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AN EFFICIENT NUMERICAL TECHNIQUE FOR A SPECIFIC FAMILY OF SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS

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Abstract. This paper deals with a particular family of singularly perturbed two-point boundary value problems characterized by the perturbation parameter $0 < \varepsilon \ll 1$, and it introduces a new numerical technique to approximate its solution. As the perturbation parameter ε decreases, the majority of classic numerical methods that utilize uniform grids necessitate significantly reduced step sizes. Consequently, we employ a non-equidistant mesh. After discretizing the problem and constructed some high order compact methods, the original problem is transformed into a linear algebraic system. Also, it is demonstrated that the present method converges with order 4 in L_∞ norm. Finally, numerical simulations will demonstrate the efficacy of the proposed method and confirm the theoretical results.

Keywords: Two-point boundary value problems, perturbation parameter, differential equation, fourth-order convergence.
2020 MSC: 65L11, 65L10, 65L70, 15A18.

1. Introduction

In recent years, singularly perturbed boundary value problems (SPBVPs) have been extensively studied. As we know, the behavior of this problem is overshadowed by a small positive value denoted as ε, which is referred to as the perturbation parameter. If the perturbation parameter $\varepsilon \to 0$, the solution has a thin layer(s) (interior and/or boundary layer(s)) with a very steep gradient. Because the small perturbation parameter ε affects on changing the width of the layer(s), these kinds of problems appear in the mathematical modeling of several problems in physics and engineering, like solid mechanics, mechanical and electrical systems, celestial mechanics, electromagnetic field problems in moving media, aerodynamics, chemical reactions, financial mathematics, quantum physics, reaction-diffusion processes, biochemical reactions and evolutionary biology (red cell system) [35], epidemics and population dynamics [11], the oscillations and chaos in physiological control systems [17], tumor growth [31]

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etc. Since, the exact solution of the SPBVPs inside the layer(s) changes very fast, and outside the layer(s) the solution has regular behavior and slow variation, these problems are difficult to solve numerically. As the perturbation parameter approaches zero, the smoothness of the solutions reduces. On one hand, the exact solutions of the SPBVPs are not generally available, so we are forced to employ numerical methods. On the other hand, due to the layers in these solutions, the numerical behavior of such problems face some difficulties. Practically, when $\varepsilon \to 0$, classical numerical methods on uniform meshes like finite element or finite difference methods to solve these kinds of problems do not attain satisfactory and accurate results and may fail to yield a stable solution. Numerous numerical techniques have been extended by researchers to solve SPBVPs, like as parameter uniform numerical methods [12, 18, 25, 28], finite difference method [32], priori mesh approach [33], exponentially fitted finite difference schemes [26, 27], second order numerical schemes [3, 7], a semi-analytic method [4], absolutely stable difference scheme [6], discrete cubic spline method [36], non-standard fitted operator scheme [29], upwind finite difference method [10], fitted operator finite difference method [24], parameter-uniform improved hybrid numerical scheme [19], etc.

Many of the methods mentioned above are effective for solving SPBVPs. However, to the best of our knowledge, the convergence rate of the aforesaid methods is less than or equal to two. Therefore, the main motivation of this article is to introduce a simple and high-order numerical method for solving a particular class of SPBVP characterized by a perturbation parameter ε. In order to achieve this goal, we introduce some fourth-order compact techniques for discretizing the associated SPBVP. The accuracy and performance of the new method are analyzed, and it is inferred that the method is capable of constructing fourth-order convergent solutions in the L_∞ norm.

The outline of the rest of the paper is structured as follows. In Section 2 some properties of the exact solution of the corresponding SPBVPs and their requirements are taken into account. In Section 3 a new numerical method based on the proposed compact schemes is developed for solving SPBVPs. The convergence and error estimation of the proposed method are also studied in Section 4. In Section 5 some numerical examples are given to demonstrate the efficiency of the presented method.

2. **Properties of the exact solution and some requirements**

First, we get more precise information about the exact solution's behavior of the SPBVPs. In this research, we consider the SPBVPs that are determined by the second-order ordinary differential equation described as follows,

$$\varepsilon y''(t) + p(t)y'(t) + q(t)y(t) = f(t), \quad t \in \Gamma = (0, 1) \tag{1a}$$

$$y(0) = \tau_0, \quad y(1) = \tau_1 \tag{1b}$$

in which ε is the perturbation parameter, $0 < \varepsilon \ll 1$ and it is assumed that given functions $p(t), q(t)$ and $f(t)$ belong to the $C^4[0,1]$. It is important to note that when $p(t)$ is positive/negative, the exact solution of equation (1) demonstrates the layer behavior at the left/right end of the interval of width $O(\varepsilon)$. For the case $p(t) = 0$, if $q(t) > 0$ then it will be seen the oscillatory behavior of the exact solution to the corresponding problem and for the case $q(t) < 0$, two layers of width $O(\sqrt{\varepsilon})$ each will appear. For more details regarding this issue, refer to [9, 23]. The solutions of SPBVPs (1) are often represented as asymptotic expansions in powers of ε, as follows:

$$y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots .$$

To clarify the characteristics of the solution, researchers examined the above-mentioned asymptotic expansions. To study the asymptotic behavior of the solutions, see [8, 22].

Assumption 1. Assume that there are constants $\lambda > 0$ and $\eta > 0$ so that the relation $p(t) \geq \lambda$ as well as $q(t) \leq -\eta$ can be established.

It should be mentioned that under Assumption 1 the solution has an interior layer of width $\mathcal{O}(\varepsilon)$ in the neighbourhood of the point $t = \varepsilon$ [19, 28]. The following lemma demonstrates that the operator $L_w(t) = \varepsilon w''(t) + p(t)w'(t) + q(t)w(t)$ satisfies the minimum principle.

Lemma 2.1. *Assume that for the smooth function $w(t)$ relations $w(0) \geq 0$ and $w(1) \geq 0$ are fulfilled, and also assumption 1 is established. If the operator $L_w(t) := \varepsilon w''(t) + p(t)w'(t) + q(t)w(t)$ satisfies the condition $L_w(t) \leq 0, \forall t \in \Gamma$, it can be concluded that $w(t) \geq 0, \forall t \in \overline{\Gamma}$.*

Proof. To prove of the lemma see [20]. □

A parameter bound for the solution $y(t)$ of (1) will be determined with the help of the above lemma.

Lemma 2.2. *Let $y(t)$ be the exact solution of the SPBVP (1), then we can conclude from Assumption 1 that*

$$\|y\|_\infty \leq \left(\frac{1}{\eta} \|f\|_\infty + \max\{|\tau_0|, |\tau_1|\} \right), \quad \forall t \in \overline{\Gamma}. \tag{2}$$

Proof. See [20]. □

Remark 2.3. Even if, under Assumption 1, the sign of the constant λ is negative and the sign of the operator $L_w(t)$ also changes in Lemma 2.1, again, relation (2) will be established. The details are provided in the Appendix.

In the following theorem, we will determine the bound of the derivatives of the exact solution to the SPBVP (1), up to the fourth order.

Theorem 2.4. *If $p(t), q(t), f(t) \in C^2(\overline{\Gamma})$, then derivatives of the exact solution $y(t)$ of SPBVP (1) satisfies*

$$\|y^{(k)}\|_\infty \leq \frac{C}{\varepsilon^k}, \quad k = 1, 2, 3, 4,$$

in which the constant C is only dependent on $\|p^{(l)}\|_\infty, \|q^{(l)}\|_\infty, \|f^{(l)}\|_\infty, l = 0, 1, 2,$ and $\|y\|_\infty$, and not on ε.

Proof. To prove the theorem see [7]. □

3. Description of the method

A novel numerical technique for solving SPBVP (1) will be introduced in this section. In this study, we utilize y_n instead of $y(t_n)$. Additionally, Y_n represents the estimated value of y_n. Taking arbitrary $t_k \in \overline{\Gamma}^N = \{t_0, t_1, \ldots, t_N\}$, in which $t_0 = 0, t_N = 1$, and then integrating (1a) over $[t_k, t]$, concludes that

$$\varepsilon(y'(t) - y'_k) + \int_{t_k}^{t} p(s)y'(s)ds + \int_{t_k}^{t} q(s)y(s)ds = \int_{t_k}^{t} f(s)ds. \tag{3}$$

Same as the procedure in [2], integrating (3) over subintervals $[t_{k-1}, t_k]$ and $[t_k, t_{k+1}]$, separately, and then subtracting the two obtained expressions from one another, conclude that

$$\begin{aligned}
&\varepsilon(y_{k+1} - 2y_k + y_{k-1}) - \varepsilon(h_{k+1} - h_k)y'_k \\
&\qquad + \int_{t_k}^{t_{k+1}} (t_{k+1} - t)p(t)y'(t)dt - \int_{t_{k-1}}^{t_k} (t_{k-1} - t)p(t)y'(t)dt \\
&\qquad + \int_{t_k}^{t_{k+1}} (t_{k+1} - t)q(t)y(t)dt - \int_{t_{k-1}}^{t_k} (t_{k-1} - t)q(t)y(t)dt \\
&\qquad = \int_{t_k}^{t_{k+1}} (t_{k+1} - t)f(t)dt - \int_{t_{k-1}}^{t_k} (t_{k-1} - t)f(t)dt, \quad k = 1, \ldots, N-1.
\end{aligned} \tag{4}$$

To estimate the integral parts of the system of Eqs. (4), we utilize the findings from the lemma provided below.

Lemma 3.1. *Let $g(t) \in C^4[0, 1]$. Applying Lagrange second degree interpolating polynomial at nodes t_{k-1}, t_k, t_{k+1} for function $g(t)$ yields that*

$$\int_{t_k}^{t_{k+1}} (t_{k+1} - t)g(t)dt - \int_{t_{k-1}}^{t_k} (t_{k-1} - t)g(t)dt = \beta_k g_{k-1} + \alpha_k g_k + \gamma_k g_{k+1} - e_k(g), \tag{5}$$

where for $k = 1, \ldots, N-1$,

$$\beta_k = \frac{h_k^4 + 2h_k^3 h_{k+1} - h_{k+1}^4}{12h_k(h_k + h_{k+1})}, \alpha_k = \frac{h_k^4 + 4h_k^3 h_{k+1} + 4h_k h_{k+1}^3 + h_{k+1}^4}{12h_k h_{k+1}}, \gamma_k = \frac{h_{k+1}^4 + 2h_k h_{k+1}^3 - h_k^4}{12h_{k+1}(h_k + h_{k+1})},$$

such that $h_k = t_k - t_{k-1}$ *and for the corresponding error term we get*

$$e_k(g) = \widehat{H}_k g^{(3)}(\zeta_k) + H_k g^{(4)}(\varsigma_k), \tag{6}$$

such that $\zeta_k \in (t_{k-1}, t_k), \varsigma_k \in (t_{k-1}, t_{k+1})$ *and*

$$\begin{cases} \widehat{H}_k = \dfrac{2}{6!}\left(5h_k h_{k+1}(h_k^3 - h_{k+1}^3) + 2(h_k^5 - h_{k+1}^5)\right), \\ H_k = \dfrac{2}{6!} h_{k+1}^4 (5h_k + 2h_{k+1})(h_k + h_{k+1})\hbar_k, \end{cases}$$

in which $0 < \hbar_k < 1$.

Proof. The proof of this lemma can be found in the Appendix. □

Now, employing Lemma 3.1 for functions $p(t)y'(t), q(t)y(t)$ and $f(t)$ and then substituting in (4) yields that

$$\begin{aligned} \varepsilon(y_{k+1} - 2y_k + y_{k-1}) - \varepsilon(h_{k+1} - h_k)y_k' + \beta_k p_{k-1} y_{k-1} + \alpha_k p_k y_k + \gamma_k p_{k+1} y_{k+1} \\ + \beta_k q_{k-1} y_{k-1}' + \alpha_k q_k y_k' + \gamma_k q_{k+1} y_{k+1}' \\ = \beta_k f_{k-1} + \alpha_k f_k + \gamma_k f_{k+1} + e_k(py') + e_k(qy) - e_k(f), k = 1, \dots, N-1, \end{aligned} \tag{7}$$

where the error terms $e_k(py'), e_k(qy)$ and $e_k(f)$ can be computed in accordance with (6). Since, there are no boundary conditions for the values of y_0' and y_N', we approximate them by the following formulas

$$\begin{cases} y_0' = v_0 y_0 + \dfrac{v_1}{h_1} y_1 + \dfrac{v_2}{h_1 + h_2} y_2 + \dfrac{v_3}{h_1 + h_2 + h_3} y_3 + \dfrac{v_4}{h_1 + h_2 + h_3 + h_4} y_4 + \bar{c}_0, \\ y_N' = w_0 y_N + \dfrac{w_1}{h_N} y_{N-1} + \dfrac{w_2}{h_N + h_{N-1}} y_{N-2} + \dfrac{w_3}{h_N + h_{N-1} + h_{N-2}} y_{N-3} \\ \qquad + \frac{w_4}{h_N + h_{N-1} + h_{N-2} + h_{N-3}} y_{N-4} + \bar{c}_N. \end{cases}$$

Expanding y_1, y_2, y_3, and y_4, around point t_0 as well as $y_{N-1}, y_{N-2}, y_{N-3}$ and , y_{N-4} around point t_N yields that

$$\begin{cases} v_0 = -\dfrac{1}{h_1} - \dfrac{1}{h_1+h_2} - \dfrac{1}{h_1+h_2+h_3} - \dfrac{1}{h_1+h_2+h_3+h_4}, \\ v_1 = \dfrac{(h_1+h_2)(h_1+h_2+h_3)(h_1+h_2+h_3+h_4)}{h_2(h_2+h_3)(h_2+h_3+h_4)}, \\ v_2 = -\dfrac{h_1(h_1+h_2+h_3)(h_1+h_2+h_3+h_4)}{h_2h_3(h_3+h_4)}, \\ v_3 = \dfrac{h_1(h_1+h_2)(h_1+h_2+h_3+h_4)}{h_3h_4(h_2+h_3)}, v_4 = -\dfrac{h_1(h_1+h_2)(h_1+h_2+h_3)}{h_4(h_3+h_4)(h_2+h_3+h_4)}, \\ w_0 = \dfrac{1}{h_N} + \dfrac{1}{h_N+h_{N-1}} + \dfrac{1}{h_N+h_{N-1}+h_{N-2}} + \dfrac{1}{h_N+h_{N-1}+h_{N-2}+h_{N-3}}, \\ w_1 = -\dfrac{(h_N+h_{N-1})(h_N+h_{N-1}+h_{N-2})(h_N+h_{N-1}+h_{N-2}+h_{N-3})}{h_{N-1}(h_{N-1}+h_{N-2})(h_{N-1}+h_{N-2}+h_{N-3})}, \\ w_2 = \dfrac{h_N(h_N+h_{N-1}+h_{N-2})(h_N+h_{N-1}+h_{N-2}+h_{N-3})}{h_{N-1}h_{N-2}(h_{N-2}+h_{N-3})}, \\ w_3 = -\dfrac{h_N(h_N+h_{N-1})(h_N+h_{N-1}+h_{N-2}+h_{N-3})}{h_{N-2}h_{N-3}(h_{N-1}+h_{N-2})}, \\ w_4 = \dfrac{h_N(h_N+h_{N-1})(h_N+h_{N-1}+h_{N-2})}{h_{N-3}(h_{N-2}+h_{N-3})(h_{N-1}+h_{N-2}+h_{N-3})}, \end{cases}$$

and

$$\begin{cases} \bar{c}_0 = \dfrac{1}{5!}\left(h_1^4 v_1 y(\zeta_{0,1}) + (h_1+h_2)^4 v_2 y(\zeta_{0,2}) + (h_1+h_2+h_3)^4 v_3 y(\zeta_{0,3}) + (h_1+h_2+h_3+h_4)^4 v_4 y(\zeta_{0,4})\right), \\ \bar{c}_N = -\dfrac{1}{5!}\left(h_N^4 w_1 y(\varsigma_{0,1}) + (h_N+h_{N-1})^4 w_2 y(\varsigma_{0,2}) + (h_N+h_{N-1}+h_{N-2})^4 w_3 y(\varsigma_{0,3})\right) \\ \qquad\qquad -\dfrac{1}{5!}\left((h_N+h_{N-1}+h_{N-2}+h_{N-3})^4 w_4 y(\varsigma_{0,4})\right), \end{cases}$$

in which $\zeta_{0,k} \in (t_0, t_4), \varsigma_{0,k} \in (t_{N-4}, t_N)$, for $k = 1, 2, 3, 4$.

The system of equations (7) can be summarized in the following matrix equation,

$$(8) \quad \left(\varepsilon\mathbf{A} + \mathbf{C}D_p + \bar{\mathbf{A}}\right)\mathbf{y} + \left(\mathbf{C}D_q - \varepsilon D_h\right)\mathbf{y}' = \mathbf{Cf} + \mathbf{r} + \widetilde{\mathbf{E}} + D_{\widehat{H}}\,\widehat{\mathbf{E}} + D_H\,\overline{\mathbf{E}},$$

where $\mathbf{A} = \text{tridiag}(1, -2, 1)$ and $\mathbf{C}$ are $(N-1)$-dimensional tridiagonal matrices, with

$$C(l,k) = \begin{cases} \alpha_k, & l = k, \quad 1 \le k \le N-1, \\ \beta_{k+1}, & l = k+1, \quad 1 \le k \le N-2, \\ \gamma_{k-1}, & l = k-1, \quad 2 \le k \le N-1, \\ 0 & o.w, \end{cases}$$

and also

$$\bar{\mathbf{A}}(l,k)=\begin{cases}\dfrac{\beta_1 q_0 v_k}{\sum_{j=1}^{k} h_j}, & l=1,k=1,\dots,4,\\ \dfrac{\gamma_{N-1}q_N w_{N-k}}{\sum_{j=0}^{N-k-1} h_{N-j}}, & l=N-1,k=N-4,\dots,N-1,\\ 0 & o.w,\end{cases}$$

and $D_h, D_p, D_q, D_{\widehat{H}}, D_H$ are $(N-1)$-dimensional diagonal matrices, with

$D_h(k,k)=h_{k+1}-h_k, D_p(k,k)=p(t_k), D_q(k,k)=q(t_k), D_{\widehat{H}}(k,k)=\widehat{H}_k, D_H(k,k)=H_k$,

and $\mathbf{y}=[y_1,y_2,\dots,y_{N-1}]^\top, \mathbf{y}'=[y_1',y_2',\dots,y_{N-1}']^\top, \mathbf{f}=[f_1,f_2,\dots,f_{N-1}]^\top, \mathbf{r}=[r_1,0,\dots,0,r_{N-1}]^\top$, and $\widetilde{\mathbf{E}}=[\widetilde{\mathbf{E}}_1,0,\dots,0,\widetilde{\mathbf{E}}_{N-1}]^\top, \widehat{\mathbf{E}}=[\widehat{\mathbf{E}}_1,\widehat{\mathbf{E}}_2,\dots,\widehat{\mathbf{E}}_{N-1}]^\top, \overline{\mathbf{E}}=[\overline{\mathbf{E}}_1,\overline{\mathbf{E}}_2,\dots,\overline{\mathbf{E}}_{N-1}]^\top$, in which,

$$\begin{cases} r_1=\beta_1 f_0-(\varepsilon+\beta_1 p_0+\beta_1 q_0 v_0)y_0, \quad r_{N-1}=\gamma_{N-1}f_N-(\varepsilon+\gamma_{N-1}p_N+\gamma_{N-1}q_N w_0)y_N,\\ \widetilde{\mathbf{E}}_1=-\beta_1 q_0\bar{c}_0, \quad \widetilde{\mathbf{E}}_{N-1}=-\gamma_{N-1}q_N\bar{c}_N,\\ \widehat{\mathbf{E}}_k=\dfrac{d^3}{dt^3}(p(t)y'(t))|_{t=\zeta_{k,1}}+\dfrac{d^3}{dt^3}(q(t)y(t))|_{t=\zeta_{k,2}}-\dfrac{d^3}{dt^3}(f(t))|_{t=\zeta_{k,3}}, k=1,2,\dots,N-1,\\ \overline{\mathbf{E}}_k=\dfrac{d^4}{dt^4}(p(t)y'(t))|_{t=\varsigma_{k,1}}+\dfrac{d^4}{dt^4}(q(t)y(t))|_{t=\varsigma_{k,2}}-\dfrac{d^4}{dt^4}(f(t))|_{t=\varsigma_{k,3}}, k=1,2,\dots,N-1,\end{cases}$$

where $\zeta_{k,j},\varsigma_{k,j}\in(t_{k-1},t_{k+1}), k=1,\dots,N-1, j=1,2,3$. As we can observe, $\mathbf{y}$ and $\mathbf{y}'$ are the two sets of unknowns in the system of equations (7). So, in the subsequent stage, an attempt is made to approximate the unknowns $\mathbf{y}'$ by representing them in the form of $\mathbf{y}$. To achieve this, we propose a compact method that demonstrates high-order accuracy as follows:
At first, expanding y_2', y_0, y_2, and y_3, around point t_1 leads to

$$\begin{cases} y'(t_2)=y'(t_1+h_2)=y'(t_1)+h_2y''(t_1)+\frac{1}{2}h_2^2y^{(3)}(t_1)+\frac{1}{6}h_2^3y^{(4)}(t_1)+\frac{1}{4!}h_2^4y^{(5)}(\zeta_{1,4}),\\ y(t_0)=y(t_1-h_1)=y(t_1)-h_1y'(t_1)+\frac{1}{2}h_1^2y''(t_1)-\frac{1}{6}h_1^3y^{(3)}(t_1)+\frac{1}{4!}h_1^4y^{(4)}(t_1)-\frac{1}{5!}h_1^5y^{(5)}(\varsigma_{1,4}),\\ y\left(t_2\right)=y\left(t_1+h_2\right)=y\left(t_1\right)+h_2y'\left(t_1\right)+\frac{1}{2}h_2^2y''\left(t_1\right)+\frac{1}{6}h_2^3y^{(3)}\left(t_1\right)+\frac{1}{4!}h_2^4y^{(4)}(t_1)+\frac{1}{5!}h_2^5y^{(5)}(\bar{\varsigma}_{1,4}),\\ y\left(t_3\right)=y(t_1+(h_2+h_3))=y(t_1)+(h_2+h_3)y'(t_1)+\frac{1}{2}(h_2+h_3)^2y''(t_1)+\frac{1}{6}(h_2+h_3)^3y^{(3)}(t_1)\\ \qquad +\frac{1}{4!}(h_2+h_3)^4y^{(4)}(t_1)+\frac{1}{5!}(h_2+h_3)^5y^{(5)}(\varsigma_{1,5}),\end{cases}$$

such that $\bar{\varsigma}_{1,4},\zeta_{1,4},\varsigma_{1,4},\varsigma_{1,5}\in(t_0,t_3)$. Therefore, taking

$$y_1'+\mu_1y_2'=a_1y_0+b_1y_1+c_1y_2+d_1y_3+\bar{c}_1, \tag{9}$$

yields that

$$\begin{cases}\mu_1=\dfrac{h_1\left(h_2+h_3\right)}{h_3\left(h_1+h_2\right)}, a_1=-\dfrac{h_2^2\left(h_2+h_3\right)}{h_1\left(h_1+h_2\right)^2\left(h_1+h_2+h_3\right)}, b_1=\dfrac{1}{h_1}-\dfrac{2}{h_2}-\dfrac{1}{h_2+h_3},\\ c_1=-\dfrac{h_1\left(h_2+h_3\right)\left(h_1\left(h_2-2h_3\right)+h_2\left(h_2-3h_3\right)\right)}{h_2\left(h_1+h_2\right)^2h_3^2}, d_1=\dfrac{h_1h_2^2}{h_3^2\left(h_2+h_3\right)\left(h_1+h_2+h_3\right)},\\ \bar{c}_1=\dfrac{1}{5!}\left((c_1h_2y^{(5)}(\bar{\varsigma}_{1,4})-5\mu_1y^{(5)}(\zeta_{1,4}))h_2^4-a_1h_1^5y^{(5)}(\varsigma_{1,4})+d_1(h_2+h_3)^5y^{(5)}(\varsigma_{1,5})\right).\end{cases}$$

In the next step, through the expansion of the $y'_{k-1}, y'_{k+1}, y_{k-1}$, and y_{k+1}, around point $t_k, k = 2, 3, \dots, N-2$, it is deduced that,

$$\begin{cases} y'(t_{k-1}) = y'(t_k - h_k) = y'(t_k) - h_k y''(t_k) + \frac{1}{2}h_k^2 y^{(3)}(t_k) - \frac{1}{6}h_k^3 y^{(4)}(t_k) + \frac{1}{4!}h_k^4 y^{(5)}(\zeta_{k,4}), \\ y'(t_{k+1}) = y'(t_k + h_{k+1}) = y'(t_k) + h_{k+1} y''(t_k) + \frac{1}{2}h_{k+1}^2 y^{(3)}(t_k) + \frac{1}{6}h_{k+1}^3 y^{(4)}(t_k) + \frac{1}{4!}h_{k+1}^4 y^{(5)}(\bar{\zeta}_{k,4}), \\ y(t_{k-1}) = y(t_k - h_k) = y(t_k) - h_k y'(t_k) + \frac{1}{2}h_k^2 y''(t_k) - \frac{1}{6}h_k^3 y^{(3)}(t_k) + \frac{1}{4!}h_k^4 y^{(4)}(t_k) - \frac{1}{5!}h_k^5 y^{(5)}(\varsigma_{k,4}), \\ y(t_{k+1}) = y(t_k + h_{k+1}) = y(t_k) + h_{k+1} y'(t_k) + \frac{h_{k+1}^2}{2} y''(t_k) + \frac{h_{k+1}^3}{6} y^{(3)}(t_k) + \frac{h_{k+1}^4}{4!} y^{(4)}(t_k) + \frac{h_{k+1}^5}{5!} y^{(5)}(\bar{\varsigma}_{k,4}). \end{cases}$$

where $\zeta_{k,4}, \varsigma_{k,4}, \bar{\varsigma}_{k,4}, \bar{\zeta}_{k,4} \in (t_{k-1}, t_{k+1}), k = 2, \dots, N-2$. The difference equations below can be taken into consideration in this case

$$(10)\ \ \delta_k y'_{k-1} + y'_k + \mu_k y'_{k+1} = a_k y_{k-1} + b_k y_k + c_k y_{k+1} + \bar{c}_k,\ \ k = 2, \dots, N-2,$$

such that

$$\begin{cases} \delta_k = \dfrac{h_{k+1}^2}{(h_k + h_{k+1})^2}, \mu_k = \dfrac{h_k^2}{(h_k + h_{k+1})^2}, \\ a_k = -\dfrac{2h_{k+1}^2 (2h_k + h_{k+1})}{h_k (h_k + h_{k+1})^3}, b_k = 2\left(\dfrac{1}{h_k} - \dfrac{1}{h_{k+1}}\right), c_k = \dfrac{2h_k^2 (h_k + 2h_{k+1})}{h_{k+1} (h_k + h_{k+1})^3}, \\ \bar{c}_k = \dfrac{1}{5!}\left(-(5\delta_k y^{(5)}(\zeta_{k,4}) + a_k h_k y^{(5)}(\varsigma_{k,4}))h_k^4 + (c_k h_{k+1} y^{(5)}(\bar{\varsigma}_{k,4}) - 5\mu_k y^{(5)}(\bar{\zeta}_{k,4}))h_{k+1}^4\right). \end{cases}$$

Finally, by expanding $y'_{N-2}, y_{N-3}, y_{N-2}$, and y_N, around point t_{N-1} we conclude that there exist
$\zeta_{N-1,4}, \varsigma_{N-1,5}, \bar{\varsigma}_{N-1,4}, \varsigma_{N-1,4} \in (t_{N-3}, t_N)$ such that

$$\begin{cases} y'(t_{N-2}) = y'(t_{N-1} - h_{N-1}) = y'(t_{N-1}) - h_{N-1} y''(t_{N-1}) + \frac{h_{N-1}^2}{2} y^{(3)}(t_{N-1}) - \frac{h_{N-1}^3}{6} y^{(4)}(t_{N-1}) \\ \qquad + \frac{h_{N-1}^4}{4!} y^{(5)}(\zeta_{N-1,4}), \\ y(t_{N-3}) = y(t_{N-1} - (h_{N-1} + h_{N-2})) = y(t_{N-1}) - (h_{N-1} + h_{N-2}) y'(t_{N-1}) + \frac{(h_{N-1}+h_{N-2})^2}{2} y''(t_{N-1}) \\ \qquad - \frac{(h_{N-1}+h_{N-2})^3}{6} y^{(3)}(t_{N-1}) + \frac{(h_{N-1}+h_{N-2})^4}{4!} y^{(4)}(t_{N-1}) - \frac{(h_{N-1}+h_{N-2})^5}{5!} y^{(5)}(\varsigma_{N-1,5}), \\ y(t_{N-2}) = y(t_{N-1} - h_{N-1}) = y(t_{N-1}) - h_{N-1} y'(t_{N-1}) + \frac{h_{N-1}^2}{2} y''(t_{N-1}) - \frac{h_{N-1}^3}{6} y^{(3)}(t_{N-1}) \\ \qquad + \frac{h_{N-1}^4}{4!} y^{(4)}(t_{N-1}) - \frac{h_{N-1}^5}{5!} y^{(5)}(\bar{\varsigma}_{N-1,4}), \\ y(t_N) = y(t_{N-1} + h_N) = y(t_{N-1}) + h_N y'(t_{N-1}) + \frac{h_N^2}{2} y''(t_{N-1}) + \frac{h_N^3}{6} y^{(3)}(t_{N-1}) + \frac{h_N^4}{4!} y^{(4)}(t_{N-1}) \\ \qquad + \frac{h_N^5}{5!} y^{(5)}(\varsigma_{N-1,4}). \end{cases}$$

So, we can obtain the following difference equation
(11)
$$\delta_{N-1} y'_{N-2} + y'_{N-1} = c_{N-1} y_{N-3} + a_{N-1} y_{N-2} + b_{N-1} y_{N-1} + d_{N-1} y_N + \bar{c}_{N-1},$$

such that

$$
\begin{cases}
\delta_{N-1} = \dfrac{(h_{N-2}+h_{N-1})\,h_N}{h_{N-2}\,(h_{N-1}+h_N)},\ c_{N-1} = -\dfrac{h_{N-1}^2 h_N}{h_{N-2}^2\,(h_{N-2}+h_{N-1})\,(h_{N-2}+h_{N-1}+h_N)},\\
a_{N-1} = -\dfrac{h_N\,(h_{N-2}+h_{N-1})\,(h_{N-2}\,(3h_{N-1}+2h_N)-h_{N-1}\,(h_{N-1}+h_N))}{h_{N-2}^2 h_{N-1}\,(h_{N-1}+h_N)^{\,2}},\\
b_{N-1} = \dfrac{1}{h_{N-2}+h_{N-1}} - \dfrac{1}{h_N} + \dfrac{2}{h_{N-1}},\ d_{N-1} = \dfrac{h_{N-1}^2\,(h_{N-2}+h_{N-1})}{h_N\,(h_{N-1}+h_N)^{\,2}\,(h_{N-2}+h_{N-1}+h_N)},\\
\bar{c}_{N-1} = -\dfrac{1}{5!}\left(5\delta_{N-1}y^{(5)}(\zeta_{N-1,4}) - a_{N-1}h_{N-1}y^{(5)}(\bar{\varsigma}_{N-1,4})\right)h_{N-1}^4\\
\qquad\qquad + \dfrac{1}{5!}\left(-c_{N-1}(h_{N-1}+h_{N-2})^5 y^{(5)}(\varsigma_{N-1,5}) + d_{N-1}h_N^5 y^{(5)}(\varsigma_{N-1,4})\right).
\end{cases}
$$

Consequently, the proposed compact finite difference scheme (9), (10) and (11) can be represented as the following matrix form

$$
M\mathbf{y}' = B\mathbf{y} + \mathbf{r}_0 + \bar{c}, \tag{12}
$$

where M and $\mathbf{C}$ are tri/penta-diagonal matrices, respectively, with

$$
M(l,k) = \begin{cases}
1, & l = k, 1 \le k \le N-1,\\
\delta_{k+1}, & l = k+1, 1 \le k \le N-2,\\
\mu_k, & l = k-1, 2 \le k \le N-1,\\
0 & o.w,
\end{cases}
\quad
B(l,k) = \begin{cases}
d_1, & l = 1, k = 3,\\
b_k, & l = k, 1 \le k \le N-1,\\
a_k, & l = k+1, 1 \le k \le N-2,\\
c_k, & l = k-1, 2 \le k \le N-1,\\
c_{N-1}, & l = N-1, k = N-3,\\
0 & o.w,
\end{cases}
$$

and $\mathbf{r}_0 = [a_1 y_0, 0, \ldots, 0, d_{N-1}y_N]^\top, \bar{c} = [\bar{c}_1, \ldots, \bar{c}_{N-1}]^\top$.

It should be mentioned that an irreducible matrix is nonsingular [30]. Also, according to the results of the [5], all we need for irreducibility of the tridiagonal matrix (L_{ij}) is $L_{ij}L_{ji} \neq 0$ for all $j = i, i+1$. Since, δ_k, μ_k are nonzero and all diagonal elements of the matrix M are equal to 1 then we conclude that this matrix is irreducible and accordingly is nonsingular. So, from (12) $\mathbf{y}'$ can be represented in terms of $\mathbf{y}$ as follows

$$
\mathbf{y}' = \bar{B}\mathbf{y} + \bar{\mathbf{r}}_0 + \check{\mathbf{E}}, \tag{13}
$$

where, $\bar{B} = M^{-1}B, \bar{\mathbf{r}}_0 = M^{-1}\mathbf{r}_0$ and $\check{\mathbf{E}} = M^{-1}\bar{c}$. Consequently, if we put

$$
\begin{cases}
\mathbf{K} = \varepsilon\mathbf{A} + \mathbf{C}D_p + \bar{\mathbf{A}} + (\mathbf{C}D_q - \varepsilon D_h)\,\bar{B},\\
\mathbf{b} = \mathbf{C}\mathbf{f} + \mathbf{r} - (\mathbf{C}D_q - \varepsilon D_h)\,\bar{\mathbf{r}}_0,\\
\mathbf{E} = \widetilde{\mathbf{E}} + D_H\overline{\mathbf{E}} + D_{\widehat{H}}\,\widehat{\mathbf{E}} - (\mathbf{C}D_q - \varepsilon D_h)\,\check{\mathbf{E}},
\end{cases}
$$

then from (8) and (13) we can obtain

$$
\mathbf{K}\mathbf{y} = \mathbf{b} + \mathbf{E}. \tag{14}
$$

Finally, if the discrete system of equations,

$$
\mathbf{K}\mathbf{Y} = \mathbf{b}, \tag{15}
$$

solved, then the numerical solution of the SPBVP (1), denoted by

$$\mathbf{Y} = [Y_1, Y_2, \dots, Y_{N-1}]^\top$$

will be determined. The system of equations (15) describes a linear system of algebraic equations that can be solved by performing an LU factorization. It's worth noting that the above-mentioned matrix formulation is straightforward to implement.

4. **Error estimation**

Convergence properties of the proposed method will be discussed in this section. In this section, it will be exhibited that the numerical solution obtained using our method converges with order 4 concerning to L_∞ norm. As mentioned before, for small perturbation parameter ε, the layer(s) (interior or left/right/twin boundary) appear in the solution of the SPBVP (1). Therefore, to find an appropriate solution when $\varepsilon \to 0$, it is necessary to utilize small step-sizes when implementing an equidistant mesh for discretizing $\overline{\Gamma}$. So, employing an equidistant mesh leads to an increase in computational costs. Hence, we utilize a specialized non-equidistant mesh known as Bakhvalov-Shishkin-type [13–15], which has particularly a high concentration within the boundary layers. This discretization is as follows

$$t_k = \begin{cases} -\dfrac{2\varepsilon}{\lambda} \ln\left(1 - 2\left(1 - \dfrac{1}{N}\right)\dfrac{k}{N}\right), & 0 \le k \le \frac{N}{2}, \\ 1 - 2\left(1 - \dfrac{2\varepsilon}{\lambda}\ln(N)\right)\left(1 - \dfrac{k}{N}\right), & \frac{N}{2} + 1 \le k \le N. \end{cases} \tag{16}$$

In Fig. 1 the Bakhvalov-Shishkin-type mesh (16) is plotted with $\lambda = 1, N = 16$ and $\varepsilon = 10^{-l}, l = 8, \dots, 20$.

Corollary 4.1. *If Bakhvalov-Shishkin discretization* (16) *utilizes for* $\overline{\Gamma} = [0, 1]$ *then one concludes that, for* $k = 1, \dots, N$, *there exist* $\frac{k-1}{N} \le \theta_k \le \frac{k}{N}$, *such that,*

$$h_k = t_k - t_{k-1} = \begin{cases} \dfrac{2\varepsilon}{\lambda} \times \dfrac{2(1 - \frac{1}{N})}{1 - 2(1 - \frac{1}{N})\theta_k} \times \dfrac{1}{N}, & 0 \le k \le \frac{N}{2}, \\ 2\left(1 - \dfrac{2\varepsilon}{\lambda}\ln(N)\right) \times \dfrac{1}{N}, & \frac{N}{2} + 1 \le k \le N. \end{cases}$$

Hence, if we consider $\mathbf{E}_Y(N) = \mathbf{y} - \mathbf{Y}$, then from (14) and (15) we have

$$\mathbf{K}\ \mathbf{E}_Y(N) = \mathbf{E}, \tag{17}$$

and therefore, we conclude that

$$\|\mathbf{K}\ \mathbf{E}_Y(N)\|_2 = \|\mathbf{E}\|_2 . \tag{18}$$

From Lemma 1 in [1], we can infer that

$$\|\mathbf{E}_Y(N)\|_2 \leqslant \frac{\|\mathbf{E}\|_2}{\sigma_{min}(\mathbf{K})}, \tag{19}$$

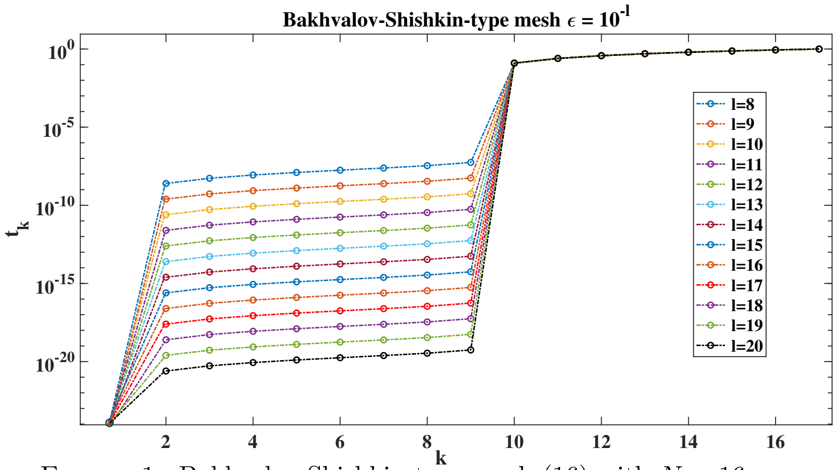


FIGURE 1. Bakhvalov-Shishkin-type mesh (16) with $N = 16$.

in which $\sigma_{min}(\mathbf{K})$ is the smallest singular value of matrix $\mathbf{K}$.
Some useful results will be given for obtaining the bound of the error terms (19).

Remark 4.2. Based on the Bakhvalov-Shishkin mesh type (16), it is possible to conclude that,

$$\begin{cases} h_k \leqslant h_{k+1}, & k = 1, \dots, \frac{N}{2} - 1, \\ \dfrac{1}{N} \leqslant h_k \leqslant \dfrac{2}{N}, & k = \frac{N}{2} + 1, \dots, N, \end{cases}$$

and $\forall m \in \mathbb{N}$,

$$h_{k+1}^m - h_k^m = \begin{cases} \mathcal{O}\left(\dfrac{1}{N^{m+1}}\right), & 1 \leq k \leq \frac{N}{2}, \\ 0, & \frac{N}{2} + 1 < k \leq N - 1. \end{cases}$$

and also $h_k = \mathcal{O}\left(\frac{1}{N}\right),\ 1 \leqslant k \leqslant N$.

Remark 4.3. Since $h_k = \mathcal{O}\left(\frac{1}{N}\right), 1 \leqslant k \leqslant N$, then we can infer that

$$\begin{cases} \alpha_k = \mathcal{O}\left(\frac{1}{N^2}\right),\ \beta_k = \mathcal{O}\left(\frac{1}{N^2}\right),\ \gamma_k = \mathcal{O}\left(\frac{1}{N^2}\right),\ k = 1, \dots, N-1, \\ D_h(k,k) = \lfloor \frac{N}{N/2+k} \rfloor \mathcal{O}\left(\frac{1}{N^2}\right),\ D_{\widehat{H}}(k,k) = \widehat{H}_k = \lfloor \frac{N}{N/2+k} \rfloor \mathcal{O}\left(\frac{1}{N^6}\right), \\ D_H(k,k) = H_k = \mathcal{O}\left(\frac{1}{N^6}\right), k = 1, \dots, N-1, \\ \widetilde{\mathbf{E}}_1 = \mathcal{O}\left(\frac{1}{N^6}\right),\ \widetilde{\mathbf{E}}_{N-1} = \mathcal{O}\left(\frac{1}{N^6}\right), \\ \bar{c}_k = \mathcal{O}\left(\frac{1}{N^4}\right), k = 0, \dots, N. \end{cases}$$

On account of Remark 4.3, it can be yielded that the components of matrices $\mathbf{C}, \bar{\mathbf{A}}$ and D_h possess order $\mathcal{O}\left(\frac{1}{N^2}\right)$, while the components of matrices $\widetilde{\mathbf{E}}, D_H$, and $D_{\widehat{H}}$ exhibit order $\mathcal{O}\left(\frac{1}{N^6}\right)$. Additionally, the components of matrix $\breve{\mathbf{E}}$ are characterized by order $\mathcal{O}\left(\frac{1}{N^4}\right)$. Hence, we can infer that $\|\mathbf{E}\|_2 = \mathcal{O}\left(\frac{1}{N}\right)^{5+\frac{1}{2}}$. On the other hand, while $h_k \to 0$ (i.e $N \to \infty$) we can easily deduce that $\sigma_{min}(\mathbf{K}) \sim \sigma_{min}(\varepsilon A)$. Since matrix A is a nonsingular positive semidefinite tridiagonal then $\sigma_{min}(\varepsilon A) = \varepsilon\,|\lambda_{min}(A)|$, where $\lambda_{min}(A)$ is the smallest eigenvalue of matrix A. Now, from Lemma 3 in [2], it can be inferred that $\lambda_{min}(A) = 4\sin^2\left(\frac{\pi}{2N}\right) = \mathcal{O}\left(\frac{1}{N^2}\right)$. So, we conclude that $\|\mathbf{E}_Y(N)\|_2 = \mathcal{O}\left(\frac{1}{N}\right)^{3+\frac{1}{2}}$. As we know the relations $\|\mathbf{E}_Y(N)\|_\infty \leqslant \|\mathbf{E}_Y(N)\|_2 \leqslant \sqrt{N}\|\mathbf{E}_Y(N)\|_\infty$ are established. Consequently, based on the equivalence of all norms in the finite-dimensional spaces we can attain

$$\|\mathbf{E}_Y(N)\|_\infty = \mathcal{O}\left(\frac{1}{N}\right)^4. \tag{20}$$

From Eq. (20), it should be noted that the proposed method has uniform convergence with respect to the perturbation parameter ε.

5. **Simulation results**

The effectiveness of the introduced method is demonstrated through the examination of several test examples in this section. In the numerical examples, for various values of $N = 2^k, k \in \mathbb{N}$, discrete maximum norm $\mathbf{E}_Y(N) = \|\mathbf{Y} - \mathbf{y}\|_\infty$ will be computed. Also, using multiple values of ε and several numbers of mesh intervals N, the methods' order of convergence will be calculated as $\log_2\left(\frac{\mathbf{E}_Y(N)}{\mathbf{E}_Y(2N)}\right)$.

Example 5.1. *Take the following SPBVP into consideration as a first example*

$$\varepsilon y''(t) + ty'(t) - 4ty(t) = f(t), \quad t \in (0,1), y(0) = 1, \quad y(1) = e^{-\frac{4}{\varepsilon}},$$

where $f(t) = -\frac{4e^{-\frac{4t}{\varepsilon}}(t(\varepsilon+1)-4)}{\varepsilon}$ *and* $y(t) = e^{-\frac{4t}{\varepsilon}}$ *is the exact solution.*

Since, $p(t) = t$ and $q(t) = -4t$ then it can be inferred that $p(t) > 0, \forall t \in (0,1)$, and therefore the left boundary layer at $t = 0$ occurs for this problem. We apply the proposed method (15) for several values of $\varepsilon = 2^{-2k}, k = 1, \ldots, 6$, and mesh intervals $N = 2^k, k = 4, \ldots, 10$, to solve this problem, numerically. The simulation results of this problem are presented in Table 1. Based on the exhibited results in this table and the calculated L_∞ errors, it can be concluded that the desired fourth order of convergence in the L_∞ norm has been attained. In addition, the table indicates that the method being discussed is effective and highly accurate when used for the SPBVP (1). Also, utilizing a mesh interval of $N = 2^{11}$ and varying perturbation parameter values for ε, the numerical solution and corresponding absolute errors derived from the present method are depicted in Fig. 2. Although a boundary layer at $t = 0$ occurs for this problem, based on the results shown in this figure, we can assert that the current method is precise and effective.

Table 1. Errors and convergence order of the proposed method for Example 5.1.

	$\varepsilon = 2^{-2}$		$\varepsilon = 2^{-4}$		$\varepsilon = 2^{-6}$		$\varepsilon = 2^{-8}$		$\varepsilon = 2^{-10}$		$\varepsilon = 2^{-12}$	
N	L_∞ error	Order	L_∞ error	Order	L_∞ error	Order	L_∞ error	Order	L_∞ error	Order	L_∞ error	Order
16	6.5322e-03	–	9.5506e-03	–	1.1923e-02	–	1.4621e-02	–	5.0359e+00	–	1.2375e+01	–
32	4.1142e-04	4.0	5.8932e-04	4.0	7.1812e-04	4.0	8.0278e-04	4.2	8.7799e-04	1.2	2.8234e-01	5.4
64	2.6175e-05	4.0	3.7255e-05	4.0	4.5296e-05	4.0	5.0385e-05	4.0	5.3451e-05	4.0	5.5676e-05	1.2
128	1.6642e-06	4.0	2.3677e-06	4.0	2.8749e-06	4.0	3.1965e-06	4.0	3.3880e-06	4.0	3.4986e-06	4.0
256	1.0524e-07	4.0	1.4970e-07	4.0	1.8174e-07	4.0	2.0205e-07	4.0	2.1411e-07	4.0	2.2106e-07	4.0
512	6.6230e-09	4.0	9.4197e-09	4.0	1.1435e-08	4.0	1.2713e-08	4.0	1.3471e-08	4.0	1.3908e-08	4.0
1024	4.1547e-10	4.0	5.9065e-10	4.0	7.1700e-10	4.0	7.9741e-10	4.0	8.4477e-10	4.0	8.7231e-10	4.0

Example 5.2. *In the second example, the following SPBVP is considered [21, 34]*

$$\varepsilon y''(t) + y'(t) - 2y(t) = -e^{t-1}, \quad t \in (0,1), \quad y(0) = 0, \quad y(1) = 0,$$

where $y(t) = \frac{e^{s^- t}(1-e^{s^+ -1}) - e^{s^+ t}(1-e^{s^- -1})}{(1-\varepsilon)(e^{s^+} - e^{s^-})} + \frac{e^{t-1}}{1-\varepsilon}$, *is the exact solution and* $s^\pm = \frac{-1 \pm \sqrt{1+8\varepsilon}}{2\varepsilon}$.

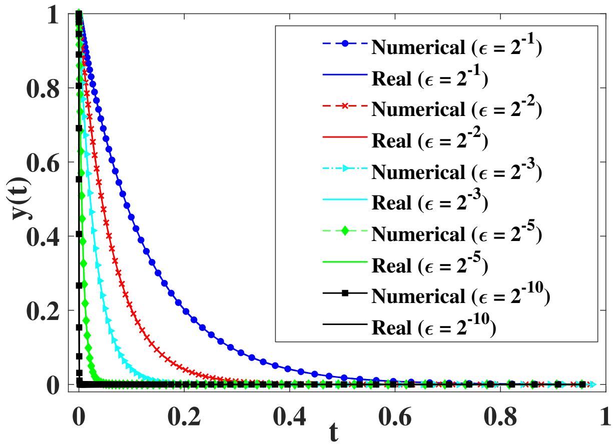
Table 2 exhibits the L_∞ errors and convergence order of the present method (15) and preconditioning methods [21,34] for $N = 2^5, 2^6, \ldots, 2^{10}$, mesh intervals and $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$. Based on the computational results outlined in this table, we can conclude that fourth-order convergence with respect to the L_∞ norm is reached by the numerical solution obtained from the proposed method. Whereas the convergence rates achieved for the other methods [34] and [21], are 1.5 and 1.6, respectively. Moreover, we can observe that when $\varepsilon = 10^{-4}$, the L_∞ error associated with the present method is 9.99×10^{-10}, while for methods [34] and [21], the corresponding error is 6.59×10^{-4} and 3.15×10^{-5}, respectively. As a result, the present technique demonstrates superior accuracy compared to the methods provided by [34] and [21].

Example 5.3. *Consider the following SPBVP [16]*

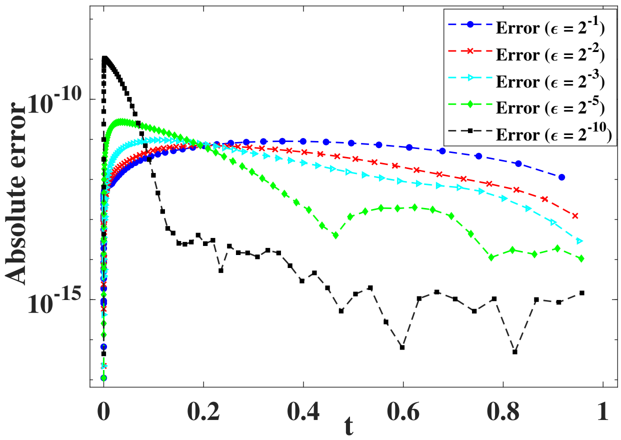
$$\varepsilon y''(t) + t y'(t) = -\varepsilon \pi^2 \cos(\pi t) - \pi t \sin(\pi t), \quad t \in (0,1), \quad y(0) = 1, \quad y(1) = 0.$$

For this problem $y(t) = \cos(\pi t) + \frac{efr\left(\frac{1}{\sqrt{2\varepsilon}} t\right)}{efr\left(\frac{1}{\sqrt{2\varepsilon}}\right)}$ *is the exact solution and* efr *refers to the error function.*

For this example, the computational results of the presented method and Legendre neural network method [16] with $\varepsilon = 10^{-3}$ and $N = 10^2$, for knots $t_k = \frac{k}{10}, k = 0, 1, \ldots, 9$, are shown in Table 3. It can be observed for the interior points that, the maximum error obtained by the Legendre neural network method [16] is of order $\mathcal{O}(10^{-6})$ while the present method is $\mathcal{O}(10^{-13})$. Hence, the current technique demonstrates more accuracy in comparison to the Legendre neural network method [16].



(a) The plots of the exact and numerical solutions.



(b) The log plots of the absolute errors.

FIGURE 2. Numerical results of the proposed method (15) to solve the Example 5.1 with $N = 2^{11}$.

Example 5.4. *As a final example, consider the following SPBVP [29]*

$$\varepsilon y''(t) + y'(t) - y(t) = 0, \quad t \in (0,1), \quad y(0) = 1, \quad y(1) = 1,$$

where $y(t) = \frac{e^{m_1 t}(e^{m_2}-1)+e^{m_2 t}(1-e^{m_1})}{e^{m_2}-e^{m_1}}$, *such that* $m_{1,2} = \frac{-1\pm\sqrt{1+4\varepsilon}}{2\varepsilon}$.

Here, the computational results of two techniques, the present technique (15) and the non-standard fitted operator method [29] are exhibited in Table 4 while $[0,1]$ is broken up into $N = 2^k, k = 5, 6, \ldots, 10$, mesh intervals. It can be concluded that the accuracy order of the proposed technique and the method provided by [29] is 4 and one in the L_∞ norm, respectively. Furthermore, in

Table 2. Maximum norm of the errors and order of the methods for Example 5.2.

		$\varepsilon = 10^{-1}$		$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-4}$	
	N	L_∞ error	Order	L_∞error	Order	L_∞ error	Order	L_∞ error	Order
Method of [34]:	32	1.56e-02	–	5.30e-02	–	4.74e-02	–	4.69e-02	–
	64	4.66e-03	1.7	3.01e-02	0.8	2.62e-02	0.8	2.59e-02	0.8
	128	1.73e-03	1.4	1.44e-02	1.1	1.22e-02	1.1	1.20e-02	1.1
	256	8.64e-04	1.0	6.06e-03	1.2	5.02e-03	1.3	4.94e-03	1.3
	512	4.32e-04	1.0	2.34e-03	1.4	1.90e-03	1.4	1.86e-03	1.4
	1024	2.16e-04	1.0	8.53e-04	1.4	6.73e-04	1.5	6.59e-04	1.5
Method of [21]:	32	6.88e-03	–	5.01e-03	–	4.95e-03	–	4.94e-03	–
	64	3.45e-03	1.0	2.22e-03	1.2	2.19e-03	1.2	2.18e-03	1.2
	128	1.73e-03	1.0	8.65e-04	1.3	8.49e-04	1.4	8.47e-04	1.4
	256	8.64e-04	1.0	3.07e-04	1.5	3.00e-04	1.5	3.00e-04	1.5
	512	4.32e-04	1.0	1.09e-04	1.5	9.96e-05	1.6	9.94e-05	1.6
	1024	2.16e-04	1.0	5.19e-05	1.1	3.16e-05	1.6	3.15e-05	1.6
Present method:	32	1.13e-04	–	6.78e-04	–	1.00e-03	–	1.5e-03	–
	64	1.87e-05	2.6	4.62e-05	3.9	1.09e-05	6.5	4.68e-05	5.0
	128	1.33e-06	3.8	3.25e-06	3.8	7.89e-07	3.9	3.58e-06	3.7
	256	7.98e-08	4.0	2.04e-07	4.0	5.08e-08	4.0	2.42e-07	3.9
	512	5.01e-09	4.0	1.27e-08	4.0	3.46e-09	3.9	1.57e-08	4.0
	1024	3.02e-10	4.0	7.95e-10	4.0	2.16e-10	4.0	9.99e-10	4.0

Table 3. Maximum norm of the errors of the methods for Example 5.3.

t	Method [16] ($\varepsilon = 0.001$ and $N = 100$ nodes)	Present method ($\varepsilon = 0.001$ and $N = 100$ nodes)
0	8.4657e-07	0
0.1	1.9697e-06	4.5063e-08
0.2	6.6230e-06	1.4615e-09
0.3	5.6222e-06	1.3941e-09
0.4	6.9552e-06	1.1928e-09
0.5	7.8209e-06	7.1508e-10
0.6	4.5348e-06	2.598e-10
0.7	2.4616e-06	6.5609e-11
0.8	9.2330e-06	8.7768e-12
0.9	9.0752e-06	2.8193e-13

the case $N = 2^{10}$ mesh intervals and $\varepsilon = 2^{-10}$, the L_∞ errors of the proposed method and non-standard fitted operator method [29] are of order $\mathcal{O}(10^{-8})$ and $\mathcal{O}(10^{-3})$, respectively. So, the present technique is much more accurate than method [29].

6. Conclusions

In this article, some numerical techniques were used to approximate the solution of the SPBVPs. Because of the small perturbation parameter $0 < \varepsilon \ll 1$ and boundary layers we utilize a non-equidistant partition named Bakhvalov-Shishkin-type. We employed some appropriate techniques such as Lagrange

TABLE 4. Maximum norm of the errors and order of the methods for Example 5.4.

		$\varepsilon = 2^{-5}$		$\varepsilon = 2^{-6}$		$\varepsilon = 2^{-7}$		$\varepsilon = 2^{-8}$		$\varepsilon = 2^{-9}$		$\varepsilon = 2^{-10}$	
	N	L_∞ error	Order	L_∞error	Order	L_∞ error	Order	L_∞ error	Order	L_∞ error	Order	L_∞ error	Order
Method of [29]:	2^5	4.72e-02	–	3.48e-02	–	3.06e-02	–	2.99e-02	–	2.99e-02		2.98e-02	
	2^6	3.83e-02	0.30	2.42e-02	0.52	1.77e-02	0.79	1.56e-02	0.94	1.53e-02	0.97	1.53e-02	0.97
	2^7	3.42e-02	0.16	1.95e-02	0.31	1.22e-02	0.54	8.95e-03	0.80	7.88e-03	0.96	7.73e-03	0.98
	2^8	3.22e-02	0.09	1.74e-02	0.17	9.84e-03	0.31	6.14e-03	0.54	4.50e-03	0.81	3.96e-03	0.96
	2^9	3.13e-02	0.04	1.64e-02	0.09	8.76e-03	0.17	4.94e-03	0.31	3.08e-03	0.54	2.25e-03	0.81
	2^{10}	3.08e-02	0.02	1.59e-02	0.04	8.24e-03	0.09	4.40e-03	0.17	2.48e-03	0.31	1.54e-03	0.55
Present method:	2^5	3.76e-04	–	7.36e-04	–	1.48e-03	–	2.91e-03	–	5.81e-03	–	3.99e-02	–
	2^6	2.35e-05	4.0	4.64e-05	4.0	9.27e-05	4.0	1.86e-04	4.0	3.73e-04	4.0	7.46e-04	5.7
	2^7	1.47e-06	4.0	2.90e-06	4.0	5.81e-06	4.0	1.16e-05	4.0	2.33e-05	4.0	4.68e-05	4.0
	2^8	9.18e-08	4.0	1.81e-07	4.0	3.63e-07	4.0	7.29e-07	4.0	1.46e-06	4.0	2.93e-06	4.0
	2^9	5.74e-09	4.0	1.13e-08	4.0	2.27e-08	4.0	4.56e-08	4.0	9.14e-08	4.0	1.83e-07	4.0
	2^{10}	3.59e-10	4.0	7.11e-10	4.0	1.42e-09	4.0	2.85e-09	4.0	5.71e-09	4.0	1.14e-08	4.0

second-degree interpolating, and then constructed some fourth-order compact approaches to discretize the corresponding SPBVP. Subsequently, discretizing the original SPBVP leads to a linear algebraic system. By determining the truncation errors and using certain matrix calculations, it is proven that the current method has fourth-order accuracy in the L_∞ norm when applied to SPBVP (1). The performance of the suggested approach is demonstrated through comparative test examples that have been recently examined using other methods.

Appendix

Proof of Remark 2.3

Assume that there are constants $\lambda < 0$ and $\eta > 0$ such that the relations $p(t) \leq \lambda$ and $q(t) \leq -\eta$ are fulfilled. Also, assume that for the smooth function $w(t)$ we have $w(0) \leq 0$ and $w(1) \leq 0$, and $L_w(t) \geq 0, \forall t \in \Gamma$. Then we want to prove that

$$\|y\|_\infty \leq \left(\frac{1}{\eta}\|f\|_\infty + \max\{|\tau_0|, |\tau_1|\}\right), \quad \forall t \in \overline{\Gamma}.$$

Initially, we demonstrate that $w(t) \leq 0, \forall t \in \overline{\Gamma}$. In the same way as Lemma 2.1, it is sufficient to assume that $w(s) = \max_{t\in[0,1]} w(t) > 0$ in which $s \in [0,1]$. Therefore, we infer that $s \notin \{0,1\}, w'(s) = 0, w''(s) \leq 0$ and then

$$L_w(s) = \varepsilon w''(s) + p(s)w'(s) + q(s)w(s) = \varepsilon w''(s) + q(s)w(s) < 0,$$

that contradicts assumptions. So, we can attain $w(t) \geq 0, \forall t \in \overline{\Gamma}$. Now, it is enough to define

$$w^{\pm}(t) = -\frac{1}{\eta}\|f\|_\infty - \max\{|\tau_0|, |\tau_1|\} \pm y(t),$$

and then apply Lemma 2.2 under consideration Remark 2.3.

Proof of Lemma 3.1
Taking $L_2(t)$ as Lagrange second degree interpolating polynomial at nodes t_{k-1}, t_k, t_{k+1} for function $g(t) \in C^4[0,1]$ yields that

$$\begin{cases} g(t) = \sum_{i=0}^{2} L_i(t) g(t_{k-1+i}) + l_2(t) \dfrac{g^{'''}(\xi)}{6}, & \xi \in (t_{k-1}, t_{k+1}), \\ l_2(t) = \prod_{i=0}^{2} (t - t_{k-1+i}), \quad L_i(t) = \prod_{\substack{j=0 \\ j \neq i}}^{2} \left(\dfrac{t - t_{k-1+j}}{t_{k-1+i} - t_{k-1+j}} \right), & i = 0,1,2. \end{cases}$$

Putting the above relationships on the left-hand side of the equation (5) results in:

$$\begin{cases} \displaystyle\int_{t_k}^{t_{k+1}} (t_{k+1} - t) g(t) dt = \sum_{i=0}^{2} \left(g(t_{k-1+i}) \int_{t_k}^{t_{k+1}} (t_{k+1} - t) L_i(t) dt \right) \\ \qquad\qquad + \displaystyle\int_{t_k}^{t_{k+1}} \left((t_{k+1} - t) l_2(t) \frac{g^{'''}(\xi)}{6} \right) dt \\ \displaystyle\int_{t_{k-1}}^{t_k} (t_{k-1} - t) g(t) dt = \sum_{i=0}^{2} \left(g(t_{k-1+i}) \int_{t_{k-1}}^{t_k} (t_{k-1} - t) L_i(t) \right) dt \\ \qquad\qquad + \displaystyle\int_{t_{k-1}}^{t_k} \left((t_{k-1} - t) l_2(t) \frac{g^{'''}(\xi)}{6} \right) dt. \end{cases}$$

Since, the functions $(t_{k+1} - t) l_2(t)$ and $(t_{k-1} - t) l_2(t)$ have no change of sign in intervals (t_k, t_{k+1}) and (t_{k-1}, t_k), respectively, then we conclude there exist $\bar{\zeta}_k \in (t_k, t_{k+1}), \bar{\varsigma}_k \in (t_{k-1}, t_k)$ such that

$$\begin{aligned} \int_{t_k}^{t_{k+1}} (t_{k+1} - t) g(t) dt - \int_{t_{k-1}}^{t_k} (t_{k-1} - t) g(t) dt &= \beta_k g_{k-1} + \alpha_k g_k + \gamma_k g_{k+1} \\ &- \frac{1}{360} \left(g^{'''}(\bar{\zeta}_k) h_k^4 (2h_k + 5h_{k+1}) - g^{'''}(\bar{\varsigma}_k) h_{k+1}^4 (2h_{k+1} + 5h_k) \right), \end{aligned}$$

where the coefficients β_k, α_k and γ_k are given in Lemma 3.1. It should be pointed out that $\bar{\varsigma}_k < \bar{\zeta}_k$, so, taking $\hbar_k = \frac{\bar{\zeta}_k - \bar{\varsigma}_k}{h_k + h_{k+1}}$ yields that $0 < \hbar_k < 1$ and $g^{'''}(\bar{\zeta}_k) - g^{'''}(\bar{\varsigma}_k) = \hbar_k (h_k + h_{k+1}) g^{(4)}(\varsigma_k)$ in which $\varsigma_k \in (t_{k-1}, t_{k+1})$. Utilizing the last relation, Eq. (6) will be obtained.

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