also an injective module. □

Corollary 2.5. *Let M be an R-module. If $\Gamma_{d,\mathfrak{b}}(M) = M$, then $H^i_{d,\mathfrak{b}}(M) = 0$ for all $i \geq 1$.*

Proof. Using Theorem 2.3, we construct an injective resolution

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \cdots$$

of M such that $\Gamma_{d,\mathfrak{b}}(E^i) = E^i$ for all $i \geq 0$. We have

$$H^i_{d,\mathfrak{b}}(M) = \frac{\ker(E^i \to E^{i+1})}{\mathrm{Im}(E^{i-1} \to E^i)} = 0,$$

for all $i \geq 1$. □

Proposition 2.6. *Let M be an R-module. Then the following statements hold:*
(1) If $\Gamma_{d,\mathfrak{b}}(M) = M$, then $D^i_{d,\mathfrak{b}}(M) = 0$ for all $i \geq 0$. Moreover, for any R-module X, $D^i_{d,\mathfrak{b}}(H^j_{d,\mathfrak{b}}(X)) = 0$ for all $i \geq 0$ and all $j \geq 0$.
(2) $D_{d,\mathfrak{b}}(M) \cong D_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M))$.
(3) $D_{d,\mathfrak{b}}(M) \cong D_{d,\mathfrak{b}}(D_{d,\mathfrak{b}}(M))$.
(4) $H^i_{d,\mathfrak{b}}(D_{d,\mathfrak{b}}(M)) = 0$ for $i = 0, 1$.
(5) For each $i \geq 2$, $H^i_{d,\mathfrak{b}}(M) \cong H^i_{d,\mathfrak{b}}(D_{d,\mathfrak{b}}(M))$.
(6) $D_{d,\mathfrak{b}}(M) \cong \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} D_{\mathfrak{a}}(M)$. Moreover,

$$D^i_{d,\mathfrak{b}}(M) \cong \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} D^i_{\mathfrak{a}}(M) \cong \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} H^{i+1}_{\mathfrak{a}}(M)$$

for all $i \geq 1$.
(7) $D_{d,\mathfrak{b}}(M) \cong \varinjlim_{\mathfrak{a}\in\Sigma} D_{\mathfrak{a},\mathfrak{b}}(M)$, where $D_{\mathfrak{a},\mathfrak{b}}(M) = \varinjlim_{\mathfrak{c}\in\tilde{W}(\mathfrak{a},\mathfrak{b})} D_{\mathfrak{c}}(M)$. Moreover,

$$D^i_{d,\mathfrak{b}}(M) \cong \varinjlim_{\mathfrak{a}\in\Sigma} \varinjlim_{\mathfrak{c}\in\tilde{W}(\mathfrak{a},\mathfrak{b})} D^i_{\mathfrak{c}}(M) \cong \varinjlim_{\mathfrak{a}\in\Sigma} \varinjlim_{\mathfrak{c}\in\tilde{W}(\mathfrak{a},\mathfrak{b})} H^{i+1}_{\mathfrak{c}}(M)$$

for all $i \geq 1$. Note that $\tilde{W}(\mathfrak{a},\mathfrak{b}) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \exists n \in \mathbb{N}; \mathfrak{a}^n \subseteq \mathfrak{p} + \mathfrak{b}\}$, where $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}(R)$ (see [5, p. 582]).
(8) If $L_d(M) = M$, then $D^i_{d,\mathfrak{b}}(M) = 0$ for all $i \geq 0$. Moreover, for any R-module X, $D^i_{d,\mathfrak{b}}(H^j_d(X)) = 0$ for all $i \geq 0$ and all $j \geq 0$.
(9) If M is a $\mathfrak{b}$-torsion module, then $D^i_{d,\mathfrak{b}}(M) = 0$ for all $i \geq 0$. Moreover, for any R-module X, $D^i_{d,\mathfrak{b}}(H^j_{\mathfrak{a}}(X)) = 0$ for all $i \geq 0$ and all $\mathfrak{a} \in \mathcal{I}(R)$.
(10) If $\Gamma_{d,\mathfrak{b}}(M) \subseteq \Gamma_{\mathfrak{b}}(M)$, then $D^i_{d,\mathfrak{b}}(M) = D^i_d(M)$ for all $i \geq 0$.

(11) If $\mathfrak{a} \in \tilde{W}(d, \mathfrak{b})$, then $\operatorname{Hom}_R(R/\mathfrak{a}, D_{d,\mathfrak{b}}(M)) = 0$.

Proof. (1) By Corollary 2.5, $H^1_{d,\mathfrak{b}}(M) = 0$. By Lemma 2.2, we obtain $D_{d,\mathfrak{b}}(M) = 0$. By Lemma 2.2 and Corollary 2.5, we have $D^i_{d,\mathfrak{b}}(M) \cong H^{i+1}_{d,\mathfrak{b}}(M) = 0$ for all $i \geq 1$. The second statement follows because $\Gamma_{d,\mathfrak{b}}(H^i_{d,\mathfrak{b}}(M)) = H^i_{d,\mathfrak{b}}(M)$ for all $i \geq 0$.
(2) From the short exact sequence

$$0 \longrightarrow \Gamma_{d,\mathfrak{b}}(M) \longrightarrow M \longrightarrow M/\Gamma_{d,\mathfrak{b}}(M) \longrightarrow 0 \qquad (\sharp)$$

we conclude that the sequence

$$0 \longrightarrow D_{d,\mathfrak{b}}(\Gamma_{d,\mathfrak{b}}(M)) \longrightarrow D_{d,\mathfrak{b}}(M) \longrightarrow D_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M)) \longrightarrow D^1_{d,\mathfrak{b}}(\Gamma_{d,\mathfrak{b}}(M))$$

is exact. Since by part (1), $D_{d,\mathfrak{b}}(\Gamma_{d,\mathfrak{b}}(M)) = D^1_{d,\mathfrak{b}}(\Gamma_{d,\mathfrak{b}}(M)) = 0$, it follows that $D_{d,\mathfrak{b}}(M) \cong D_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M))$.
(3) By Lemma 2.2,

$$0 \to M/\Gamma_{d,\mathfrak{b}}(M) \to D_{d,\mathfrak{b}}(M) \to H^1_{d,\mathfrak{b}}(M) \to 0$$

is a short exact sequence. This gives rise to the following exact sequence

$$0 \longrightarrow D_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M)) \longrightarrow D_{d,\mathfrak{b}}(D_{d,\mathfrak{b}}(M)) \longrightarrow D_{d,\mathfrak{b}}(H^1_{d,\mathfrak{b}}(M)).$$

By parts (1) and (2), we obtain

$$D_{d,\mathfrak{b}}(D_{d,\mathfrak{b}}(M)) \cong D_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M)) \cong D_{d,\mathfrak{b}}(M).$$

(4) By Lemma 2.2, we obtain the exact sequence

$$0 \to \Gamma_{d,\mathfrak{b}}(D_{d,\mathfrak{b}}(M)) \to D_{d,\mathfrak{b}}(M) \to D_{d,\mathfrak{b}}(D_{d,\mathfrak{b}}(M)) \to H^1_{d,\mathfrak{b}}(D_{d,\mathfrak{b}}(M)) \to 0.$$

Thus, by part (3), the claim follows.
(5) Applying $H^i_{d,\mathfrak{b}}$ to exact sequence $(\sharp)$, we obtain the exact sequence

$$H^i_{d,\mathfrak{b}}(\Gamma_{d,\mathfrak{b}}(M)) \longrightarrow H^i_{d,\mathfrak{b}}(M) \longrightarrow H^i_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M)) \longrightarrow H^{i+1}_{d,\mathfrak{b}}(\Gamma_{d,\mathfrak{b}}(M))$$

for all $i \geq 0$. Hence, by Corollary 2.5, $H^i_{d,\mathfrak{b}}(M) \cong H^i_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M))$ for all $i \geq 1$. Now, applying $H^i_{d,\mathfrak{b}}$ to the short exact sequence

$$0 \to M/\Gamma_{d,\mathfrak{b}}(M) \to D_{d,\mathfrak{b}}(M) \to H^1_{d,\mathfrak{b}}(M) \to 0$$

the proof is complete, since by Corollary 2.5, we have $H^i_{d,\mathfrak{b}}(H^1_{d,\mathfrak{b}}(M)) = 0$ for all $i \geq 1$.
(6) Similar to [6, Lemma 2.1(6)], we have $H^i_{d,\mathfrak{b}}(M) \cong \varinjlim_{\mathfrak{a} \in \tilde{W}(d,\mathfrak{b})} H^i_{\mathfrak{a}}(M)$ for

all $i \geq 0$. Now, consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \longrightarrow & \Gamma_{d,\mathfrak{b}}(M) & \longrightarrow & M & \longrightarrow & D_{d,\mathfrak{b}}(M) & \longrightarrow & H^1_{d,\mathfrak{b}}(M) & \longrightarrow 0 \\ & \downarrow\cong & & \downarrow = & & \downarrow & & \downarrow\cong & \\ 0 \to & \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} \Gamma_{\mathfrak{a}}(M) & \to & M & \to & \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} D_{\mathfrak{a}}(M) & \to & \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} H^1_{\mathfrak{a}}(M) & \to 0. \end{array}$$

By the Five lemma, the result follows. The second part is obtained from [2, Theorem 1.3.5].

(7) First, for an R-module X, we show that $H^i_{d,\mathfrak{b}}(X) \cong \varinjlim_{\mathfrak{a}\in\Sigma} H^i_{\mathfrak{a},\mathfrak{b}}(X)$, where $H^i_{\mathfrak{a},\mathfrak{b}}(X) \cong \varinjlim_{\mathfrak{c}\in\tilde{W}(\mathfrak{a},\mathfrak{b})} H^i_{\mathfrak{c}}(X)$, which was proved in [5, Theorem 3.2]. To establish this, we begin by proving that $\Gamma_{d,\mathfrak{b}}(X) \cong \varinjlim_{\mathfrak{a}\in\Sigma} \Gamma_{\mathfrak{a},\mathfrak{b}}(X)$. Since Σ is a system of ideals, it is enough to show that $\Gamma_{d,\mathfrak{b}}(X) = \cup_{\mathfrak{a}\in\Sigma}\Gamma_{\mathfrak{a},\mathfrak{b}}(X)$. Let $x \in \Gamma_{d,\mathfrak{b}}(X)$. Then there exists $\mathfrak{a} \in \Sigma$ such that $\mathfrak{a}x \subseteq \mathfrak{b}x$. Hence $x \in \Gamma_{\mathfrak{a},\mathfrak{b}}(X)$, which implies $x \in \cup_{\mathfrak{a}\in\Sigma}\Gamma_{\mathfrak{a},\mathfrak{b}}(X)$. For the inverse inclusion, we note that for each $\mathfrak{a} \in \Sigma$ and each $i \in \mathbb{N}_0$, we have $\mathfrak{a}^i \in \Sigma$. Now, since both $H^i_{d,\mathfrak{b}}(-)$ and $\varinjlim_{\mathfrak{a}\in\Sigma} H^i_{\mathfrak{a},\mathfrak{b}}(-)$, $i \in \mathbb{N}_0$, are strongly connected, it follows from [2, Theorem 1.3.5] that the result holds.

Again, using the commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \Gamma_{d,\mathfrak{b}}(M) & \longrightarrow & M & \longrightarrow & D_{d,\mathfrak{b}}(M) & \longrightarrow & H^1_{d,\mathfrak{b}}(M) & \longrightarrow 0 \\ & \downarrow\cong & & \downarrow = & & \downarrow & & \downarrow\cong & \\ 0 \to & \varinjlim_{\mathfrak{a}\in\Sigma} \Gamma_{\mathfrak{a},\mathfrak{b}}(M) & \to & M & \to & \varinjlim_{\mathfrak{a}\in\Sigma} D_{\mathfrak{a},\mathfrak{b}}(M) & \to & \varinjlim_{\mathfrak{a}\in\Sigma} H^1_{\mathfrak{a},\mathfrak{b}}(M) & \to 0 \end{array}$$

with exact rows, we apply the Five lemma to conclude the proof of the first part. The second part then follows directly from [2, Theorem 1.3.5].

(8) Clearly, $L_d(M) \subseteq \Gamma_{d,\mathfrak{b}}(M)$. Also $\Gamma_{d,\mathfrak{b}}(M) \subseteq M = L_d(M)$. Thus, $\Gamma_{d,\mathfrak{b}}(M) = M$ and so by part (1), the result follows. The next part follows directly from the fact that $L_d(H^j_d(M)) = H^j_d(M)$ for all $j \geq 0$.

(9) Clearly, $L_d(M) \subseteq \Gamma_{d,\mathfrak{b}}(M)$. Now, let $x \in \Gamma_{d,\mathfrak{b}}(M)$. Hence there exists $\mathfrak{a} \in \Sigma$ such that $\mathfrak{a}x \subseteq \mathfrak{b}x$. On the other hand, since $x \in M = \Gamma_{\mathfrak{b}}(M)$, there exists $t \in \mathbb{N}_0$ such that $\mathfrak{b}^t x = 0$. Consequently, $\mathfrak{a}^t x = 0$, and since $\mathfrak{a}^t \in \Sigma$, it follows that $x \in L_d(M)$. Hence, we have $L_d(M) = \Gamma_{d,\mathfrak{b}}(M)$ and the proof is completed similarly to part (8). This result also follows from the fact that $\Gamma_{\mathfrak{b}}(H^j_{\mathfrak{b}}(M)) = H^j_{\mathfrak{b}}(M)$ for all $j \geq 0$.

(10) It is easy to see that $\Gamma_{d,\mathfrak{b}}(M) = L_d(M)$. Then by [2, Theorem 1.3.5], $H^i_{d,\mathfrak{b}}(M) = H^i_d(M)$ for all $i \geq 0$. Now, consider the commutative diagram of

R-modules and R-homomorphisms with exact rows

$$\begin{array}{ccccccccc} 0 \longrightarrow & \Gamma_{d,\mathfrak{b}}(M) & \longrightarrow & M & \longrightarrow & D_{d,\mathfrak{b}}(M) & \longrightarrow & H^1_{d,\mathfrak{b}}(M) & \longrightarrow 0 \\ & \big\downarrow{=} & & \big\downarrow{=} & & \big\downarrow & & \big\downarrow{\cong} & \\ 0 \longrightarrow & L_d(M) & \longrightarrow & M & \longrightarrow & T_d(M) & \longrightarrow & H^1_d(M) & \longrightarrow 0. \end{array}$$

Applying the Five Lemma, we obtain $D_{d,\mathfrak{b}}(M) \cong D_d(M)$. Finally, by reapplying [2, Theorem 1.3.5], the result follows.
(11) Let E be an injective R-module. Then $H^1_{d,\mathfrak{b}}(E) = 0$ and so by Lemma 2.2, $E/\Gamma_{d,\mathfrak{b}}(E) \cong D_{d,\mathfrak{b}}(E)$. Consequently, for all $\mathfrak{a} \in \tilde{W}(d,\mathfrak{b})$, we have

$$\mathrm{Hom}_R(R/\mathfrak{a}, E/\Gamma_{d,\mathfrak{b}}(E)) \cong \mathrm{Hom}_R(R/\mathfrak{a}, D_{d,\mathfrak{b}}(E)).$$

Now, let $\mathfrak{I} \in \Sigma$ such that $\mathfrak{I} \subseteq \mathfrak{a} + \mathfrak{b}$, and let $f \in \mathrm{Hom}_R(R/\mathfrak{a}, E/\Gamma_{d,\mathfrak{b}}(E))$. Then $f(1+\mathfrak{a}) \in E/\Gamma_{d,\mathfrak{b}}(E)$, and so there exists $e \in E$ such that $f(1+\mathfrak{a}) = e+\Gamma_{d,\mathfrak{b}}(E)$. This implies that $\mathfrak{a}e \subseteq \Gamma_{d,\mathfrak{b}}(E)$, and consequently, there exists $\mathfrak{I}' \in \Sigma$ such that $\mathfrak{I}'\mathfrak{a}e \subseteq \mathfrak{b}\mathfrak{a}e$. Clearly $\mathfrak{I}\mathfrak{I}' \in \Sigma$. Now, since $\mathfrak{I} \subseteq \mathfrak{a} + \mathfrak{b}$, we have

$$\mathfrak{I}\mathfrak{I}'e \subseteq (\mathfrak{a} + \mathfrak{b})\mathfrak{I}'e \subseteq \mathfrak{a}\mathfrak{I}'e + \mathfrak{b}\mathfrak{I}'e \subseteq \mathfrak{b}\mathfrak{a}e + \mathfrak{b}\mathfrak{I}'e \subseteq \mathfrak{b}e.$$

Hence $\mathfrak{I}\mathfrak{I}'e \subseteq \mathfrak{b}e$, which implies that $e \in \Gamma_{d,\mathfrak{b}}(E)$, and thus $f(1+\mathfrak{a}) = 0$. It follows that $\mathrm{Hom}_R(R/\mathfrak{a}, E/\Gamma_{d,\mathfrak{b}}(E)) = 0$, and consequently $\mathrm{Hom}_R(R/\mathfrak{a}, D_{d,\mathfrak{b}}(E)) = 0$. □

3. $\mathbf{B_{d,\mathfrak{b}}(M)}$ and its results on $\mathbf{D_{d,\mathfrak{b}}(M)}$

Definition 3.1. Let M be an R-module, and let $S = R \setminus Z_R(M)$. For $\mathfrak{a} \in \tilde{W}(d,\mathfrak{b})$, it is clear that $\{(M :_{S^{-1}M} \mathfrak{a}), \varphi^M_{\mathfrak{a},\mathfrak{c}}\}_{\mathfrak{a},\mathfrak{c}\in\tilde{W}(d,\mathfrak{b})}$ is a direct system of R-modules indexed by $\tilde{W}(d,\mathfrak{b})$, where $\mathfrak{c} \subseteq \mathfrak{a}$ and the morphisms $\varphi^M_{\mathfrak{a},\mathfrak{c}} : (M :_{S^{-1}M} \mathfrak{a}) \to (M :_{S^{-1}M} \mathfrak{c})$ are the natural inclusion maps. We define the R-module $B_{d,\mathfrak{b}}(M)$ as follows:

$$B_{d,\mathfrak{b}}(M) := \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} (M :_{S^{-1}M} \mathfrak{a}).$$

For each $\mathfrak{a} \in \tilde{W}(d,\mathfrak{b})$, let $\varphi^M_{\mathfrak{a}} : (M :_{S^{-1}M} \mathfrak{a}) \to B_{d,\mathfrak{b}}(M)$ denote the canonical map.

Theorem 3.2. *Let M be an R-module. Then*
(1) *If $\Gamma_{d,\mathfrak{b}}(M) = 0$, then $B_{d,\mathfrak{b}}(M) \cong D_{d,\mathfrak{b}}(M)$.*
(2) *$B_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M)) \cong D_{d,\mathfrak{b}}(M)$.*

Proof. (1) Let $\mathfrak{a} \in \tilde{W}(d,\mathfrak{b})$. Then there exists $\mathfrak{I} \in \Sigma$ such that $\mathfrak{I} \subseteq \mathfrak{a} + \mathfrak{b}$. We show that the map $\psi_{\mathfrak{a}} : (M :_{S^{-1}M} \mathfrak{a}) \to \mathrm{Hom}_R(\mathfrak{a}, M)$ defined by $\psi_{\mathfrak{a}}(x)(a) = ax$ for all $x \in (M :_{S^{-1}M} \mathfrak{a})$ and $a \in \mathfrak{a}$, is an R-isomorphism. Clearly, $\psi_{\mathfrak{a}}$ is an R-homomorphism. Now, suppose $\psi_{\mathfrak{a}}(m/s) = 0$, for some $m/s \in (M :_{S^{-1}M} \mathfrak{a})$. This means that $a \cdot m/s = 0$ for all $a \in \mathfrak{a}$. Equivalently, $\mathfrak{a}m = 0$ for some $m \in M$, which implies that $\mathfrak{I}m \subseteq \mathfrak{b}m$. Therefore $m \in \Gamma_{d,\mathfrak{b}}(M) = 0$, which implies that $m=0$. Thus $\mathfrak{a}\nsubseteq Z_R(M)$. Let $y\in\mathfrak{a}\setminus Z_R(M)$. Then $y\cdot m/s=0$ and so $ym/s=0$. Hence there exists $u\in R\setminus Z_R(M)$ such that $uym=0$. Since $uy\in R\setminus Z_R(M)$, it follows that $m=0$, and therefore $m/s=0$. Thus, $\psi_{\mathfrak{a}}$ is an R-monomorphism. Now, it is enough to prove that $\psi_{\mathfrak{a}}$ is an R-epimorphism. Let $f\in\operatorname{Hom}_R(\mathfrak{a},M)$. Since $y\in\mathfrak{a}\setminus Z_R(M)$, we have $f(y)/y\in S^{-1}M$. We show that $x\cdot f(y)/y\in M$ for all $x\in\mathfrak{a}$. We have

$$x\cdot f(y)/y=f(xy)/y=yf(x)/y=f(x)/1\in M.$$

This implies that $a\cdot f(y)/y\subseteq M$, which leads to $f(y)/y\in(M:_{S^{-1}M}\mathfrak{a})$. Therefore $\psi_{\mathfrak{a}}(f(y)/y)=f$. It follows that $\psi_{\mathfrak{a}}$ is an R-epimorphism.
(2) It is straightforward to verify that $\Gamma_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M))=0$. Thus, by part (1), we have $B_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M))\cong D_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M))$, and by Proposition 2.6(2), $B_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M))\cong D_{d,\mathfrak{b}}(M)$. □

Theorem 3.3. *Let M be an R-module. Then* $\operatorname{Ass}(D_{d,\mathfrak{b}}(M))=\operatorname{Ass}(M/\Gamma_{d,\mathfrak{b}}(M))$.

Proof. From the short exact sequence

$$0\to M/\Gamma_{d,\mathfrak{b}}(M)\to D_{d,\mathfrak{b}}(M)\to H^1_{d,\mathfrak{b}}(M)\to 0,$$

we conclude that $\operatorname{Ass}(M/\Gamma_{d,\mathfrak{b}}(M))\subseteq\operatorname{Ass}(D_{d,\mathfrak{b}}(M))$. For the inverse inclusion, by Theorem 3.2(1), $B_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M))\cong D_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M))$ and so by Proposition 2.6(2), $B_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M))\cong D_{d,\mathfrak{b}}(M)$. Hence

$$\operatorname{Ass}(B_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M)))=\operatorname{Ass}(D_{d,\mathfrak{b}}(M)).$$

Let $\mathfrak{p}\in\operatorname{Ass}(D_{d,\mathfrak{b}}(M))$ and set $N=M/\Gamma_{d,\mathfrak{b}}(M)$. Then $\mathfrak{p}\in\operatorname{Ass}(B_{d,\mathfrak{b}}(N))$. Thus there exists $0\neq x\in B_{d,\mathfrak{b}}(N)$ such that $\mathfrak{p}=(0:_R x)$. Hence, there exist $\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})$, $\varphi_{\mathfrak{a}}\in\operatorname{Hom}_R((N:_{S^{-1}N}\mathfrak{a}),B_{d,\mathfrak{b}}(N))$, and $y\in(N:_{S^{-1}N}\mathfrak{a})$ such that $x=\varphi_{\mathfrak{a}}(y)$, where $S=R\setminus Z_R(N)$ and $\{(N:_{S^{-1}N}\mathfrak{a}),\varphi_{\mathfrak{a}}\}_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})}$ forms the desired direct system. Clearly $\mathfrak{a}y\subseteq N$. Since $\mathfrak{p}x=0$, we have $\mathfrak{p}\cdot\varphi_{\mathfrak{a}}(y)=\varphi_{\mathfrak{a}}(\mathfrak{p}y)=0$. Then there exists $\mathfrak{c}\in\tilde{W}(d,\mathfrak{b})$ such that $\mathfrak{c}\subseteq\mathfrak{a}$ and $\varphi^{\mathfrak{a}}_{\mathfrak{c}}(\mathfrak{p}y)=0$. Since $\varphi^{\mathfrak{a}}_{\mathfrak{c}}$ is the inclusion map, it follows that $\mathfrak{p}y=0$ and thus $\mathfrak{p}=(0:_R y)$. Moreover, we have $\Gamma_{d,\mathfrak{b}}(S^{-1}N)=0$, because $\Gamma_{d,\mathfrak{b}}(N)=\Gamma_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M))=0$. Now, if $\mathfrak{p}\in\tilde{W}(d,\mathfrak{b})$, then there is $\mathfrak{I}\in\Sigma$ such that $\mathfrak{I}\subseteq\mathfrak{p}+\mathfrak{b}$. Since $\mathfrak{p}y=0$, we obtain $\mathfrak{I}y\subseteq\mathfrak{b}y$, which implies $y\in\Gamma_{d,\mathfrak{b}}(S^{-1}N)=0$. Therefore $\mathfrak{p}=(0:_R y)=R$, which is a contradiction. Thus $p\notin\tilde{W}(d,\mathfrak{b})$. Furthermore, $\mathfrak{a}\nsubseteq\mathfrak{p}$, because otherwise we would have $\mathfrak{a}y=0$. On the other hand $\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})$, then there exists $\mathfrak{d}\in\Sigma$ such that $\mathfrak{d}\subset\mathfrak{a}+\mathfrak{b}$. This implies $\mathfrak{d}y\subseteq\mathfrak{b}y$. Thus $y\in\Gamma_{d,\mathfrak{b}}(S^{-1}N)=0$, which is also a contradiction. Now, let $a\in\mathfrak{a}\setminus\mathfrak{p}$. Then $\mathfrak{p}=(0:_R ay)$. Hence $\mathfrak{p}\in\operatorname{Ass}(N)$. Thus, $\operatorname{Ass}(D_{d,\mathfrak{b}}(M))\subseteq\operatorname{Ass}(N)$, completing the proof. □

Corollary 3.4. *Let M be an R-module. Then*

$$\operatorname{Supp}(D_{d,\mathfrak{b}}(M))=\operatorname{Supp}(M/\Gamma_{d,\mathfrak{b}}(M)).$$

Proof. From the short exact sequence

$$0 \to M/\Gamma_{d,\mathfrak{b}}(M) \to D_{d,\mathfrak{b}}(M) \to H^1_{d,\mathfrak{b}}(M) \to 0,$$

we conclude that $\mathrm{Supp}(D_{d,\mathfrak{b}}(M)) = \mathrm{Supp}(M/\Gamma_{d,\mathfrak{b}}(M)) \cup \mathrm{Supp}(H^1_{d,\mathfrak{b}}(M))$. Then

$$\mathrm{Supp}(M/\Gamma_{d,\mathfrak{b}}(M)) \subseteq \mathrm{Supp}(D_{d,\mathfrak{b}}(M)).$$

Now, let $\mathfrak{p} \in \mathrm{Supp}(D_{d,\mathfrak{b}}(M))$. Then there exists $\mathfrak{q} \in \mathrm{Ass}(D_{d,\mathfrak{b}}(M))$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. Thus, by Theorem 3.3, $\mathfrak{q} \in \mathrm{Ass}(M/\Gamma_{d,\mathfrak{b}}(M))$ and so there exists $m \in M \backslash \Gamma_{d,\mathfrak{b}}(M)$ such that $\mathfrak{q} = (0 :_R m + \Gamma_{d,\mathfrak{b}}(M))$. Now, $0 \neq (m + \Gamma_{d,\mathfrak{b}}(M))/1 \in (M/\Gamma_{d,\mathfrak{b}}(M))_{\mathfrak{p}}$ and it follows that $\mathfrak{p} \in \mathrm{Supp}(M/\Gamma_{d,\mathfrak{b}}(M))$. Thus, $\mathrm{Supp}(D_{d,\mathfrak{b}}(M)) \subseteq \mathrm{Supp}(M/\Gamma_{d,\mathfrak{b}}(M))$, which completes the proof. □

Corollary 3.5. *Let M be an R-module. We have*

$$\mathrm{Supp}(H^1_{d,\mathfrak{b}}(M)) \subseteq \mathrm{Supp}(M/\Gamma_{d,\mathfrak{b}}(M)).$$

Proof. Consider the short exact sequence

$$0 \to M/\Gamma_{d,\mathfrak{b}}(M) \to D_{d,\mathfrak{b}}(M) \to H^1_{d,\mathfrak{b}}(M) \to 0.$$

Applying Corollary 3.4 to this sequence yields the desired result. □

Theorem 3.6. *Let M be an R-module and set $S = R \backslash Z_R(M)$. For each prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$, define $U_{\mathfrak{p}} = R_{\mathfrak{p}} \setminus Z_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$. Let $\lambda_{\mathfrak{p}} : S^{-1}M \to U_{\mathfrak{p}}^{-1}M_{\mathfrak{p}}$ be the canonical homomorphism at $\mathfrak{p}$. Then*

$$B_{d,\mathfrak{b}}(M) \cong \cap_{\mathfrak{p} \notin W(d,\mathfrak{b})} \lambda_{\mathfrak{p}}^{-1}(M_{\mathfrak{p}}).$$

Proof. Let $y \in B_{d,\mathfrak{b}}(M)$. Then there exist $\mathfrak{a} \in \tilde{W}(d,\mathfrak{b})$ and an element $m/s \in (M :_{S^{-1}M} \mathfrak{a})$ such that $\varphi^M_{\mathfrak{a}}(m/s) = y$, where $\varphi^M_{\mathfrak{a}} : (M :_{S^{-1}M} \mathfrak{a}) \to B_{d,\mathfrak{b}}(M)$ is the natural map. It is easy to see that $\mathfrak{a} \nsubseteq \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(R) \setminus W(d,\mathfrak{b})$. Let $a \in \mathfrak{a} \setminus \mathfrak{p}$, where $\mathfrak{p} \in \mathrm{Spec}(R) \setminus W(d,\mathfrak{b})$. Hence $am/s \in M$, and consequently

$$\lambda_{\mathfrak{p}}(m/s) = (m/1)/(s/1) = ((am/s)/a)/(1/1) \in M_{\mathfrak{p}}.$$

Then $m/s \in \lambda_{\mathfrak{p}}^{-1}(M_{\mathfrak{p}})$. We show that the map $\varphi : B_{d,\mathfrak{b}}(M) \to \cap_{\mathfrak{p} \notin W(d,\mathfrak{b})} \lambda_{\mathfrak{p}}^{-1}(M_{\mathfrak{p}})$ defined by $\varphi(y) = \varphi(\varphi^M_{\mathfrak{a}}(m/s)) = m/s$ is an R-isomorphism. Clearly, φ is an R-monomorphism. Let $m/s \in \lambda_{\mathfrak{p}}^{-1}(M_{\mathfrak{p}})$ for all $\mathfrak{p} \in \mathrm{Spec}(R) \setminus W(d,\mathfrak{b})$. Then there exists $s_{\mathfrak{p}} \in R \setminus \mathfrak{p}$ such that $s_{\mathfrak{p}} m/s \in M$ for all $\mathfrak{p} \in \mathrm{Spec}(R) \setminus W(d,\mathfrak{b})$. Put $\mathfrak{I} = \sum_{\mathfrak{p} \notin W(d,\mathfrak{b})} Rs_{\mathfrak{p}}$. We claim that $\mathfrak{I} \in \tilde{W}(d,\mathfrak{b})$. Assume that $\mathfrak{q} \in \mathrm{Spec}(R)$ satisfies $\mathfrak{I} + \mathfrak{b} \subseteq \mathfrak{q}$ and $\dim(R/(\mathfrak{I} + \mathfrak{b})) = \dim(R/\mathfrak{q})$. If $\mathfrak{q} \notin W(d,\mathfrak{b})$, then $Rs_{\mathfrak{q}} \subseteq \mathfrak{I} \subseteq \mathfrak{I} + \mathfrak{b} \subseteq \mathfrak{q}$, which implies $s_{\mathfrak{q}} \in \mathfrak{q}$, contradicting the choice of $s_{\mathfrak{q}}$. Thus, we conclude that $\mathfrak{q} \in W(d,\mathfrak{b})$. Then there exists $\mathfrak{d} \in \Sigma$ such that $\mathfrak{d} \subseteq \mathfrak{q} + \mathfrak{b}$. Hence, we have

$$\mathfrak{d} \subseteq \mathfrak{q} + \mathfrak{b} \subseteq \mathfrak{q} + \mathfrak{I} + \mathfrak{b} \subseteq \mathfrak{q}.$$

This implies that $\dim(R/\mathfrak{q}) \leq \dim(R/\mathfrak{d}) \leq d$. Consequently, $\dim(R/(\mathfrak{I} + \mathfrak{b})) \leq d$, which shows that $\mathfrak{I} + \mathfrak{b} \in \Sigma$. Now, since $\mathfrak{I} + \mathfrak{b} \subseteq \mathfrak{I} + \mathfrak{b}$, it follows that $\mathfrak{I} \in \tilde{W}(d,\mathfrak{b})$. Thus $m/s \in (M :_{S^{-1}M} \mathfrak{I})$ and $\varphi(\varphi^M_{\mathfrak{I}}(m/s)) = m/s$. This proves that φ is an R-epimorphism, thereby completing the proof. □

Theorem 3.7. *Let R be a domain. Then we have*

$$D_{d,\mathfrak{b}}(R) = \cap_{\mathfrak{p}\notin W(d,\mathfrak{b})} R_{\mathfrak{p}}.$$

Proof. Under the assumptions of Theorem 3.6, it is evident that $\lambda_{\mathfrak{p}}^{-1}(R_{\mathfrak{p}}) = R_{\mathfrak{p}}$ (note that R is a domain). Suppose $\Gamma_{d,\mathfrak{b}}(R) \neq 0$ and let $0 \neq r \in R$ be an element of $\Gamma_{d,\mathfrak{b}}(R)$. Then there exists $\mathfrak{I} \in \Sigma$ such that $\mathfrak{I} \subseteq (0 :_R r) + \mathfrak{b}$. Since $(0 :_R r) = 0$, $\mathfrak{I} \subseteq \mathfrak{b}$, and so $\mathfrak{b} \in \Sigma$. Therefore, $\Gamma_{d,\mathfrak{b}}(R) = R$, which leads to $D_{d,\mathfrak{b}}(R) = 0$ and $\mathrm{Spec}(R) = W(d, \mathfrak{b})$. This implies that $\cap_{\mathfrak{p}\notin W(d,\mathfrak{b})} R_{\mathfrak{p}} = 0$, completing the proof in this case. Now if $\Gamma_{d,\mathfrak{b}}(R) = 0$, then by Theorem 3.2(2) and Theorem 3.6, the result follows. □

Theorem 3.8. *Let M be an R-module such that $\Gamma_{d,\mathfrak{b}}(M) = 0$. Then*

$$\Omega_{d,\mathfrak{b}}(M) := \cup_{s\in R-Z_R(M)}(W(d, \mathfrak{b}) \cap \mathrm{Ass}_R(M/sM)) = \emptyset$$

if and only if $M \cong D_{d,\mathfrak{b}}(M)$.

Proof. ($\Rightarrow$) First, assume that $\Omega_{d,\mathfrak{b}}(M) = \emptyset$. Since $\Gamma_{d,\mathfrak{b}}(M) = 0$, it follows from Theorem 3.2(2), $D_{d,\mathfrak{b}}(M) \cong B_{d,\mathfrak{b}}(M)$. Hence, we use $B_{d,\mathfrak{b}}(M)$ instead of $D_{d,\mathfrak{b}}(M)$. Let $\Omega_{d,\mathfrak{b}}(M) = \emptyset$ and $\mathfrak{a} \in \tilde{W}(d, \mathfrak{b})$. It is easy to see that $\varphi : M \to B_{d,\mathfrak{b}}(M)$ defined by $\varphi(m) = \varphi_{\mathfrak{a}}(m)$ is an R-monomorphism for all $m \in M$. Now, we prove that φ is an R-epimorphism. Suppose $y \in B_{d,\mathfrak{b}}(M)$. Then there exist $\mathfrak{c} \in \tilde{W}(d, \mathfrak{b})$, $\varphi_{\mathfrak{c}} \in \mathrm{Hom}_R((M :_{S^{-1}M} \mathfrak{c}), B_{d,\mathfrak{b}}(M))$ and $m/s \in (M :_{S^{-1}M} \mathfrak{c})$ such that $\varphi_{\mathfrak{c}}(m/s) = y$. It suffices to show that $m/s \in M$. Since $\mathfrak{c}m/s \subseteq M$, we have $\mathfrak{c}m \subseteq sM$, which implies $\mathfrak{c}(m + sM) = 0$. If $\mathfrak{c} \subseteq Z_R(M/sM)$, then there exists $\mathfrak{p} \in \mathrm{Ass}(M/sM)$ such that $\mathfrak{c} \subseteq p$. Consequently, $\mathfrak{p} \in W(d, \mathfrak{b})$, leading to $\mathfrak{p} \in \Omega_{d,\mathfrak{b}}(M)$, which contradicts our assumption. Then $\mathfrak{c} \nsubseteq Z_R(M/sM)$, which means that $m \in sM$, implying $m/s \in M$.
($\Leftarrow$) Let $M \cong D_{d,\mathfrak{b}}(M)$. Assume that $\mathfrak{p} \in \mathrm{Ass}(M/sM)$ for some $s \in R\backslash Z_R(M)$. Then there exists $m \in M \setminus sM$ such that $\mathfrak{p} = (sM :_R m)$. It follows that $\mathfrak{p}m \subseteq sM$ and so $\mathfrak{p}.m/s \subseteq M$, implying that $m/s \in (M :_{S^{-1}M} \mathfrak{p})$. If $m/s \in M$, then $m \in sM$ and so $\mathfrak{p} = R$, which is a contradiction. Then $m/s \in (M :_{S^{-1}M} \mathfrak{p})\backslash M$. Now, we show that $\mathfrak{p} \notin W(d, \mathfrak{b})$. Let $\mathfrak{p} \in W(d, \mathfrak{b})$. Then $\mathfrak{p} \in \tilde{W}(d, \mathfrak{b})$. Since $m/s \in (M :_{S^{-1}M} \mathfrak{p})$, $\varphi_{\mathfrak{p}}(m/s) \in B_{d,\mathfrak{b}}(M)$, which implies $\varphi^{-1}(\varphi_{\mathfrak{p}}(m/s)) \in M$. Thus $\varphi^{-1}(\varphi(m)) \in sM$ and so $m \in sM$, which is a contradiction. Therefore $\mathfrak{p} \notin W(d, \mathfrak{b})$ and we conclude that $\Omega_{d,\mathfrak{b}}(M) = \emptyset$. □

Corollary 3.9. *Assume that M is an R-module.*
(1) If $\Gamma_{d,\mathfrak{b}}(M) = H^1_{d,\mathfrak{b}}(M) = 0$, then $\Omega_{d,\mathfrak{b}}(M) = \emptyset$.
(2) $\Omega_{d,\mathfrak{b}}(D_{d,\mathfrak{b}}(M)) = \emptyset$.
(3) If M is an injective module, then $\Omega_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M)) = \emptyset$.

Proof. (1) Since $\Gamma_{d,\mathfrak{b}}(M) = H^1_{d,\mathfrak{b}}(M) = 0$, by Lemma 2.2, $M \cong D_{d,\mathfrak{b}}(M)$. Thus, by Theorem 3.8, $\Omega_{d,\mathfrak{b}}(M) = \emptyset$.
(2) By Proposition 2.6(4), $\Gamma_{d,\mathfrak{b}}(D_{d,\mathfrak{b}}(M)) = 0$. Moreover, by Proposition 2.6(3), $D_{d,\mathfrak{b}}(M) \cong D_{d,\mathfrak{b}}(D_{d,\mathfrak{b}}(M))$. Hence, by Theorem 3.8, $\Omega_{d,\mathfrak{b}}(D_{d,\mathfrak{b}}(M)) =$

$\emptyset$.
(3) By Theorem 2.3, $\Gamma_{d,\mathfrak{b}}(M)$ is injective. Then, from the short exact sequence

$$0 \to \Gamma_{d,\mathfrak{b}}(M) \to M \to M/\Gamma_{d,\mathfrak{b}}(M) \to 0,$$

it follows that $M/\Gamma_{d,\mathfrak{b}}(M)$ is also injective. Therefore $H^1_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M)) = 0$ and so by Lemma 2.2, $M/\Gamma_{d,\mathfrak{b}}(M) \cong D_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M))$. Now, the result follows from part (2). □

4. **Localization theorem**

We recall that a finite-dimensional Noetherian ring R is said to be *biequidimensional* if:
(i) $\dim R/\mathfrak{p} = \dim R$, for all $\mathfrak{p} \in \mathrm{Ass}(R)$;
(ii) $\dim R/\mathfrak{p} + \dim R_{\mathfrak{p}} = \dim R$, for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

Theorem 4.1. *Let R be a catenary and biequidimensional ring and let M be an R-module. Then for each $\mathfrak{p} \in Spec(R)$ and each $i \geq 0$, we have*

$$H^i_{d,\mathfrak{b}}(M)_{\mathfrak{p}} \cong H^i_{d-dim(R/(\mathfrak{p}+\mathfrak{b})),\mathfrak{b}_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

Proof. By the standard argument of homology theory (see [2, Exercise 1.3.4]), it is sufficient to prove that

$$\Gamma_{d,\mathfrak{b}}(M)_{\mathfrak{p}} \cong \Gamma_{d-\dim(R/(\mathfrak{p}+\mathfrak{b})),\mathfrak{b}_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

First, we show that $\Gamma_{d,\mathfrak{b}}(M)_{\mathfrak{p}} \subseteq \Gamma_{d-\dim(R/(\mathfrak{p}+\mathfrak{b})),\mathfrak{b}_{\mathfrak{p}}}(M_{\mathfrak{p}})$. Let $m/s \in \Gamma_{d,\mathfrak{b}}(M)_{\mathfrak{p}}$. Then we have $\dim(R/((0:_R m)+\mathfrak{b})) \leq d$. Moreover, $s \in R \setminus \mathfrak{p}$. We show that $\dim(R/((0:_R m)+\mathfrak{b}))_{\mathfrak{p}} \leq d - \dim(R/(\mathfrak{p}+\mathfrak{b}))$. Let $\mathfrak{q}_{\mathfrak{p}}$ be a minimal prime in $\mathrm{Supp}(R/((0:_R m)+\mathfrak{b})_{\mathfrak{p}})$. Hence, we have

$$\begin{aligned}
\dim(R_{\mathfrak{p}}/\mathfrak{q}_{\mathfrak{p}}) &= \dim((R/\mathfrak{q})_{\mathfrak{p}/\mathfrak{q}}) \\
&= ht(\mathfrak{p}/\mathfrak{q}) = ht(\mathfrak{p}) - ht(\mathfrak{q}) \\
&= [\dim(R) - \dim(R/\mathfrak{p})] - [\dim(R) - \dim(R/\mathfrak{q})] \\
&= \dim(R/\mathfrak{q}) - \dim(R/\mathfrak{p}).
\end{aligned}$$

Then $\dim(R_{\mathfrak{p}}/\mathfrak{q}_{\mathfrak{p}}) \leq d - \dim(R/\mathfrak{p})$ and so $\dim(R_{\mathfrak{p}}/\mathfrak{q}_{\mathfrak{p}}) \leq d - \dim(R/(\mathfrak{p}+\mathfrak{b}))$. This implies that

$$\dim(R/((0:_R m)+\mathfrak{b}))_{\mathfrak{p}} \leq d - \dim(R/(\mathfrak{p}+\mathfrak{b})).$$

Consequently, we obtain $m/s \in \Gamma_{d-\dim(R/(\mathfrak{p}+\mathfrak{b})),\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}})$, leading to the inclusion

$$\Gamma_{d,\mathfrak{b}}(M)_{\mathfrak{p}} \subseteq \Gamma_{d-\dim(R/(\mathfrak{p}+\mathfrak{b})),\mathfrak{b}_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

Now, assume that $m/s \in \Gamma_{d-\dim(R/(\mathfrak{p}+\mathfrak{b})),\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}})$. Then $m \in M$, $s \in S$, and also

$$\dim(R/((0:_R m)+\mathfrak{b}))_{\mathfrak{p}} \leq d - \dim(R/(\mathfrak{p}+\mathfrak{b})).$$

We show that $\dim(R/((0:_R m)+\mathfrak{b})) \leq d$. To prove this, let $\mathfrak{q}$ be a minimal prime in $\mathrm{Supp}(R/((0:_R m)+\mathfrak{b}))$. If $\mathfrak{q} \subseteq \mathfrak{p}$, then $\mathfrak{q}R_\mathfrak{p}$ is a minimal prime in $\mathrm{Supp}(R/((0:_R m)+\mathfrak{b})_\mathfrak{p})$. In this case, we have

$$\dim(R_\mathfrak{p}/\mathfrak{q}_\mathfrak{p}) = \dim((R/\mathfrak{q})_{\mathfrak{p}/\mathfrak{q}}) \leq d - \dim(R/(\mathfrak{p}+\mathfrak{b})),$$

which implies

$$ht(\mathfrak{p}/\mathfrak{q}) \leq d - \dim(R/(\mathfrak{p}+\mathfrak{b})).$$

Then $ht(\mathfrak{p}) - ht(\mathfrak{q}) \leq d - \dim(R/(\mathfrak{p}+\mathfrak{b}))$, which gives

$$\dim(R/\mathfrak{q}) - \dim(R/\mathfrak{p}) \leq d - \dim(R/(\mathfrak{p}+\mathfrak{b})).$$

Since $\mathfrak{b} \subseteq (0:_R m)+\mathfrak{b} \subseteq \mathfrak{q} \subseteq \mathfrak{p}$, it follows that $\mathfrak{b} \subseteq \mathfrak{p}$. Thus $\mathfrak{p} = \mathfrak{b}+\mathfrak{p}$, which implies $\dim(R/\mathfrak{q}) \leq d$. Hence $\dim(R/((0:_R m)+\mathfrak{b})) \leq d$. Now, suppose $\mathfrak{q} \nsubseteq \mathfrak{p}$. Let $x \in \mathfrak{q} \setminus \mathfrak{p}$. The $R_\mathfrak{q}$-module $(R/((0:_R m)+\mathfrak{b}))_\mathfrak{q}$ is finitely generated and concentrated in $\mathfrak{q}_\mathfrak{q}$, hence it is annihilated by a suitable power $\mathfrak{q}^n R_\mathfrak{q}$. In particular, $x^n m = 0$ in $M_\mathfrak{q}$. Therefore, we have $\mathrm{Supp}(R/((0:_R x^n m)+\mathfrak{b})) \subsetneq \mathrm{Supp}(R/((0:_R m)+\mathfrak{b}))$, but $\mathfrak{q} \notin \mathrm{Supp}(R/((0:_R x^n m)+\mathfrak{b}))$. Let $\mathfrak{q}_1$ be a minimal prime in $\mathrm{Supp}(R/((0:_R x^n m)+\mathfrak{b}))$. If $\mathfrak{q}_1 \subseteq \mathfrak{p}$, then by a similar argument as above, the result follows. Otherwise, we can construct an element $t \notin \mathfrak{p}$ such that $\mathrm{Supp}(R/((0:_R tm)+\mathfrak{b})) \leq d$, completing the proof. □

Corollary 4.2. *Let R be a catenary and biequidimensional ring, $\mathfrak{p} \in Spec(R)$ and let M be an R-module. Then for each $i \geq 0$,*

$$D^i_{d,\mathfrak{b}}(M)_\mathfrak{p} \cong D^i_{d-dim(R/(\mathfrak{p}+\mathfrak{b})),\mathfrak{b}_\mathfrak{p}}(M_\mathfrak{p}).$$

Proof. First, we show that $D_{d,\mathfrak{b}}(M)_\mathfrak{p} \cong D_{d-\dim(R/(\mathfrak{p}+\mathfrak{b})),\mathfrak{b}_\mathfrak{p}}(M_\mathfrak{p})$. By Theorem 4.1, $H^i_{d,\mathfrak{b}}(M)_\mathfrak{p} \cong H^i_{d-\dim(R/(\mathfrak{p}+\mathfrak{b})),\mathfrak{b}_\mathfrak{p}}(M_\mathfrak{p})$ for all $i \geq 0$. Put $d_1 = d - \dim(R/(\mathfrak{p}+\mathfrak{b}))$. Now, using the following commutative diagram

$$\begin{array}{ccccccccc}
\Gamma_{d,\mathfrak{b}}(M)_\mathfrak{p} & \longrightarrow & M_\mathfrak{p} & \longrightarrow & D_{d,\mathfrak{b}}(M)_\mathfrak{p} & \longrightarrow & H^1_{d,\mathfrak{b}}(M)_\mathfrak{p} & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow = & & \downarrow & & \downarrow \cong & & \downarrow = \\
\Gamma_{d_1,\mathfrak{b}_\mathfrak{p}}(M_\mathfrak{p}) & \longrightarrow & M_\mathfrak{p} & \longrightarrow & D_{d_1,\mathfrak{b}_\mathfrak{p}}(M_\mathfrak{p}) & \longrightarrow & H^1_{d_1,\mathfrak{b}_\mathfrak{p}}(M_\mathfrak{p}) & \longrightarrow & 0,
\end{array}$$

where both rows are exact by Lemma 2.2, this follows from the Five Lemma. Now, assume that $i \geq 1$. Hence, by Lemma 2.2, $D^i_{d,\mathfrak{b}}(M) \cong H^{i+1}_{d,\mathfrak{b}}(M)$. Then

$$\begin{aligned}
D^i_{d,\mathfrak{b}}(M)_\mathfrak{p} &\cong H^{i+1}_{d,\mathfrak{b}}(M)_\mathfrak{p} \\
&\cong H^{i+1}_{d-\dim(R/(\mathfrak{p}+\mathfrak{b})),\mathfrak{b}_\mathfrak{p}}(M_\mathfrak{p}) \\
&\cong D^i_{d-\dim(R/(\mathfrak{p}+\mathfrak{b})),\mathfrak{b}_\mathfrak{p}}(M_\mathfrak{p}).
\end{aligned}$$

This concludes the proof. □

5. **Vanishing theorems**

In this section, we establish a vanishing theorem for the module $D^i_{d,\mathfrak{b}}(M)$ and demonstrate that the integer n, which serves as an upper bound for the vanishing of $D^i_{d,\mathfrak{b}}(M)$, is related to the dimension of a certain set of prime ideals.

Throughout this section, let d' be a fixed integer defined by

$$d' := \max\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in W(d,\mathfrak{b})\}.$$

Definition 5.1. Let M be a finitely generated R-module. For any integer $k \geq 0$, we define the *singular set* of M as

$$\mathrm{S}^*_k(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) + \dim(R/\mathfrak{p}) \leq k\}$$

(see [1]). Additionally, assuming $M \neq 0$, we define the grade of M as

$$\mathrm{grade}(M) = \inf\{i \in \mathbb{N}_0 \mid \mathrm{Ext}^i_R(M,R) \neq 0\}$$

(see [3, p. 131]). Moreover, for a local ring $(R,\mathfrak{m})$, we simply write $\mathrm{depth}(\mathfrak{m},M)$ as $\mathrm{depth}(M)$(see [3, p. 130]).

Theorem 5.2. *Let M be a finitely generated R-module and let $n \geq 2$ and k be two non-negative integers such that $\dim(\mathrm{S}^*_{n+k}(M)) \leq k$ for all $k < d'$. Then $D_{d,\mathfrak{b}}(M) \cong M$ and $D^i_{d,\mathfrak{b}}(M) = 0$ for all $0 < i < n-1$.*

Proof. By Lemma 2.2, it is sufficient to prove that $H^i_{d,\mathfrak{b}}(M) = 0$ for all $i < n$. Let $\mathfrak{a} \in \tilde{W}(d,\mathfrak{b})$. Hence there exists $\mathfrak{I} \in \Sigma$ such that $\mathfrak{I} \subseteq \mathfrak{a} + \mathfrak{b}$. Assume that $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$. Then $\mathfrak{a} \subseteq \mathfrak{p}$. Consequently, we obtain $\mathfrak{I} \subseteq \mathfrak{p} + \mathfrak{b}$ and so $\mathfrak{p} \in W(d,\mathfrak{b})$. Put $k := \dim(R/\mathfrak{p}) - 1$. Since $k < d'$, the given assumption ensures that $\mathfrak{p} \notin \mathrm{S}^*_{n+k}(M)$. Therefore, by Definition 5.1, $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) + \dim(R/\mathfrak{p}) > k+n$. This implies that $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) + k + 1 > k + n$, which further simplifies to $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) + 1 > n$. Thus, we conclude that $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq n$. Consequently, $\mathrm{depth}(\mathfrak{a},M) \geq n$. By [3, Theorem 16.7], it follows that for each $\mathfrak{a} \in \tilde{W}(d,\mathfrak{b})$, we have $\mathrm{Ext}^i_R(R/\mathfrak{a},M) = 0$ for all $i < n$. Then $H^i_{d,\mathfrak{b}}(M) = 0$ for all $i < n$. □

Theorem 5.3. *Let $(R,\mathfrak{m})$ be a local ring, and let M be a finitely generated R-module. If $t \in \mathbb{N}$, then for all $i \geq t$, the module $D^i_{d,\mathfrak{b}}(M)$ is finitely generated if and only if $D^i_{d,\mathfrak{b}}(M) = 0$.*

Proof. ($\Rightarrow$) First, we assume that $D^i_{d,\mathfrak{b}}(M)$ is finitely generated for all $i \geq t$ and show that $D^i_{d,\mathfrak{b}}(M) = 0$. To establish this, we proceed by induction on $n = \dim(M)$. For the case $n = 0$, then by Grothendieck's Vanishing Theorem([2, Theorem 6.1.2]), together with Lemma 2.2 and Proposition 2.6(6), implies that $D^i_{d,\mathfrak{b}}(M) = 0$ for all $i \geq t$. Now, assume inductively that $n > 0$ and that the result has been proved for all finitely generated R-modules of dimension $n-1$. Consider the exact sequence

$$0 \to \Gamma_{d,\mathfrak{b}}(M) \to M \to M/\Gamma_{d,\mathfrak{b}}(M) \to 0$$

which gives rise to the following long exact sequence:

$$\cdots \to D^i_{d,\mathfrak{b}}(\Gamma_{d,\mathfrak{b}}(M)) \to D^i_{d,\mathfrak{b}}(M) \to D^i_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M)) \to D^{i+1}_{d,\mathfrak{b}}(\Gamma_{d,\mathfrak{b}}(M)) \to \cdots .$$

Since, by Proposition 2.6(1), $D^i_{d,\mathfrak{b}}(\Gamma_{d,\mathfrak{b}}(M)) = 0$ for all $i \geq 0$, it follows that $D^i_{d,\mathfrak{b}}(M) \cong D^i_{d,\mathfrak{b}}(M/\Gamma_{d,\mathfrak{b}}(M))$ for all $i \geq 0$. Thus, by replacing M with $M/\Gamma_{d,\mathfrak{b}}(M)$, we may assume that $\Gamma_{d,\mathfrak{b}}(M) = 0$. Hence $\mathfrak{m} \nsubseteq Z_R(M)$. Thus, there exists an element $x \in \mathfrak{m} \setminus Z_R(M)$. The exact sequence

$$0 \to M \to M \to M/xM \to 0$$

induces the long exact sequence

$$\cdots \to D^i_{d,\mathfrak{b}}(M) \to D^i_{d,\mathfrak{b}}(M) \to D^i_{d,\mathfrak{b}}(M/xM) \to D^{i+1}_{d,\mathfrak{b}}(M) \to \cdots$$

which implies that $D^i_{d,\mathfrak{b}}(M/xM)$ is finitely generated for all $i \geq t$. Since M/xM is finitely generated with $\dim(M/xM) = n-1$, the inductive hypothesis implies that $D^i_{d,\mathfrak{b}}(M/xM) = 0$ for all $i \geq t$. Therefore $D^i_{d,\mathfrak{b}}(M) \cong xD^i_{d,\mathfrak{b}}(M)$ for all $i \geq t$. Then, by Nakayama's Lemma, it follows that $D^i_{d,\mathfrak{b}}(M) = 0$ for all $i \geq t$.
($\Leftarrow$) The converse is obvious. □

6. Conclusion

In this paper, we introduced and studied the generalized local cohomology modules $H^i_{d,\mathfrak{b}}(M)$ and their derived functors $D^i_{d,\mathfrak{b}}(M)$ over a commutative Noetherian ring R. We established key properties, including vanishing conditions, localization theorems, and structural results. Specifically, we proved that when R is a catenary and biequidimensional ring, the localization of these modules satisfies:

$$H^i_{d,\mathfrak{b}}(M)_\mathfrak{p} \cong H^i_{d-\dim(R/(\mathfrak{p}+\mathfrak{b})),\mathfrak{b}_\mathfrak{p}}(M_\mathfrak{p}).$$

Additionally, we provided criteria for the vanishing of $D^i_{d,\mathfrak{b}}(M)$ and explored connections between these modules and torsion theories. Our results extend and unify previous work in local cohomology, offering new insights into the behavior of these functors in algebraic geometry and commutative algebra.

7. Data Availability Statement

Not applicable

8. Aknowledgement

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9. Ethical considerations

Not applicable

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11. **Conflict of interest**

The author declares no competing interests.

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